cesifo WORKING PAPERS

7487
2019

January 2019

Wealth Management and Uncertain Tipping Points

Steinar Strøm, Jon Vislie

CESifo

Impressum:

CESifo Working Papers
ISSN 2364-1428 (electronic version)
Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH
The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute
Poschingerstr. 5, 81679 Munich, Germany
Telephone +49 (0)89 2180-2740, Telefax +49 (0)89 2180-17845, email office@cesifo.de
Editor: Clemens Fuest
www.cesifo-group.org/wp

An electronic version of the paper may be downloaded
· from the SSRN website: www.SSRN.com
· from the RePEc website: www.RePEc.org
· from the CESifo website: www.CESifo-group.org/wp

CESifo *Working Paper No. 7487*
Category 9: Resource and Environment Economics

Wealth Management and Uncertain Tipping Points

Abstract

We analyze optimal wealth management, within a global setting, where accumulation of GHGs caused by extraction of fossil resources affects the probability distribution for hitting a threshold or tipping point, indicating a climate change. We derive an optimal strategy for overall wealth management, within a Ramsey-Hotelling-framework. We have two assets; one being reproducible (reversible capital equipment) and another being non-reproducible (stock of exhaustible natural resources – fossil fuels). Resources, along with capital equipment, are inputs in the production of an aggregate output allocated to consumption and net investment. Resource extraction adds to a stock of GHGs that affects the likelihood for a catastrophic event. If, and when, such an event occurs there is a downscaling of production opportunities. We derive a first-best precautionary global tax on using fossil fuel, which internalizes the present value of (conditional) expected welfare loss of hitting a threshold, as well as a set of risk-modified optimality conditions for overall wealth management, as long as no catastrophe has occurred.

JEL-Codes: E210, O440, Q320.

Keywords: wealth management, stochastic tipping points, catastrophic outcome, precautionary taxation, social rates of discount.

Steinar Strøm
Department of Economics
University of Oslo & The
Ragnar Frisch Centre of Economic Research
Oslo / Norway
steinar.strom@econ.uio.no

Jon Vislie
Department of Economics
University of Oslo and CREE & The
Ragnar Frisch Centre of Economic Research
Oslo / Norway
jon.vislie@econ.uio.no

January 23, 2019

1. Introduction

What is the impact of a possible, manmade random catastrophe on the overall decision rules for saving? As a benchmark for answering this question we apply the "Ramsey-Hotelling"- condition (in continuous time) for optimal saving with two capital categories – reproducible and malleable real capital equipment and non-renewable fossil fuels (natural resources), both entering as inputs in a neoclassical macro production function. Without taking the environmental cost of stock pollution into account, and no risk, we know that at the intertemporal optimum the required rate of return from deferring consumption ("saving"), should be equal to the rate of return on any type of asset or investment (the investment discount rate).[1]

To this model, we add another element. We incorporate emissions of GHGs from resource extraction and in such a way that a global planner's probability beliefs for a climate change will depend on the stock of GHGs in the atmosphere. With this supplement, we should be able to assess the impact on optimal saving (from a global point of view), and how the social cost of resource extraction should be internalized through an emission tax or a "carbon" tax.

A large number of scholars has discussed the climate issue, especially in relation to the "Stern Review" from 2007 (Stern (2007)). The present paper adds another argument to this important question by combining the standard Ramsey-model for optimal saving, with one including exhaustible resources; the Hotelling-model, along with a stochastic future climate change caused by accumulated emissions of GHGs or fossil fuels. At a very high level of aggregation, we then derive conditions for an intertemporal optimum or optimal wealth management. We find that the standard optimality conditions; cf. footnote 1, is modified so as to take into account the current (conditional) probability beliefs for a catastrophic event, caused by accumulated

---

[1] With a neoclassical macro production function $F(K,R)$, with real capital services (K) and use of fossil fuels (R), as inputs (no extraction costs), with r being a positive utility discount rate, $\widehat{\omega}(c)$, the absolute value of the elasticity of marginal utility of consumption, c, and δ, the rate of capital depreciation, intertemporal optimum obeys the "Ramsey-Hotelling"-condition, expressed as

