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May 2018

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CESifo

Impressum:

CESifo Working Papers
ISSN 2364-1428 (electronic version)
Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH
The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute
Poschingerstr. 5, 81679 Munich, Germany
Telephone +49 (0)89 2180-2740, Telefax +49 (0)89 2180-17845, email office@cesifo.de
Editors: Clemens Fuest, Oliver Falck, Jasmin Gröschl
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CESifo *Working Paper No. 7062*
Category 1: Public Finance

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Abstract

This study shows the uniqueness of Nash equilibrium in the model of multiple voluntarily supplied public goods with potential contributors possessing different Cobb-Douglas preferences. This study provides a sufficient condition for uniqueness using graph theory. This sufficient condition allows us to use the replacement function approach of Cornes and Hartley (2007) not only to develop an algorithm for identifying free riders, but also to provide an alternative proof for the uniqueness of a Nash equilibrium in multiple public goods models.

JEL-Codes: H410, F130, D010.

Keywords: public good, voluntary provision, uniqueness, aggregate game, Nash equilibrium, algorithm.

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May 7, 2018
An early version of this paper was presented in seminars at Hokkaido University and the Japan Society of International Economics Spring Meeting (2012). We are very grateful to Yoshihiro Suga, Yoshihro Nishimura, and other participants for their helpful comments. The research of the second author was supported by Grant-in-Aid 23530367 from the Ministry of Education, Culture, Sport, Science and Technology of Japan. The usual disclaimer applies.

1 Introduction

We find many examples from day-to-day life in which individuals or groups contribute simultaneously to more than one pure public good. The voluntary contributions of individuals are simultaneously made to several charitable trusts or non-governmental organization (NGOs). Families simultaneously make voluntary contributions to multiple household public goods, such as caring for the old, children, and the sick, housework, and gardening. It can also be the case at the macro level whereby national governments choose to allocate their budgets toward provision of several national public goods, such as public healthcare, infrastructure, as well as several international public goods, including global pollution abatement, military alliances, and foreign aid.

Nevertheless, there are few explicit analyses of voluntary contribution models with many public goods. Notable exceptions are Kemp (1984), Bergstrom, Blume, and Varian (1986), Cornes and Schweinberger (1996), and Cornes and Itaya (2010). Kemp (1984) establishes a neutrality proposition on the assumption that all players are priori positive contributors to every public good. Bergstrom, Blume, and Varian (1986) note that this assumption is problematic. They establish equilibrium existence in the presence of many public goods, and present a neutrality proposition. Cornes and Schweinberger (1996) simultaneously develop several extensions of the multiple public provision model, making it difficult to pinpoint the implications of assuming many public goods. Cornes and Itaya (2010) have also shown that in the multiple public provision model, voluntary contribution equilibrium typically generates, not only a level of public good provision that is too low, but also the wrong mix of public goods in such a way that variations of the existing combination of public goods lead to Pareto improvement.

In this study, we are interested in a more fundamental question: *when there is more than one public good, under what condition is a Nash equilibrium unique*? We first provide a sufficient condition for a unique equilibrium in the presence of many heterogeneous potential contributors and many public goods under specific preferences. In the standard voluntary provision model with a *single* public good, several researchers have explored the conditions under which a Nash equilibrium is unique. In particular, the existence of a Nash equilibrium is invariably established with the assistance of Brower's fixed-point theorem, and its uniqueness is proved using various transformations of the best response functions of individual contributors (e.g., Bergstrom et al., 1986; Bergstrom, Blume, and Varian, 1992; Fraser, 1992; Glazer and Konrad, 1993; Nett and Wolfgang, 1993; Andreoni and Bergstrom, 1996). Furthermore, Cornes, Hartley, and Sandler (1999) employ the concept of contraction mapping, and shown that if private and public goods are both normal goods, the existence and uniqueness of equilibrium are simultaneously guaranteed. More recently, Cornes and Hartley (2005, 2007) exploit the aggregative structure of the public good provision model together with the replacement function approach to prove the existence of a unique Nash equilibrium. By conditioning every player's behavior on a common aggregate, rather than conditioning each player's behavior on that of all others, their study avoids many dimensions that are associated with the use of many best-reaction functions associated with many heterogeneous players involved in that game. The use of a single aggregate for multiple-player games provides a transparent representation of Nash equilibrium in terms of a function from the real line to the real line, compared with the multi-dimensional best-reaction functions employed in the standard literature mentioned above, lending itself to a simple graphical treatment.

Although the replacement function is very useful and powerful (e.g., Cornes and Hartley, 2005, 2007, 2012; Kotchen, 2007), this approach is no longer valid in the presence of *multiple public goods*. This is mainly because the contribution function of each individual to public goods might not be point-valued so that it might become a correspondence rather than a function, which ends up preventing application of the replacement function approach of Cornes and Hartley (2005, 2007, 2012) as well as the intermediate value theorem to prove the existence of a unique Nash equilibrium in the non-cooperative voluntary provision game with multiple public goods. Although it would be possible still to prove the only equilibrium existence using Kakutani's fixed-point theorem, even under such a replacement correspondence (more precisely, an upper-hemi continuous correspondence), the use of the replacement correspondence does not in general guarantee uniqueness, thereby losing the graphical transparency in demonstrating the existence of a unique Nash equilibrium as well as predictive power of comparative statics analysis (see Cornes and Hartley, 2005, 2007, 2012).

The novelty of this study is that it identifies a **sufficient condition** to ensure the uniqueness of a Nash equilibrium in multiple public good provision games with Cobb–Douglas preferences using *graph theory*. Although the results might be limited because of the use of specific utility functions, we find not only that the normality assumption is not sufficient to ensure the uniqueness of a Nash equilibrium in multiple public good provision models, but also that the non-unique Nash equilibrium is **not generic**, at least under Cobb–Douglas preferences, in the sense that this non-unique property vanishes when perturbing just slightly the parameters of preferences or wealth. More importantly, the sufficient condition for uniqueness we find transforms the mapping from the space of public goods to each individual's contribution to a function (which is called a *replacement function*) under *heterogeneous* Cobb–Douglas preferences across individuals. This transformation enables us to use the replacement function approach in single voluntarily provided public good models that has a variety of potential applications (see Cornes and Hartley, 2005, 2007, 2012). The replacement function approach allows us not only to provide an alternative proof for the uniqueness of a Nash equilibrium, but also to conduct a comparative static analysis *even in the multiple public good provision model with arbitrary finite numbers of individuals.* In this study, we focus on how to identify free riders in multiple public goods models. Bergstrom et al. (1986), Andreoni and McGuire (1993), Shrestha and Cheong (2007), and Yildirim (2014) develop an algorithm that can identify who is a non-contributor (i.e., free rider) or a contributor in a Nash equilibrium of the voluntary provision model given knowledge of the incomes and preferences of all individuals. These authors' algorithms are valid only in the model of a **single** voluntarily supplied public good. By contrast, with the help of the replacement function coupled with the sufficient condition for uniqueness, we can provide an alternative algorithm that makes it possible to identify free riders in multiple public goods models with heterogeneous Cobb–Douglas preferences.

The reminder of the paper is organized as follows. Section 2 outlines the model. Section 3 proves the existence of a Nash equilibrium and provides a a sufficient condition for the uniqueness of equilibrium. Section 4 provides an algorithm for identifying free riders and provides an alternative proof for the existence of a unique Nash equilibrium. Section 5 concludes and provides possible extensions of the model. Mathematical details are relegated to the appendix.

2 The model

Consider a model in which there are two types of pure public goods, one private good, and n individuals. Individual i consumes the private good and voluntarily supplies the public goods. The total supply of these two types of public goods, denoted by G and H, is the sum of voluntary contributions, g_i and h_i, respectively, provided by individual i. Individual i's preferences are represented by the Cobb–Douglas utility function:

$$U_i\left(x_i, G, H\right) \equiv x_i^{\alpha_i} G^{\beta_i} H^{\gamma_i}, \quad i = 1, 2, ..., n, \tag{1}$$

where $x_i \geq 0$ is individual i's consumption of the private good. We assume that $(\alpha_i, \beta_i, \gamma_i) > 0$ and $\alpha_i + \beta_i + \gamma_i = 1$.

Individual i's budget constraint is expressed by

$$x_i + p_i g_i + q_i h_i = w_i, \tag{2}$$

where $w_i > 0$ is the exogenously given income of individual i, and $p_i > 0$ and $q_i > 0$ stand for the relative prices (unit costs of production) of the public goods G and H, respectively, relative to the (numeraire) private good. Under a linear production frontier, as in Bergstrom et al. (1986), low (high) p_i and q_i reflect high (low) marginal costs in producing the public goods or taxes.

For later reference, it is convenient to define the sum of supply provided by all individuals except i by

$$G_{-i} \equiv \sum_{j \neq i} g_j = G - g_i, \tag{3}$$

$$H_{-i} \equiv \sum_{j \neq i} h_j = H - h_i. \tag{4}$$

When individual i contributes to neither of the two public goods, that is, $g_i = h_i = 0$, he or she is called a *non-contributor*. When he or she makes a positive contribution to either (or both) of the public goods, he or she is called a *contributor*. Accordingly, each contributor belongs to one of the following three sets of individuals:

$$\begin{aligned} C^G &\equiv \{ i \mid g_i > 0, h_i = 0 \} \\ C^H &\equiv \{ i \mid g_i = 0, h_i > 0 \}, \text{ and} \\ C^{Both} &\equiv \{ i \mid g_i > 0, h_i > 0 \}. \end{aligned} \tag{5}$$

Denote C^N as the set of non-contributors. Note that these four sets C^N, C^G, C^H, and C^{Both} are mutually *exclusive.*

Individual i maximizes (1) by his or her choice of x_i, g_i and h_i subject to budget constraint (2) and the non-negativity constraints $x_i \geq 0$, $g_i \geq 0$ and $h_i \geq 0$, given the contributions G_{-i} and H_{-i} of the others, with the assumption that they will be unaffected by their own choices. Hence, a Nash equilibrium of the corresponding contribution game played by n individuals is defined as follows.