$r + \widehat{\omega}(c(t)) \cdot \frac{\dot{c}(t)}{c(t)} = F_K(K(t),R(t)) - \delta = \frac{d}{dt} \ln F_R(K(t),R(t))$. This result has been shown in numerous places; e.g., by the early contributions by Samuelson and Solow (1956), Dorfman et al. (1958) and by Dasgupta and Heal (1979; chapter 10).

extraction of the non-renewable natural resource, as well as the expected downgrading beliefs the global planner holds for the post-catastrophic regime or the continuation period. From our model, we derive a precautionary tax on resource extraction or the use of fossil fuels. An upward shift in the hazard rate of passing a threshold for a catastrophic event, given that no such event has happened so far, will shift the tax upwards. This shift will induce less extraction of fossil fuels and may imply less capital investments, depending on the substitution possibilities between fossil fuels and capital in the production of goods and services. The consumption path is affected; in fact, it becomes steeper, indicating less consumption "now".

The paper proceeds as follows: In Section 2 we review some of the literature we find most relevant for our problem. In Section 3 we present a model, whereas we in Section 4 outline conditions for an optimal saving strategy. In Section 5 we discuss the catastrophic-risk modified Ramsey-Hotelling condition (cf. footnote 1), whereas Section 6 concludes.

2. A brief review of some relevant literature

There is a large and still growing literature on the economics of climate change and tipping points.[2] In this section, we will review some of the literature most relevant for us.

A seminal contribution to analyzing stochastic thresholds in environmental regulation is Cropper (1976). In her paper, both temporary and irreversible catastrophes are studied, with the latter one being relevant to our study. When some stock pollution (like atmospheric concentrations of CO_2 and other GHGs) exceeds (or some resource stock falls below) some critical threshold, a catastrophe is triggered, leading to a new regime with lower utility, say with zero consumption. Cropper characterizes an optimal consumption path (identical to an extraction path of a non-renewable resource of unknown size), as long as no catastrophe yet has been triggered.[3] Optimal long-run (steady state) consumption-pollution with irreversible stochastic catastrophic outcome,

---

[2] Reed and Heras (1992) provide a very useful survey of an optimization technique (based on the seminal paper by Kamien and Schwartz (1971)), for a derivation of optimal exploitation of a biological or renewable resource vulnerable to a catastrophe. See also the special issues on tipping points of Environmental and Resource Economics 2016, and Journal of Economic Behavior and Organization 2016.
[3] Se Loury (1978) for an analysis of this question.

caused by stock pollution (with some decay and pollution control), has been characterized by Clarke and Reed (1994), with a focus on the role playing by the conditional probability (the hazard rate) as well as the exogenous size of the catastrophe. Gjerde et al. (1998) take this problem a step further, and formulate a multi-region simulation model with a stochastic relationship between temperature, accumulated emissions and a catastrophic outcome.

Because our objective is to analyze overall optimal savings, we will relate our work to other Ramsey-like models. One approach, close to ours, is Aronsson et al. (1998). They study a standard model for optimal economic growth, with nuclear power waste that will generate a risk for a catastrophic outcome causing utility to drop to zero permanently, but with no extraction of exhaustible resources. They characterize an optimal consumption path, as we do, and identify factors that can push consumption (saving) upwards or downwards, as compared to the risk-free case. Hence, there is a precautionary effect as well as an opposing impatience effect on saving due to catastrophic risk. They derive a dynamic *Pigouvian* tax on nuclear energy so that a decentralized economy implements the optimal solution. This tax will capture all sources through which the use of nuclear energy will affect the probability for a catastrophe. Similar questions are raised by Tsur and Zemel (2008) within a model with clean and dirty energy, with the last one giving rise to emissions that accumulate to a stock that affects the probability distribution for a regime shift (climate change). By assumption, there is no current damage from emissions or from the stock itself; a feature shared by our model. A paper that has several features in common with ours, is van der Ploeg and Withagen (2014), but without modelling stochastic thresholds. They present a Ramsey growth model for the global economy, with capital accumulation and extraction of exhaustible oil reserves (as well as having some clean renewable energy available), but with current environmental cost determined by the stock of accumulated emissions. They characterize optimal saving regimes, and derive a carbon tax. (See also the DSGE-model by Golosov et. al (2014).) In a recently published paper van der Ploeg and de Zeeuw (2018) explicitly model stochastic tipping points within the framework of the Ramsey growth model, along the line suggested by Tsur and Zemel (op. cit.).[4] However, they do not incorporate extraction and, hence, the intertemporal management of exhaustible oil reserves into their overall saving decisions, but focus on capital

[4] See also Lemoine and Traeger (2014).

accumulation, with the use of abundant fossil fuel along with clean renewable energy, and how emissions from using fossil fuel will impact on catastrophic risk. Within a model with a specified utility function, they derive detailed decision rules for capital accumulation that incorporates the additional required rate of return on saving due to catastrophic risk, which bears strong resemblance to our decision rule. They also characterize the catastrophic-adjusted social cost of carbon (SCC) when taking into account the Present discounted value (PDV) of expected marginal and non-marginal damages, close to our precautionary tax on fossil fuels. Their SCC has a rich flavor capturing a large number of parameters characterizing the environment, preferences and technology. Explicit formulation of an exhaustibility constraint on natural resources, within a context similar to van der Ploeg and de Zeeuw, is to our knowledge analyzed rather sparsely. One recent contribution is however, a paper by Engström and Gars (2016) – in the discrete DSGE-tradition – identifying a "Green paradox effect" caused by a lower post-catastrophic value of remaining resources. Without a carbon tax, a similar "Green paradox effect" is identified within our continuous-time framework, but without imposing the ordinary DSGE-assumptions on utility and production functions.

3. A model

Below we present a model for optimal saving, with consumption, capital accumulation and resource extraction (fossil fuels), and combine this model with the possibility of an irreversible regime switch, induced by hitting a threshold or tipping point of random position. Our main contribution, in addition to the derivation of a precautionary tax on fossil fuel, is the explicit formulation of an intertemporal optimality condition – a synthesis of the Keynes-Ramsey condition for optimal saving and the Hotelling Rule for optimal management of an exhaustible natural resource – along with a catastrophic risk-adjusted social rate of discount.

In order to find an optimal strategy, we have to distinguish between the periods before after a catastrophe. A catastrophe can take place through a substantial rise in sea level caused by ice melting due to a rise in global temperature, with the loss of land and other essential resources. This will normally take a long time, but we ignore, for ease of exposition, any lags in the model, as discussed by van der Ploeg and de Zeeuw (op. cit.).) If a catastrophe should occur, the economy will adapt to the new situation, and start from "scratch" (the outset of a continuation regime) with a new set of downgraded stocks of reproducible real capital (k) and non-reproducible natural resources (s), with no further environmental consequences caused by emissions of GHGs. This is of course a very simplified representation of the continuation regime, but makes the model transparent, and helps to focus on the main problem for human mankind: what to do before a threshold is hit, and how to avoid it.

For values of capital equipment and natural resources at the start of a continuation period, given by a pair (k,s), the expected continuation payoff is simply $W(k,s)$, which is the stationary value function derived from an ordinary dynamic optimization problem after a catastrophe has occurred, from that date to infinity. Within a convex environment as ours, we know that this value function is increasing and concave in (k,s). We ignore all demographic aspects, despite the fact that population growth and fighting poverty around the world will have a significant impact on emissions in the future; see Hoel and Holtsmark (2012) for a discussion. We also rule out any abatement activity and no technical progress; of course highly unrealistic, but will help making the model more tractable.

We apply the utilitarian welfare criterion, given as the present discounted utility of all future consumption flows, with a felicity function of current flow consumption for a representative consumer with infinite lifetime. Instead of incorporating current disutility from a stock pollutant, as was done in the seminal paper by Keeler et al. (1971) and by Hoel and Kverndokk (1996), we let this stock affect the probability distribution for hitting a threshold triggering a regime shift. From an ex ante perspective, what matters is *if* and *when* the threshold is hit as well as the *expected downgrading* of future production and consumption possibilities. Thus, the stock pollutant does not itself cause any direct harm prior to a catastrophe, but affects the probability beliefs of hitting a threshold, causing a more or less severe switch in living conditions, given by the *expected* downgrading of future production capacity.