Definition 1 *A Nash equilibrium in this model is a collection of strategies* $\{ (g_i, h_i) \mid i = 1, 2, ..., n \}$ *such that* (g_i, h_i) *is a solution for the following problem for all* i*:*

$$\begin{aligned} \max_{x_i, g_i, h_i} \; & U_i \left(w_i - p_i g_i - q_i h_i, g_i + G_{-i}, h_i + H_{-i} \right) \\ s.t. \; & x_i \geq 0, \; g_i \geq 0, \; h_i \geq 0. \end{aligned} \tag{6}$$

Note that the budget constraint (2) always holds with equality for a solution by virtue of a strictly increasing function of utility with respect to every element; in addition, there is no corner solution for private consumption (i.e., $x_i = w_i - p_i g_i - p_i g_i > 0$) owing to Cobb–Douglas preferences (1).[1]

3 Individual i's Optimal Choice

The strategy of proofs for the existence and uniqueness of a Nash equilibrium in the multiple public goods model is as follows. In Section 3, we focus on an economy with *two* individuals and *two* public goods for illustrative purposes. Then, we construct a map from $(G,\ H)$ to (g_1, g_2, h_1, h_2) as well as an inverse map under the preferences (1) defined in the previous section. In Subsection 4.1 we prove the existence of the fixed point $(G^1, G^2, ..., G^m) \in R_+^m$ and of a Nash equilibrium $(g_1^1, g_2^1, ..., g_n^1, g_1^2, g_2^2, ..., g_n^2, ..., g_1^m, g_2^m, ..., g_n^m) \in R_+^{n\times m}$ in an economy with n potential contributors and m public goods under the generalized Cobb–Douglas utility function (30). In Subsection 4.2, we provide a sufficient condition for the replacement correspondence to be a one-to-one mapping (i.e., a replacement function) from $(G^1, G^2, ..., G^m) \in R_+^m$ to $(g_1^1, g_2^1, ..., g_n^1, g_1^2, g_2^2, ..., g_n^2, ..., g_1^m, g_2^m, ..., g_n^m) \in R_+^{n\times m}$, and then the uniqueness of the profile $(G^1, G^2, ..., G^m) \in R_+^m$, which together imply the uniqueness of a Nash equilibrium.

To solve the problem (6), we construct the Lagrangian expression associated with (6):

$$L\left(g_i, h_i, \xi_i, \zeta_i\right) \equiv U_i\left(w_i - p_i g_i - q_i h_i,\ g_i + G_{-i},\ h_i + H_{-i}\right) + \xi_i g_i + \zeta_i h_i. \quad (7)$$

The Lagrangian multipliers (ξ_i, ζ_i) satisfy the following *Kuhn–Tucker* condition:

$$g_i \geq 0,\ \xi_i g_i = 0,\ \xi_i \geq 0,\ h_i \geq 0,\ \zeta_i h_i = 0,\ \zeta_i \geq 0, \quad (8)$$

$$\frac{\partial U_i\left(g_i, h_i, G_{-i}, H_{-i}\right)}{\partial g_i} + \xi_i = 0,\ \frac{\partial U_i\left(g_i, h_i, G_{-i}, H_{-i}\right)}{\partial h_i} + \zeta_i = 0. \quad (9)$$

[1]The assumption of Cobb–Douglas preferences rules out the possibility that the two individuals' expansion paths for the contribution function g and h intersect. Consequently, the results we obtain here are valid over the whole (G,H) space. In addition, Cobb–Douglas utility functions are weakly separable. Consequently, individual i's indifference map in (G, H) space can be drawn independently of the precise realized value of x_i and is homothetic. These features greatly simplify the subsequent analysis without making preferences unusual or idiosyncratic in any relevant respect.

Using (1), (9) can be rewritten as

$$-p_i\alpha_i x_i^{\alpha_i-1}G^{\beta_i}H^{\gamma_i} + \beta_i x_i^{\alpha_i}G^{\beta_i-1}H^{\gamma_i} + \xi_i = 0, \tag{10}$$

$$-q_i\alpha_i x_i^{\alpha_i-1}G^{\beta_i}H^{\gamma_i} + \gamma_i x_i^{\alpha_i}G^{\beta_i}H^{\gamma_i-1} + \zeta_i = 0. \tag{11}$$

Next, we divide (10) and (11) by $\beta_i x_i^{\alpha_i-1}G^{\beta_i-1}H^{\gamma_i}$ and $\gamma_i x_i^{\alpha_i-1}G^{\beta_i}H^{\gamma_i-1}$, respectively, to yield

$$-\frac{p_i\alpha_i}{\beta_i}G + x_i + \frac{\xi_i}{\beta_i x_i^{\alpha_i-1}G^{\beta_i-1}H^{\gamma_i}} = 0, \tag{12}$$

$$-\frac{q_i\alpha_i}{\gamma_i}H + x_i + \frac{\zeta_i}{\gamma_i x_i^{\alpha_i-1}G^{\beta_i}H^{\gamma_i-1}} = 0. \tag{13}$$

By substituting (12) and (13) into x_i in the budget constraint (2) and denoting $\lambda_i^1 \equiv \xi_i/\beta_i x_i^{\alpha_i-1}G^{\beta_i-1}H^{\gamma_i}$ and $\lambda_i^2 \equiv \zeta_i/\gamma_i x_i^{\alpha_i-1}G^{\beta_i}H^{\gamma_i-1}$, we finally arrive at the following *Kuhn–Tucker* conditions (8) and (9):

$$g_i \geq 0,\ \lambda_i^1 g_i = 0,\ \lambda_i^1 \geq 0, \tag{14}$$

$$h_i \geq 0,\ \lambda_i^2 h_i = 0,\ \lambda_i^2 \geq 0, \tag{15}$$

$$p_i g_i + q_i h_i - \lambda_i^1 = w_i - \frac{\alpha_i}{\beta_i}p_i G, \tag{16}$$

$$p_i g_i + q_i h_i - \lambda_i^2 = w_i - \frac{\alpha_i}{\gamma_i}q_i H. \tag{17}$$

To construct a correspondence from $(G,\ H)$ to $(g_i,\ h_i)$ using (14)–(17), we first consider the given values of G and H that satisfy

$$0 \leq G < \frac{\beta_i w_i}{\alpha_i p_i}, \tag{18}$$

$$\frac{H}{G} > \frac{p_i\gamma_i}{q_i\beta_i} \quad (\equiv \pi_i). \tag{19}$$

To facilitate the exposition, the right-hand side of (19) is often denoted by π_i.[2]

Using (16), (18), and (19), we can show that the following inequalities hold:

$$\frac{p_i g_i + q_i h_i - \lambda_i^1}{p_i} = \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i}G > 0, \tag{20}$$

$$\frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i}G > \frac{w_i}{p_i} - \frac{\alpha_i q_i}{\gamma_i p_i}H. \tag{21}$$

[2] The ratio π_i can be interpreted as individual i's *marginal rate of substitution* between the two public goods weighted by their prices.

Inequalities (20) and (21) together imply $g_i > 0$ and $h_i = 0$. The reason for $h_i = 0$ is as follows. Suppose that $h_i > 0$, which implies that complementarity condition (15) leads to $\lambda_i^2 = 0$. Since $\lambda_i^2 = 0$, combining (16) with (17) yields

$$w_i - \frac{\alpha_i}{\gamma_i} q_i H - \lambda_i^1 = w_i - \frac{\alpha_i}{\beta_i} p_i G.$$

Recall $\lambda_i^1 \geq 0$, the above expression can be rewritten as

$$\frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G \leq \frac{w_i}{p_i} - \frac{\alpha_i q_i}{\gamma_i p_i} H,$$

which clearly contradicts (21).

Next, consider the values of G and H satisfying

$$0 \leq G < \frac{\beta_i w_i}{\alpha_i p_i} \text{ and } \frac{H}{G} = \frac{p_i \gamma_i}{q_i \beta_i}.$$

The latter condition implies that the right-hand sides of (16) and (17) coincide. Then, either $\lambda_i^1 = \lambda_i^2 = 0$ or $\lambda_i^1 = \lambda_i^2 > 0$ holds in (16) and (17). However, if $\lambda_i^1 = \lambda_i^2 > 0$, $g_i = h_i = 0$ owing to (14) and (15). This result violates the positivity of the right-hand side of (16) by virtue of the hypothesis $G < \beta_i w_i / \alpha_i p_i$. Hence, $\lambda_i^1 = \lambda_i^2 = 0$ and so $g_i \geq 0$ and $h_i \geq 0$, but $g_i = h_i = 0$ never occurs simultaneously.

Finally, consider the last case:

$$0 \leq G < \frac{\beta_i w_i}{\alpha_i p_i} \text{ and } \frac{H}{G} < \frac{p_i \gamma_i}{q_i \beta_i}.$$

Similarly, we can demonstrate that $g_i = 0$ and $h_i > 0$.

Taken together, we can illustrate these conditions as four regions in the (G, H)-plane of Fig.1. Depending on given values of (G, H), the contribution profile of individual i, (g_i, h_i), is characterized as follows:[3]

Region I $0 \leq G < \frac{\beta_i w_i}{\alpha_i p_i}$ and $H = \frac{\gamma_i p_i}{\beta_i q_i} G = \pi_i G$, that is, $\lambda_i^1 = \lambda_i^2 = 0$. As a result,

$$p_i g_i + q_i h_i = w_i - \frac{\alpha_i}{\beta_i} p_i G = w_i - \frac{\alpha_i}{\gamma_i} q_i H > 0, \quad i \in C^{Both}. \tag{22}$$

[3]The equality $\frac{\partial U_i / \partial H}{\partial U_i / \partial G} = \frac{q_i}{p_i}$ may hold even when individual i contributes to **only one** of the goods, because the coincidence between the tangency between the indifference curve and the slope of the budget line occurs at the point where the quantity of the other good is zero.

Region II $0 \leq G < \frac{\beta_i w_i}{\alpha_i p_i}$ and $H > \pi_i G$, that is, $\lambda_i^1 = 0$, $\lambda_i^2 \geq 0$. As a result,

$$g_i = \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G > 0, \; h_i = 0, \quad i \in C^G. \tag{23}$$

Region III $0 \leq H < \frac{\gamma_i w_i}{\alpha_i q_i}$ and $H < \pi_i G$, that is, $\lambda_i^1 \geq 0$, $\lambda_i^2 = 0$. As a result,

$$g_i = 0, \; h_i = \frac{w_i}{q_i} - \frac{\alpha_i}{\gamma_i} H > 0, \quad i \in C^H. \tag{24}$$

Region IV $(G, H) \geq (\beta_i w_i/\alpha_i p_i, \gamma_i w_i/\alpha_i q_i)$, that is, $\lambda_i^1 > 0$, $\lambda_i^2 > 0$. As a result,

$$g_i = h_i = 0, \quad i \in C^N. \tag{25}$$

As illustrated in Fig.1, the (G, H)-plain is divided into Regions II, III, and IV for each i, while the segment OM_i with a thick line indicates Region I, where an individual may simultaneously make positive contributions to both public goods. The location of point $M_i = (\beta_i w_i/\alpha_i p_i,\ \gamma_i w_i/\alpha_i q_i)$ reveals a potential tendency for which type of contributor individual i would be, depending on the location of given values of G and H in Fig.1. When individual i's wealth w_i is small, when his or her preferences for private consumption relative to the public goods are stronger (i.e., α_i becomes larger), or when their productivity ($1/p_i$,$1/q_i$) of public goods production is lower, individual i is induced to be a non-contributor. Graphically, the closer the point M_i gets to the origin, the wider is the area where individual i becomes a non-contributor (i.e., Region V), as illustrated in Fig.2. On the other hand, with preference for public good H (i.e., larger γ_i), or the more expensive the price of public good G is (i.e., p_i rises), the more likely the individual is to stop contributing to public good G. This makes Region III smaller, because the slope of π_i becomes steeper, as illustrated in Fig.3.

The most noteworthy point is that Fig.1 generally illustrates the one-to-one mapping (i.e., a function) of individual i from any given pair of (G, H) to a pair of (g_i, h_i) owing to the strict quasi-concavity of Cobb–Douglas preferences. However, this feature does not necessarily imply the uniqueness of a Nash equilibrium profile $\{g_1, g_2, h_1, h_2\}$. This result can be explained as follows. Suppose that individuals 1 and 2 have the same slope of the segments M_i, $i = 1, 2$ (i.e., $(\gamma_i p_i/\beta_i q_i) = (\gamma_j p_j/\beta_j q_j)$). Although the first-order conditions (22) are given by

$$\frac{\gamma_i}{\alpha_i}\left(\frac{w_i}{q_i} - \frac{p_i}{q_i} g_i - h_i\right) = \frac{\gamma_i}{\beta_i}\frac{p_i}{q_i} G, \quad i = 1, 2, \tag{26}$$

$$\frac{\beta_i}{\alpha_i}\left(\frac{w_i}{p_i} - g_i - \frac{q_i}{p_i} h_i\right) = \frac{\beta_i}{\gamma_i}\frac{q_i}{p_i} H, \quad i = 1, 2, \tag{27}$$

the right-hand sides of (26) for $i = 1, 2$ (and (27) for $i = 1, 2$) coincide with each other because of the hypothesis $(\gamma_1 p_1/\beta_1 q_1) = (\gamma_2 p_2/\beta_2 q_2)$. Consequently, the four equations in (26) and (27) are reduced to

$$\frac{\gamma_1}{\alpha_1}\left(\frac{w_1}{q_1} - \frac{p_1}{q_1}g_1 - h_1\right) = \frac{\gamma_2}{\alpha_2}\left(\frac{w_2}{q_2} - \frac{p_2}{q_2}g_2 - h_2\right), \tag{28}$$

$$\frac{\beta_1}{\alpha_1}\left(\frac{w_1}{p_1} - g_1 - \frac{q_1}{p_1}h_1\right) = \frac{\beta_2}{\alpha_2}\left(\frac{w_2}{p_2} - g_2 - \frac{q_2}{p_2}h_2\right). \tag{29}$$

Inspection of (28) and (29) immediately reveals that the configurations of two individuals' contributions are *indeterminate*, because the number of unknown variables $\{g_1, g_2, h_1, h_2\}$ is greater than the number of the equations (i.e., (28) and (29)). Consequently, a Nash equilibrium profile $\{g_1, g_2, h_1, h_2\}$ is no longer unique. This case corresponds to the so-called indeterminate case of Cornes and Itaya (2010). Note, however, that these authors find that this indeterminate case occurs only when individual preferences are identical, whereas in our model, the case occurs even if individual preferences are *not* identical, because we introduce different prices of public goods. It is also important to note that this case remains *exceptionally unusual*; in other words, this case is *non-generic* in the sense that it vanishes when there are slightly perturbing parameter values of preferences or wealth. In summary, to guarantee the uniqueness of the profile of individual contributions associated with given values of the total provisions of G and H, we have to assume $(\gamma_1 p_1/\beta_1 q_1) \neq (\gamma_2 p_2/\beta_2 q_2)$. Moreover, to ensure the uniqueness of a Nash equilibrium (if it exists), we have to prove the uniqueness of the profile of the total provisions as well. We show in the following sections that the abovementioned assumption is sufficient to guarantee the uniqueness of individual contributions' profile as well as of the total provision profile of all public goods in the multiple public good provision model with an arbitrary number of individuals and more than two public goods.

4 Existence and Uniqueness of a Nash Equilibrium

In this section, we prove the existence and uniqueness of a Nash equilibrium in a voluntary provision model with more than two public goods and more than two potential contributors. To this end, we consider an economy in which there are n individuals indexed by $i = 1, 2, ..., n$, a single private good, and m public goods indexed by $k = 1, 2, ..., m$. We denote individuals i's contribution to public good j by g_i^j, the total supply of public good j by

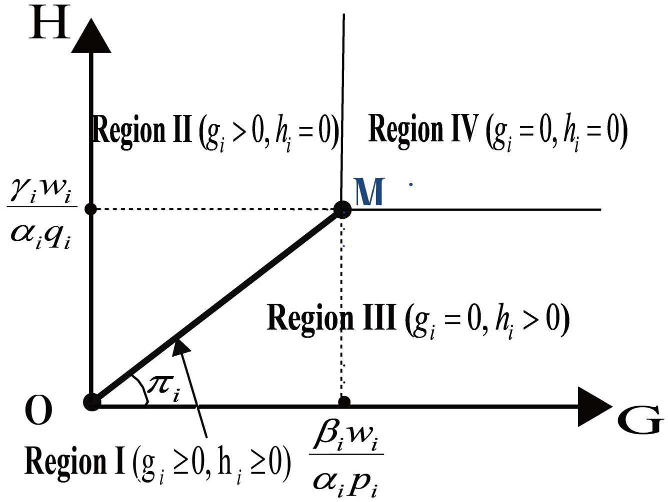

Figure 1: Individual i's contributions are determined according to Regions I, II, III, and IV of (G, H)-plane

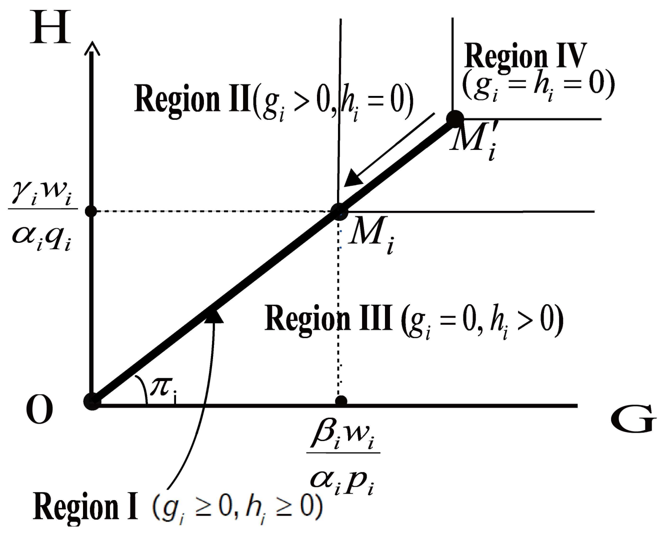

Figure 2: Effect of an Increase in Income ω_i

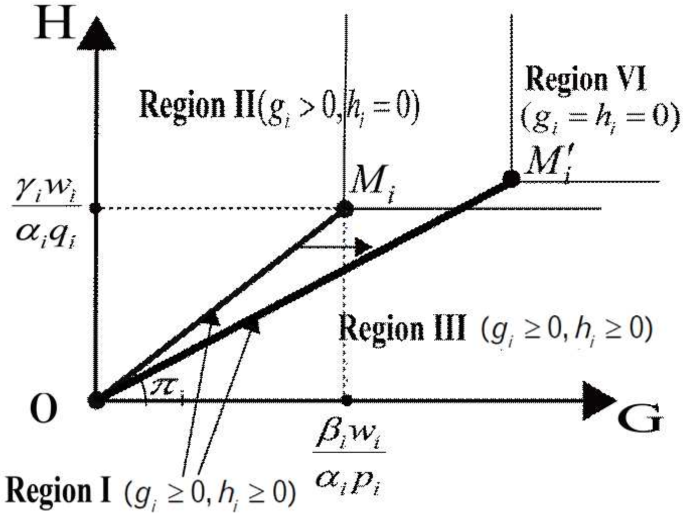

Figure 3: Effect of an Increase in β_i

$G^j \equiv \sum_{i=1}^n g_i^j$, and the profile of resource allocation by the Cartesian product $\prod_{i=1}^n (x_i, g_i^1, g_i^2, ..., g_i^m) \in R_+^{(1+m)n}$. Given $G_{-i}^1, G_{-i}^2, \ldots, G_{-i}^m$, individual i now solves the following problem:

$$\max_{\{x_i, g_i^1, g_i^2, ..., g_i^m\}} U_i(x_i, G^1, G^2, ..., G^m) = (x_i)^{\alpha_i^0} \prod_{k=1}^m (G^k)^{\alpha_i^k}, \tag{30}$$

$$\text{s.t.} \quad x_i + \sum_{j=1}^m p_i^j g_i^j = \omega_i \tag{31}$$

$$x_i \geq 0, \; g_i^j \geq 0, \quad j = 1, 2, .., m.$$

The first-order conditions for individual i's utility maximization are given by

$$\sum_{j=1}^m p_i^j g_i^j = w_i - \beta_i^k p_i^k G^k + \lambda_i^k, \quad k = 1, \ldots, m,$$

$$g_i^k \lambda_i^k = 0, \quad k = 1, \ldots, m,$$

where $\beta_i^k \equiv \alpha_i^0 / \alpha_i^k$. As in the previous section, we can rewrite those conditions as follows:

$$w_i - \sum_{j=1}^{m} p_i^j g_i^j = \beta_i^k p_i^k G^k, \quad \text{if } g_i^k > 0, \tag{32}$$

$$w_i - \sum_{j=1}^{m} p_i^j g_i^j \leq \beta_i^k p_i^k G^k, \quad \text{if } g_i^k = 0. \tag{33}$$

Since the left-hand sides of (32) and (33) are common for all $k = 1, \ldots, m$, the following conditions hold for all $k,\ h \in \{1, 2, .., m\}$ in equilibrium:

$$\beta_i^k p_i^k G^k = \beta_i^h p_i^h G^h \qquad \text{if } g_i^k > 0 \text{ and } g_i^h > 0, \tag{34}$$

$$\beta_i^k p_i^k G^k \leq \beta_i^h p_i^h G^h \qquad \text{if } g_i^k > 0 \text{ and } g_i^h = 0. \tag{35}$$

By making use of (34) and (35), we prove the existence and uniqueness of a Nash equilibrium in an economy with n individuals and m voluntarily provided public goods in the following subsections.

4.1 Existence of Equilibrium

Under the assumptions made, we can easily show by adapting the standard existence proof used in the model of a single public good that a Nash equilibrium exists in a multiple public good economy. Suppose that the profile of the total provisions $\mathbb{G} = (G^1, \ldots, G^m)$ is given. The following procedure enables us to identify which public good individual i is willing to provide.

Step 1 Find $\gamma_i(\mathbb{G}) \equiv \min\{\beta_i^j p_i^j G^j \mid j = 1, \ldots, m\}$.

Step 2 If $w_i \leq \gamma_i(\mathbb{G})$, then individual i does not provide any public good; consequently, individual i is a free rider.

Step 3 Otherwise, define the set of indexes $J_i \equiv \{k \mid \beta_i^k p_i^k G^k = \gamma_i(\mathbb{G})\}$. Individual i provides a non-negative amount of public good $k \in J_i$.

The system of equations (32) can be rewritten as

$$\sum_{j \in J_i} p_i^j g_i^j = w_i - \beta_i^k p_i^k G^k, \quad k \in J_i. \tag{36}$$

Since $\beta_i^k p_i^k G^k = \gamma_i(\mathbb{G})$ for $k \in J_i$ holds, the system of equations (36) boils down to the following single equation:

$$\sum_{j \in J_i} p_i^j g_i^j = b_i(\mathbb{G}), \tag{37}$$

where $b_i(\mathbb{G}) \equiv w_i - \gamma_i(\mathbb{G})$. As a result, if the cardinality of set J_i is **one**, say $J_i = \{k\}$, equation (37) uniquely determines the amount of provision of the public good k as a function of $\mathbb{G}$ =$(G^1, G^2, ..., G^m)$ as follows:

$$g_i^k = \frac{b_i(\mathbb{G})}{p_i^k}.$$

However, if the number of elements in the finite set J_i, denoted by $\#J_i$, is strictly greater than one, then we have an infinite number of solutions g_i^k for equations (37).

Proposition 1 *A Nash equilibrium exists.*

Proof. We denote the set of total provisions $\mathbb{G} \in R_+^m$ by X_G:

$$X_G \triangleq \left\{ \mathbb{G} = (G^1, \ldots, G^m) \;\middle|\; \begin{array}{c} G^j = \sum_i^n g_i^j, \;\; \sum_{j \in J_i} p_i^j g_i^j = b_i(\mathbb{G}), \;\; g_i^j \geq 0 \\ i = 1, 2, ..., n; \;\; j = 1, 2, ..., m. \end{array} \right\}.$$

Recalling the definition of $b_i(\mathbb{G})$, it turns out that X_G is a compact and non-empty convex set.

Define a vector-valued mapping $\varphi_i : \Re_+^m \to 2^{\Re_+^m}$, $i = 1, \ldots, n$.

$$\varphi_i(\mathbb{G}) \triangleq \left\{ (g_i^1, \ldots, g_i^m) \;\middle|\; \sum_{j \in J_i} p_i^j g_i^j = b_i(\mathbb{G}), g_i^j \geq 0 \right\}.$$

Note also that if $\#J_i \geq 2$, $\varphi_i(\mathbb{G})$ becomes a set-value mapping (i.e., a correspondence). For any $\mathbb{G} \in X_G$, $\varphi_i(\mathbb{G})$ is convex valued. From $\varphi_i(\mathbb{G})$ $(i = 1, \ldots, n)$, the correspondence $\Phi(\mathbb{G}) : X_G \to 2^{X_G}$ is constructed as

$$\Phi(\mathbb{G}) \triangleq \left\{ \mathbb{G} = (G^1, \ldots, G^m) \;\middle|\; \begin{array}{c} G^j = \sum_i^n g_i^j \; j = 1, \ldots, m, \\ (g_1^1, \ldots, g_n^m) \in \varphi_1(\mathbb{G}) \times \cdots \times \varphi_n(\mathbb{G}) \end{array} \right\}.$$

By its construction, $\Phi(\mathbb{G})$ is closed convex valued for each $\mathbb{G} \in X_G$, and thus, is upper-hemi continuous. Therefore, by applying Kakutani's fixed-point theorem, a fixed point exists that is a Nash equilibrium. ■

Note, however, that since $\Phi(\mathbb{G})$ is a correspondence, the existence of a fixed point (i.e., a Nash equilibrium point) does not in general imply the uniqueness of a Nash equilibrium.

4.2 Sufficient Condition for a Unique Equilibrium

It is well documented in the literature that under the normality assumption, a Nash equilibrium is unique in a model with a single public good. By contrast, in light of the results presented in the previous section, *the normality assumption does not suffice to ensure the uniqueness of a Nash equilibrium in the multiple public goods model in general.* Then, what is a sufficient condition to achieve a unique equilibrium of the *multiple public good* provision model under Cobb–Douglas preferences?

Let us define the linear system that all positive provisions ($g_i^k > 0$) should satisfy. Denoting the index set for non-free riders by C, we have

$$\begin{cases} \sum_{j \in J_i} p_i^j g_i^j = b_i(\mathbb{G}), & i \in C, \\ \sum_{i \in C^k} g_i^k = G^k, & k = 1, 2, ..., m, \end{cases} \tag{38}$$

where C^k is an index set of individuals who provide public good k. Given the parameters of preferences as well as wealth of individuals, the system of equations (38) contains $\#C + m$ equations and $\sum_{i \in I} \#J_i$ variables.

Now, the problem of finding a condition that guarantees the uniqueness of a Nash equilibrium is split into two steps. The first is to find a condition under which the linear system of equations (38) has a unique solution. In general, it is not guaranteed that (38) has a unique solution. To find the condition, we need to introduce the following notations. Take any integer L from $\{2, \dots, n\}$ without duplication and make sequence $s_L = (j_1, \dots, j_L, j_1)$ where j_k takes an arbitrary integer from $\{1, \dots, m\}$. Define $\pi_k(s_L)$ as

$$\pi_k(s_L) = \begin{cases} \dfrac{\beta_{i_k}^{j_{k+1}} p_{i_k}^{j_{k+1}}}{\beta_{i_k}^{j_k} p_{i_k}^{j_k}} & \text{for } k < L, \\ \dfrac{\beta_{i_L}^{j_1} p_{i_L}^{j_1}}{\beta_{i_L}^{j_L} p_{i_L}^{j_L}} & \text{for } k = L. \end{cases}$$

Assumption 1 For all $L \in \{2, \dots, n\}$, any sequence $s_L = (j_1, \dots, j_L, j_1)$, constructed from any number $j_k \in \{1, \dots, m\}$, satisfies the following:

$$\pi_1(s_L) \times \pi_2(s_L) \times \cdots \times \pi_L(s_L) \neq 1. \tag{39}$$

To understand the meaning of Assumption 1, we provide the following illustrative examples. Table 1(a) is an example for some pattern of individual contributions that satisfies Assumption 1, while the contribution pattern displayed in Table 1 (b) does not. The bipartite graphs in Figs. 4 and 5 illustrate the pattern of individual contributions illustrated by Table 1 (a) and 1(b), respectively. The graph in Fig.4 contains no *cycle* (or sequence of links that leads from one node back to itself, as described in more details in Jackson, 2008), displaying a *tree structure*, while the graph in Fig.5 has a cycle, indicated by the thick lines. Indeed, along these thick lines in Fig.5, individual 1 provides the public goods 1 and 2:

$$\beta_1^1 p_1^1 G^1 = \beta_1^2 p_1^2 G^2,$$

individual 3 provides the public goods 2 and 5:

$$\beta_3^2 p_3^2 G^2 = \beta_3^5 p_3^5 G^5,$$

and individual 5 provides the public goods 5 and 1:

$$\beta_5^5 p_5^5 G^5 = \beta_5^1 p_5^1 G^1.$$

By sequential substitution, we obtain

$$\beta_1^1 p_1^1 G^1 = \beta_1^2 p_1^2 \frac{\beta_3^5 p_3^5}{\beta_3^2 p_3^2} \frac{\beta_5^1 p_5^1}{\beta_5^5 p_5^5} G^1,$$

which amounts to

$$1 = \frac{\beta_1^2 p_1^2}{\beta_1^1 p_1^1} \frac{\beta_3^5 p_3^5}{\beta_3^2 p_3^2} \frac{\beta_5^1 p_5^1}{\beta_5^5 p_5^5} \equiv \pi_1((1,2,5,1)) \times \pi_2((1,2,5,1)) \times \pi_5((1,2,5,1)),$$

thereby violating Assumption 1.

	Public good				
Individual	1	2	3	4	5
1	✓	✓	✓		
2	✓			✓	
3		✓			✓
4			✓		
5	✓				
6	✓				
7					

(a) Possible pattern

	Public good				
Individual	1	2	3	4	5
1	✓	✓	✓		
2	✓			✓	
3		✓			✓
4			✓		
5	✓				✓
6	✓				
7					

(b) Impossible pattern

Table 1: A check mark (✓) indicates a positive amount of individual's contribution

We are now ready to prove our main proposition of this study; namely, the uniqueness of a Nash equilibrium $\{(g_i^1, g_i^2, .., g_i^m) | \, i = 1, 2, ..., n\} \in R_+^{m \times n}$.

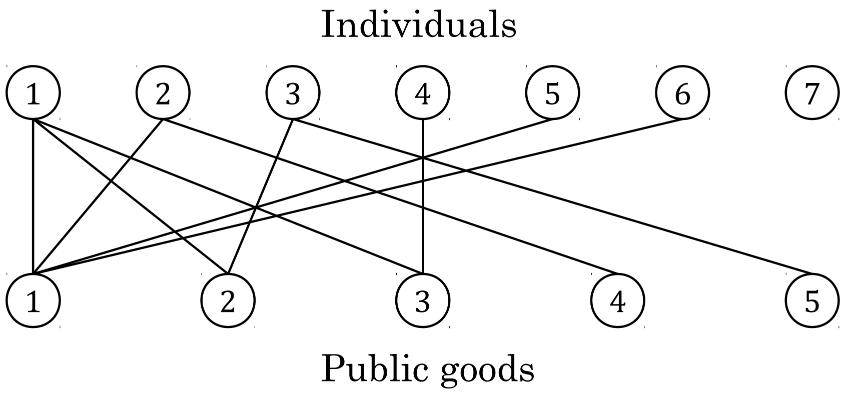

Figure 4: Profile of individual contributions displayed in Table 1(a)

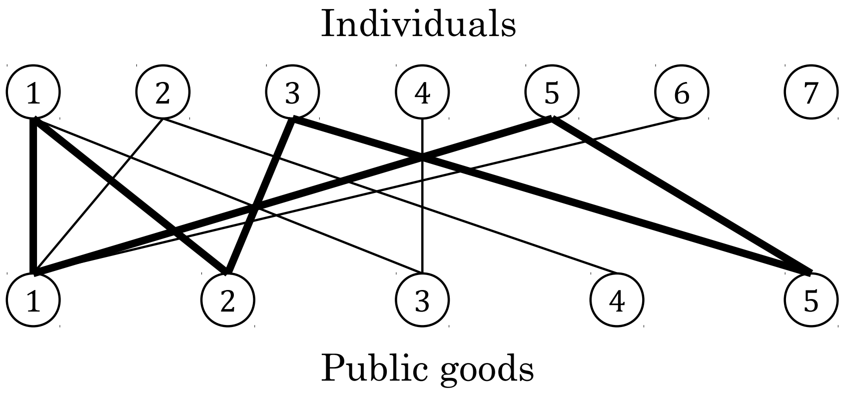

Figure 5: Profile of individual contributions displayed in Table 1(b)

Proposition 2 *If the utility function (30) satisfies Assumption 1, then there is a unique Nash equilibrium.*

The proof of Proposition 2 is relegated to the appendix. As stated before, we split the proof of the appendix into two parts. The first demonstrates that the contribution function of each individual is a point-value mapping (i.e., a function) under Assumption 1, while the second part demonstrates that there is a unique profile of the total provisions of the public goods under the same assumption. These steps together imply that the Nash equilibrium is unique. The novelty of our proof is to use *graph theory* to prove the uniqueness of a Nash equilibrium. To the best of our knowledge, we are the first to use graph theory to prove the uniqueness of a Nash equilibrium in multiple public goods models.

In a single public good provision model, a sufficient condition for uniqueness is the *only* normality assumption (see also Bergstrom et al., 1986, 1992; Fraser, 1992; Glazer and Konrad, 1993; Nett and Wolfgang, 1993; Andreoni and Bergstrom, 1996; Cornes, Hartley, and Sandler, 1999). By contrast, our result indicates that this assumption does not suffice to ensure the uniqueness of a Nash equilibrium in multiple public good provision models, even if all individuals have Cobb–Douglas preferences.

5 Application of replacement function: Algorithm for Identifying Free Riders

Recognizing the significance of identifying free riders, several authors, such as Bergstrom et al. (1986), Andreoni and McGuire (1993), Shrestha and Cheong (2007), and Yildirim (2014), provide algorithms for identifying free riders. Miyakoshi and Suzuki (2012) use Cornes and Hartley's (2007) replacement function to construct an algorithm for identifying the contributors to a single public good. We here extend their algorithm to the model of two public goods with an arbitrary number of individuals. To identify free riders in multiple public good provision models, we first find C^N, C^G, C^H, and C^{Both}, and then, a Nash equilibrium allocation (G, H), by using the algorithm based on the replacement function of Cornes and Hartley (2007), *provided Assumption 1 holds.*

From the argument in Section 3, we know that **at most** one individual, say k, may provide both public goods if $\pi_k = (p_k \gamma_k / q_k \beta_k) = H/G$, provided Assumption 1 holds. Hence, we classify two possible cases as follows:

Case 1: When $C^{Both} = \varnothing$, all contributors are members of only one C^G or C^H and the equilibrium is reduced to the structure similar to a single public good model. We can straightforwardly obtain the aggregate replacement functions for the respective public goods by summing (23) and (24):

$$G = \sum_{j \in C^G} \left[\frac{w_j}{p_j} - \frac{\alpha_i}{\beta_i} G \right] \text{ and } H = \sum_{j \in C^H} \left[\frac{w_j}{q_j} - \frac{\alpha_i}{\gamma_i} H \right]. \tag{40}$$

It is immediately observed from (40) that the right-hand sides of both expressions are decreasing in G and H, respectively. Hence, the straightforward application of the replacement function approach allows us to ensure the existence of a unique Nash equilibrium (see Cornes and Hartley, 2007).

Case 2: We consider the case in which only one individual supplies both public goods, say $C^{Both} = \{k\}$. Then, by using (22)–(24), the contribution made by each individual is written as

$$g_i = \frac{w_i}{p_i} - \frac{\alpha_i}{\beta_i} G, \qquad i \in C^G, \tag{41}$$

$$h_i = \frac{w_i}{q_i} - \frac{\alpha_i}{\gamma_i} H, \qquad i \in C^H, \tag{42}$$

$$g_k + \frac{q_k}{p_k} h_k = \frac{w_k}{p_k} - \frac{\alpha_k}{\beta_k} G \text{ and } H = \pi_k G, \quad k \in C^{Both}, \tag{43}$$

while the equilibrium values of G and H are given by

$$G = \sum_{i \in C^G} g_i + g_k \text{ and } H = \sum_{i \in C^H} h_i + h_k. \tag{44}$$

respectively. Although it follows from (41) and (42) that $\sum_{i \in C^G} g_i$ and $\sum_{i \in C^H} h_i$ in (44) are decreasing in G and H, respectively, the right-hand sides of (44) may not be decreasing in the respective public good. Hence, we cannot directly apply the replacement function approach to demonstrate the existence of a unique Nash equilibrium. Instead, multiplying the first and second equalities in (44) by p_k and q_k and summing the resultant expressions yields

$$p_k G + q_k H = p_k G_{-k} + q_k H_{-k} + p_k g_k + q_k h_k.$$

Inserting $H = \pi_k G$, $G_{-k} = \sum_{i \in C^G} g_i$, and $H_{-k} = \sum_{i \in C^H} h_i$ into the above expression yields

$$p_k G + q_k \pi_k G = p_k \sum_{i \in C^G} g_i + q_k \sum_{i \in C^H} h_i + w_k - \frac{\alpha_k}{\beta_k} p_k G. \tag{45}$$

It is easy to observe that the left-hand side of (45) is increasing in G, whereas the right-hand side is decreasing in G (recall that $\sum_{i \in C^G} g_i$ and $\sum_{i \in C^H} h_i$

in (44) are decreasing in G and H). These two loci are illustrated by Fig.6. Inspection of Fig.6 reveals that the intersection of both loci is unique, thereby yielding a unique value of G^*. From $H = \pi_k G$, we also obtain a unique value of H^*. The uniqueness of the pair (G^*, H^*) in conjunction with the unique property of the replacement function of each individual for G and H imply the uniqueness of the Nash equilibrium.

Now, we are ready to identify the free riders under **Assumption 1**. Then, we can find C^N, C^G, C^H, and C^{Both} by using the algorithm stated below. We split the procedure for identifying free riders into five steps as follows.

Step 0: Define the *dropout point* of individual i as $M_i \equiv (\hat{w}_i, \tilde{w}_i)$, where

$$\hat{w}_i = \frac{\beta_i w_i}{\alpha_i p_i}, \qquad \tilde{w}_i = \frac{\gamma_i w_i}{\alpha_i q_i}.$$

Define $\pi_i \equiv \tilde{w}_i/\hat{w}_i = p_i\gamma_i/q_i\beta_i$. Then, we arrange indexes of π_i in ascending order:

$$\pi_1 < \pi_2 < \cdots < \pi_n.$$

Note that, under Assumption 1, for any (i, j), $i \neq j$, it holds that $\pi_i \neq \pi_j$.

Set $\delta = \pi_1$ and $C^N = C^G = C^H = C^{Both} = \emptyset$.

Step 1: In this step, we have $\delta = \pi_k$, which implies that individual $i = 1, 2, \ldots, k-1$ is a potential provider of public good H, individual $i = k+1, k+2, \ldots, n$ is a potential provider of public good G, and individual k is a potential provider of both G and h.

Step 2: Define the *dropout value* of individual i as $\hat{w}_i \equiv (\beta_i w_i/\alpha_i p_i)$, which is obtained by setting $g_i = 0$ in (41) and solving the resultant equation for G. Then, we arrange those dropout values of all individuals for public good G in ascending order:

$$\hat{w}_1 \geq \hat{w}_2 \geq \cdots \geq \hat{w}_i \geq \hat{w}_{i+1} \geq \cdots \geq \hat{w}_n$$

Step 3: Start from $i = 1$. Consider first the interval $G \in [\hat{w}_1, \hat{w}_2]$. Compute the aggregate replacement function $R(G)$ such that $R(G) = g_1(G)$. Since $\hat{w}_1 > g_1(G)$ always holds, individual 1, who most prefers public good G, is willing to provide public good G. Next, if $\hat{w}_2 < g_1(G)$, individual 2 is a free rider; moreover, all other individuals $i > 2$ are free riders. Hence, we cease our search for identifying free riders. On the contrary, if $\hat{w}_2 > g_1(G)$, individual 2 is not a free rider, and we have to continue our search.

Step 4: For $i \geq 2$, consider the interval $[\hat{w}_i, \hat{w}_{i+1}]$. Compute $R(G)$ such that $R(G) \equiv \sum_{j=1}^{i} g_j(G)$ whose right-hand side is obtained from summing up the replacement function of every individual up to i. Stop the algorithm for searching for free riders when $\hat{w}_i < R(G)$, because it turns out that not only individual i but also all other individuals $j > i$ are free riders. However, so long as $\hat{w}_i > R(G)$, set $i = i + 1$ iteratively and repeat Step 2 until $\hat{w}_i < R(G)$; in other words, once $\hat{w}_i < R(G)$, the search is terminated so that all individuals $j \geq i$ are free riders.

Step 5: Repeat Steps 0–2 for public good H.

In order to explain this procedure more intuitively, we consider a game of three individuals, $\hat{w}_1 \geq \hat{w}_2 \geq \hat{w}_3$, for public good G using a simple graphical apparatus frequently adopted by Cornes and Hartley. As seen in Fig.6, the algorithm starts by choosing individual $i = 1$, and looking at the interval $G \in [\hat{w}_1, \hat{w}_2]$. Within this interval, it is immediately observed that only individual 1 makes a positive contribution. The linearity of the aggregate replacement function $R(G)$ within this interval makes it easy to observe that there is no value of G within this interval at which $\hat{w}_1 > R(G)$, as illustrated in Fig.6. Then, we do not need to move to the next interval $[\hat{w}_2, \hat{w}_3]$, since $\hat{w}_2 < R(G)$. We stop within the former interval. We know that only player 1 makes a positive contribution to public good G. This G^* is the sought equilibrium such that $G^* = R(G^*)$ (see Cornes and Hartley, 2007) and then only 1 out of these three players makes a strictly positive contribution. On the other hand, when we apply the same algorithm to each interval of public good H (i.e., $H \in [\widetilde{w}_i, \widetilde{w}_{i+1}]$ where $\widetilde{w}_i \equiv (\gamma_i w_i / \alpha_i q_i)$), we find that individual 2 is a contributor only to public good H, as Fig.7 shows. Thus, the resulting pair of total provisions (G^*, H^*) such that $G^* = R(G^*)$ and $H^* = R(H^*)$ in Figs. 6 and 7 stands for an equilibrium point and the configuration of individual contributors corresponding to this allocation is given by $1 \in C^G$, $2 \in C^H, \varnothing \in C^B$ and $3 \in C^N$.

Several remarks are in order. First, note that if there is an individual (say k) who simultaneously provides two public goods (i.e., Case 2), the above algorithm must be slightly modified as follows. We first need to find Nash equilibrium values of G^* (and thus $H^* = \pi_k G^*$) using Fig.6. Substituting these values into the (43), respectively, yields

$$g_k = \frac{w_k}{p_k} - \frac{\alpha_k}{\beta_k} G^* - \frac{q_k}{p_k} h_k, \quad k \in C^{Both},$$
$$h_k = \frac{w_k}{q_k} - \frac{\alpha_k}{\beta_k} H^* - \frac{p_k}{q_k} g_k, \quad k \in C^{Both}.$$

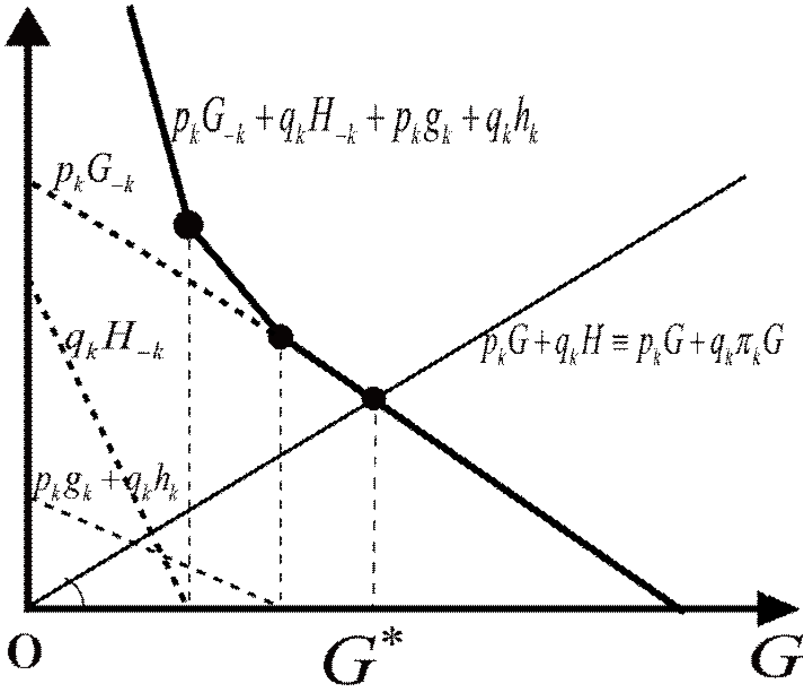

Figure 6: Uniqueness of a Nash equilibrium in Case 2

By setting the above contribution functions g_k and h_k equal to zero, we can obtain the dropout values for an individual k. The rest of the procedure is the same as in Case 1. Second, it should be stressed that the uniqueness (point-valued) property of the mapping from the total provisions of G and H to individual contributions plays a critical role in employing the replacement function approach of Cornes and Hartley (2005, 2007, 2012). The straightforward application of the replacement function enables us to identify free riders in the multiple public goods model in a way similar to Miyakoshi and Suzuki (2012), who use the replacement function to identify free riders in a *single* public good model. Third, as long as Assumption 1 is imposed, the above algorithm for identifying free riders need not to be restricted to the case of two public goods, which can be straightforwardly generalized to the case of an arbitrary number of public goods.

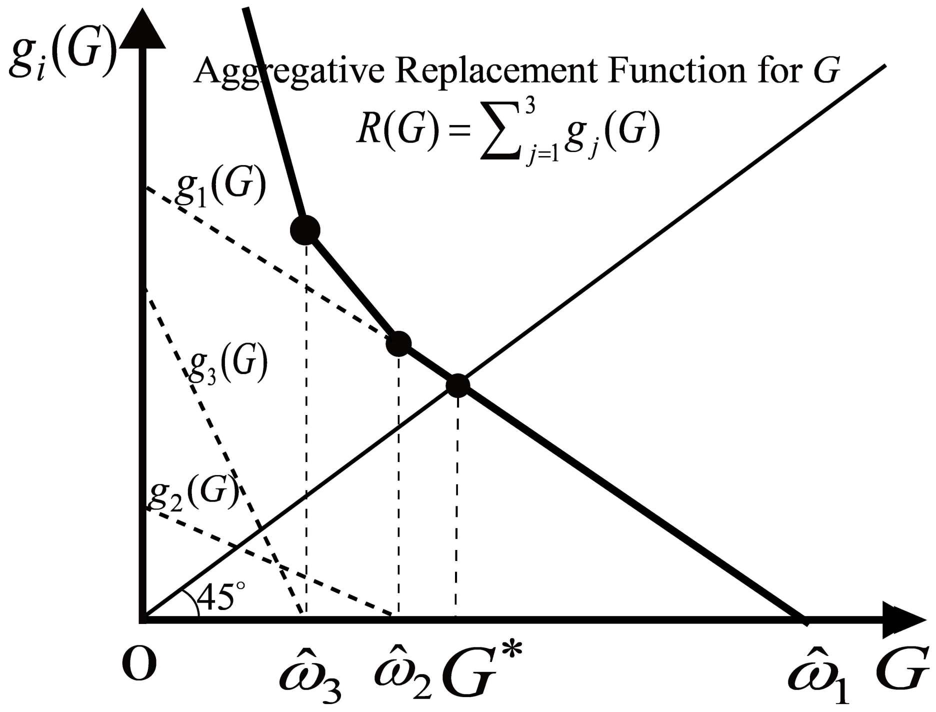

Figure 7: Algorithm for Finding a Marginal Contributor to Public Good G in Case 1

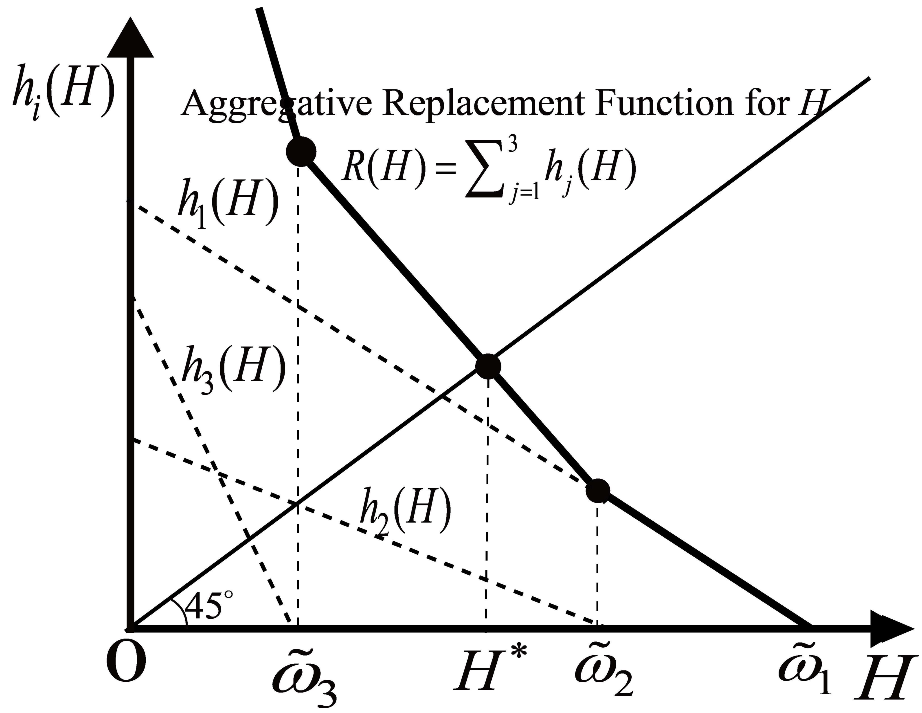

Figure 8: Algorithm for Finding a Marginal Contributor to Public Good H in Case 1

6 Concluding Remarks

Although several studies have formally incorporated the possibility of many public goods, little attention has been paid to the uniqueness of equilibrium in a world of many public goods. In the voluntary provision model with many public goods, replacement mapping from the space of public goods to individual's contributions to the respective public goods may by potentially multi-valued. Under this circumstance, we can prove the existence of a Nash equilibrium by applying Kakutani's fixed-point theorem rather than Brower's fixed-point theorem. Nevertheless, this multi-valuedness prevents us from directly applying the replacement function approach of Cornes and Hartley (2005, 2007, 2012) not only to prove the uniqueness of a Nash equilibrium in multiple public goods models, but also to perform comparative statics analysis. In this study, we have identified a sufficient condition to ensure a unique Nash equilibrium as well as a replacement function that is a point-to-point mapping from a profile of total public goods to each individual's contribution profile. Moreover, by inspection of the sufficient condition we found reveals that the non-uniqueness property is *non-generic* so long as Cobb–Douglas preferences are assumed. This finding implies, first, that so long as heterogeneous Cobb–Douglas preferences coupled with Assumption 1 are assumed, the replacement function approach remains quite robust in multiple public goods models, and second, that we can utilize a variety of results obtained from the application of the replacement function approach in a single public good provision model, such as for identifying a *marginal contributor* and demonstrate the existence of a unique Nash equilibrium.

Finally, we briefly discuss two directions in which our results may be extended. The most important extension is to address more general preferences, for instance, constant elasticity of substitution (CES) utility functions. We believe that our method of proof remains valid for CES utility functions, because (38) holds for CES utility functions as well; however, there remains an open question about the utility functions that do not satisfy (38). Second, Cornes and Hartley (2007) open up wide applicability of the replacement function approach in static Nash provision games that have an aggregative game structure, such as rent-seeking games, team's joint production games, Cournot oligopoly models, and so on. However, these applications should have been restricted only on *one aggregate*, such as a single product or a single prize. Hence, the present study opens up new possibilities of applying the replacement function approach to investigate the model with *multiple aggregates*, such as Cournot's oligopoly model with multiple products or the Tullock-type rent-seeing competition model with multiple prizes. In light of the results of this study, we need to assume specific objective functions coupled with appropriate conditions that ensure a single-valued replacement function from the space of multiple aggregates to the profiles of individual's or firm's choice functions.

Appendix: Proof of Proposition 2

Define the set of nodes, $U = \{u_1, \dots, u_n\}$, each element of which represents individual $i \in \{1, \dots, n\}$, and the set of nodes, $V = \{v_1, \dots, v_m\}$, each element of which represents public good $j \in \{1, \dots, m\}$, respectively. Denoting an edge between $u_i \in U$ and $v_j \in V$ as e_{ij} if $g_i^j > 0$, we construct an undirected bipartite graph $\mathcal{G} = (U, V, E)$, where E is a set of edges (see, e.g., Jackson, 2008 for more details).

Lemma 1 *Under Assumption 1, $\mathcal{G}$ is acyclic.*

Proof. Suppose $\mathcal{G}$ has a cycle consisting of $2L$ edges. Changing an index number, if necessary, the cycle is expressed as

$$u_1, e_{11}, v_1, e_{12}, u_2, e_{22}, v_2, \dots, v_L, e_{L1}, u_1,$$

where $e_{ij} \in E$ is an edge connecting the nodes u_i and v_j.

Without loss of generality, we suppose that $u_i = i$ and $v_j = j$. On the cycle, as individual 2 provides the public goods 1 and 2:

$$\beta_2^1 p_2^1 G^1 = \beta_2^2 p_2^2 G^2,$$

while individual 3 provides the public goods 2 and 3:

$$\beta_3^2 p_3^2 G^2 = \beta_3^3 p_3^3 G^3.$$

Combining these equalities yields

$$\beta_2^1 p_2^1 G^1 = \beta_2^2 p_2^2 \frac{\beta_3^3 p_3^3}{\beta_3^2 p_3^2} G^3.$$

Repeating the above process, we obtain

$$1 = \frac{\beta_2^2 p_2^2}{\beta_2^1 p_2^1} \frac{\beta_3^3 p_3^3}{\beta_3^2 p_3^2} \cdots \frac{\beta_L^1 p_L^1}{\beta_L^L p_L^L}.$$

This contradicts Assumption 1; hence, $\mathcal{G}$ cannot have a cycle. ■

Lemma 2 *Under Assumption 1, $\mathcal{G}$ has a unique solution.*

Proof. Lemma 1 assures that under Assumption 1, $\mathcal{G}$ is a *tree*. The upper part of system (38) corresponds to the condition that node u_i $(i = 1, \ldots, n)$ should satisfy and the lower part corresponds to the condition for node v_j $(j = 1, \ldots, m)$. Inspection of $\mathcal{G}$ immediately reveals that by traversing the tree from leafs to the root, we can uniquely determine values of g_i^j for $\forall i \in \{1, \ldots, n\}$ and $\forall j \in \{1, \ldots, m\}$. ■

So far, we have proven that given the equilibrium profile of total provisions $\mathbb{G} = (G^1, \ldots, G^m)$, the contributing patterns for all individuals are uniquely determined according to system (37). Nevertheless, the uniqueness of the total provision profile of multiple public goods remains to be proved.

Proposition 3 *The Nash equilibrium allocation of public goods* $\mathbb{G} = (G^1, \ldots, G^m)$ *is unique.*

Proof. Suppose, contrary to the proposition, that two distinctive Equilibria exist, say $\mathcal{E}_1$ and $\mathcal{E}_2$. As a result, there are two profiles of total provisions of public goods, such as $\hat{\mathbb{G}} = (\hat{G}^1, \ldots, \hat{G}^m)$ and $\tilde{\mathbb{G}} = (\tilde{G}^1, \ldots, \tilde{G}^m)$ which correspond to $\mathcal{E}_1$ and $\mathcal{E}_2$, respectively. Without loss of generality, we suppose $\hat{G}^j > \tilde{G}^j$ for some $j \in J$. As a result, we observe changes in the contributing pattern of individuals when the economy moves from $\mathcal{E}_1$ to $\mathcal{E}_2$. Without loss of generality, we can assume that individual i, who is not a provider of public good j in $\mathcal{E}_1$, may start to provide public good j in $\mathcal{E}_2$. On the other hand, individual i', who is a non-provider of good k in $\mathcal{E}_1$, may become a supplier of public good k in $\mathcal{E}_2$. We formally describe this change of provision in the following manner:

$$\begin{array}{ll} i \notin \hat{I}^j \to i \in \tilde{I}^j & i \text{ is an } \textbf{incoming individual} \text{ to public good } j, \\ i' \in \hat{I}^j \text{ and } \; i' \notin \tilde{I}^k \to i' \in \tilde{I}^k & i' \text{ is an } \textbf{outgoing individual} \text{ from public good } j, \end{array} \tag{A.1}$$

where $\hat{I}^j$ and $\tilde{I}^k$ represent the index sets of individuals who provide public goods j in $\mathcal{E}_1$ and k in $\mathcal{E}_2$, respectively.

First, we want to show that there is a public good that has no outgoing individual under Assumption 1. We call such a good "an absorbing public good". To demonstrate this assertion, suppose, on the contrary, that there is no absorbing public good. This assumption implies that all public goods should possess both incoming and outgoing individuals. Then, it follows that we can find a cycle created by the incoming and outgoing individuals such that

$$[j_1] \xrightarrow{i_1} [j_2] \xrightarrow{i_2} [j_3] \xrightarrow{i_3} \cdots \xrightarrow{i_{M-1}} [j_M] \xrightarrow{i_M} [j_1], \tag{A.2}$$

where M is the number of individuals involved in the cycle and $[j_k] \xrightarrow{i_k} [j_{k+1}]$ means that individual i_k is an outgoing individual to public good j_k and an incoming individual to public good j_{k+1}. By using the first-order condition, the relation $[j_k] \xrightarrow{i_k} [j_{k+1}]$ can be expressed by

$$\begin{aligned} \beta_{i_k}^{j_k} p_{i_k}^{j_k} \hat{G}^{j_k} &\leq \beta_{i_k}^{j_{k+1}} p_{i_k}^{j_{k+1}} \hat{G}^{j_{k+1}}, \\ \beta_{i_k}^{j_k} p_{i_k}^{j_k} \tilde{G}^{j_k} &\geq \beta_{i_k}^{j_{k+1}} p_{i_k}^{j_{k+1}} \tilde{G}^{j_{k+1}}. \end{aligned} \tag{A.3}$$

Combining the above inequalities yields

$$\frac{\hat{G}^{j_k}}{\tilde{G}^{j_k}} \leq \frac{\hat{G}^{j_{k+1}}}{\tilde{G}^{j_{k+1}}}. \tag{A.4}$$

Repeating this process in a similar manner, it follows that the cycle (A.2) entails the following relation:

$$\frac{\hat{G}^{j_1}}{\tilde{G}^{j_1}} \leq \frac{\hat{G}^{j_2}}{\tilde{G}^{j_2}} \leq \cdots \leq \frac{\hat{G}^{j_M}}{\tilde{G}^{j_M}} \leq \frac{\hat{G}^{j_1}}{\tilde{G}^{j_1}},$$

which is reduced to

$$\frac{\hat{G}^{j_1}}{\tilde{G}^{j_1}} = \frac{\hat{G}^{j_2}}{\tilde{G}^{j_2}} = \cdots = \frac{\hat{G}^{j_M}}{\tilde{G}^{j_M}}. \tag{A.5}$$

As a result, inequalities (A.3) lead to

$$\frac{\beta_{i_k}^{j_{k+1}} p_{ii_k}^{j_{k+1}}}{\beta_{i_k}^{j_k} p_{i_k}^{j_k}} = \frac{\hat{G}^{j_{k+1}}}{\hat{G}^{j_k}} = \frac{\tilde{G}^{j_{k+1}}}{\tilde{G}^{j_k}}. \tag{A.6}$$

It follows from (A.5) and (A.6) that

$$\frac{\beta_{i_1}^{j_2} p_{i_1}^{j_2}}{\beta_{i_1}^{j_1} p_{i_1}^{j_1}} \frac{\beta_{i_2}^{j_3} p_{i_2}^{j_3}}{\beta_{i_2}^{j_2} p_{i_2}^{j_2}} \cdots \frac{\beta_{i_M}^{j_1} p_{i_M}^{j_1}}{\beta_{i_M}^{j_M} p_{j_M}^{j_M}} = \frac{\hat{G}^{j_2}}{\tilde{G}^{j_1}} \frac{\hat{G}^{j_3}}{\tilde{G}^{j_2}} \cdots \frac{\hat{G}^{j_1}}{\tilde{G}^{j_M}} = 1. \tag{A.7}$$

This contradicts Assumption 1 and thus, our assertion is proven.

We divide the rest of the proof into three steps.

Step 1. We show that for absorbing public good j_*, $\hat{G}^{j_*} > \tilde{G}^{j_*}$ holds. To show this, take arbitrary j such that $\hat{G}^j > \tilde{G}^j$. If public good j is absorbing, it is done. Otherwise, we can find outgoing individual i who newly provides a public good in $\mathcal{E}_2$. Since there is no cycle under Assumption 1, we eventually reach absorbing public good j_*. When we observe $[j_k] \xrightarrow{i_k} [j_*]$, inequality (A.3) holds. Therefore, by assumption (i.e., $\hat{G}^j > \tilde{G}^j$), we have $\hat{G}^{j_k} > \tilde{G}^{j_k}$, which implies $\hat{G}^{j_*} > \tilde{G}^{j_*}$.

Step 2. Consider a case in which $\hat{I}^{j_*}$ does not include an individual who simultaneously provides multiple public goods. In other words, individual $i \in \tilde{I}^{j_*}$ provides the only public good j_*. Because j_* is absorbing, $\hat{I}^{j_*} \subset \tilde{I}^{j_*}$. For each $i \in \tilde{I}^{j_*}$, we have

$$\begin{aligned} \sum_{j \in \hat{J}_i}^{m} p_i^j \hat{g}_i^j &= w_i - \beta_i^{j_*} p_i^{j_*} \hat{G}^{j_*} \quad \text{at } \mathcal{E}_1, \\ p_i^k \tilde{g}_i^k &= w_i - \beta_i^{j_*} p_i^{j_*} \tilde{G}^{j_*} \quad \text{at } \mathcal{E}_2. \end{aligned} \tag{A.8}$$

Since $\hat{G}^{j_*} > \tilde{G}^{j_*}$ (recall Step 1), $\hat{g}_i^{j_*} < \hat{g}_i^{j_*}$ must hold. Then

$$\hat{G}^{j_*} = \sum_{i \in \hat{I}^{j_*}} \hat{g}_i^{j_*} < \sum_{i \in \hat{I}^{j_*}} \tilde{g}_i^{j_*} \leq \sum_{i \in \tilde{I}^{j_*}} \tilde{g}_i^{j_*} = \tilde{G}^{j_*}.$$

This is a contradiction, which shows that the hypothesis is false when individual $i \in \hat{I}^{j_*}$ provides the only public good j_*.

Step 3. Finally, consider the case in which $\hat{I}^{j_*}$ may include an individual who simultaneously provides multiple public goods. Even in this case, $\hat{I}^{j_*} \subset \tilde{I}^{j_*}$ is still valid, although (A.3) is replaced by

$$\begin{aligned} \sum_{j \in \hat{J}_i}^{m} p_i^j \hat{g}_i^j &= w_i - \beta_i^{j_*} p_i^{j_*} \hat{G}^{j_*} \quad \text{at } \mathcal{E}_1, \\ \sum_{j \in \tilde{J}_i}^{m} p_i^j \tilde{g}_i^j &= w_i - \beta_i^{j_*} p_i^{j_*} \tilde{G}^{j_*} \quad \text{at } \mathcal{E}_2. \end{aligned} \tag{A.9}$$

In spite of (A.9), we cannot claim that when $\hat{G}_i^{j_*} > \tilde{G}_i^{j_*}$, $\hat{g}_i^{j_*} < \tilde{g}_i^{j_*}$. If individual $i \in \hat{I}^{j_*}$ is the one who simultaneously provides multiple public goods, then indexes j_* and h are included in $\tilde{J}_i$. Hence, it follows that $h \in \tilde{I}^{j_*}$; otherwise public good j_* is not absorbing. Since j_* and $h \in \tilde{J}_i$,

$$\begin{aligned} \beta_i^{j_*} p_i^{j_*} \hat{G}^{j_*} &= \beta_i^h p_i^h \hat{G}^h, \\ \beta_i^{j_*} p_i^{j_*} \tilde{G}^{j_*} &= \beta_i^h p_i^h \tilde{G}^h. \end{aligned} \tag{A.10}$$

Next, we show that public good h is also absorbing in addition to j_*. To prove this, suppose, on the contrary, that public good h has outgoing individual i'. Then, we can find $[h] \xrightarrow{i'} [l]$, leading to

$$\begin{aligned} \beta_{i'}^h p_{i'}^h \hat{G}^h &\leq \beta_{i'}^l p_{i'}^l \hat{G}^l, \\ \beta_{i'}^h p_{i'}^h \tilde{G}^h &\geq \beta_{i'}^l p_{i'}^l \tilde{G}^l. \end{aligned} \tag{A.11}$$

It follows from (A.10) and (A.11) that

$$\beta_{i'}^{j_*} p_{i'}^{j_*} \hat{G}^{j_*} \leq \beta_{i'}^{l} p_{i'}^{l} \hat{G}^{l},$$
$$\beta_{i'}^{j_*} p_{i'}^{j_*} \tilde{G}^{j_*} \geq \beta_{i'}^{l} p_{i'}^{l} \tilde{G}^{l},$$

thereby violating the fact that public good j_* is absorbing. Hence, it follows that **at least** one of the inequalities $\hat{g}_i^{j_*} < \tilde{g}_i^{j_*}$ and $\hat{g}_i^h < \tilde{g}_i^h$ must hold. Without loss of generality, we can say that public good $j_* \in \tilde{I}^{j_*}$ always exists satisfying $\hat{g}_i^{j_*} < \tilde{g}_i^{j_*}$ (if necessary, set $h = j_*$). As a result, we can once again apply the same logic outlined in Step 2 to derive a contradiction. Taken together, these contradictions stem from the hypothesis that there are two distinct equilibria, which end up establishing the uniqueness of the profile of the total provisions $\mathbb{G} = (G^1, \ldots, G^m)$. On the other hand, if $\tilde{I}^{j_*}$ includes an individual who simultaneously provides multiple public goods other than i, it does not affect the above claim, because of the acyclic property of the individual's contributing pattern.

■

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