To derive an optimal strategy we solve the problem as a standard contingency planning problem in continuous time, similar to what we find in Dasgupta and Heal (1974), Kamien and Schwartz (1978), Davison (1978), and Dasgupta (1982). We specify a state variable, $Z(t)$, as the stock of accumulated emissions of GHGs at point in time *t*, obeying the state equation $\dot{Z}(t) := \frac{dZ(t)}{dt} = D(R(t))$, where D is the emission technology, and with D being twice differentiable, increasing and convex in the rate of extraction of a non-renewable resource (fossil fuel like oil or coal) at t, given by $R(t)$. In the following we choose units of measurement so that the D- function is linear. Hitting a threshold is modelled as reaching a specific level of Z that triggers an irreversible switching of the economy from one regime to another, as represented by a realization of a random downgrading variable, $\tilde{a}$. However, in time space, the position of a threshold is random, but it will of course depend on the level of Z.

Define the stochastic threshold by a random position Y, in the state space. Then we can derive a probability distribution $G(Z(t)) := \Pr(Y \leq Z(t))$, with $G(0) = 0$ and $\lim_{Z\to\infty} G(Z) = 1$. From this a priori distribution we get the probability distribution for *when* the threshold is hit, as represented by the random variable in time space, as defined by $T := Z^{-1}(Y)$. We rule out any natural decay. The state variable Z will be everywhere increasing; so $G(Z(t)) = \Pr(Y \leq Z(t)) = \Pr(Z^{-1}(Y) \leq t) := \Pr(T \leq t) := \Omega(t)$,

with the unconditional density for the event T to occur in a short time interval of length dt, being $G'(Z)dZ = G'(Z(t)) \cdot \dot{Z}(t)dt := \Omega'(t)dt$. From this distribution we get a conditional density or a hazard rate in time space, for the threshold to be hit during a short period of time, $\left[t, t + dt\right]$, given non-occurrence prior to t, when having a stock of GHGs equal to $Z(t)$, as $\dfrac{\Omega'(t)dt}{1-\Omega(t)} = \dfrac{G'(Z(t))\dot{Z}(t)}{1-G(Z(t))} := h(Z(t))R(t)$.

The overall ex ante planning problem is then:

$$Max_{(c,R)} \int_0^\infty \Omega'(\tau) \left[\int_0^\tau e^{-rt} U(c(t))dt + e^{-r\tau} W(S(\tau), K(\tau)) \right] d\tau$$

$s.t.$

$$\dot{K}(t) = F(K(t), R(t)) - c(t) - \delta K(t),\ K(0) = K_0 > 0,\ \lim_{t\to\infty} K(t) \geq 0$$

$$\dot{S}(t) = -R(t),\ S(0) = S_0,\ S(t) \geq 0\ \forall t$$

$$\dot{Z}(t) = R(t) \text{ with } Z(0) = Z_0,\ \textit{no conditions on } \lim_{t\to\infty} Z(t)$$

We have a standard felicity function, $U(c)$; increasing and strictly concave, and a standard neo-classical production function $F(K,R)$, with K being a reversible and fully malleable capital stock, with a constant rate of decay, δ. The felicity rate of discount is r. We assume that $\lim_{c\to 0} U'(c) = \infty$ and $\lim_{c\to\infty} U'(c) = 0$, and that F obeys the Inada-conditions. In addition, we assume that each input is essential. At any point in time *t*, the remaining stock of the exhaustible natural resource is $S(t)$.

The last term in the objective function is the expected continuation welfare – see Appendix 1 – written as a function of remaining reserves of the exhaustible resource and capital equipment, at the end of the pre-catastrophic regime. If, and when we hit a threshold, one downgrading outcome will be realized with one specific production possibility set for the corresponding continuation regime. What to do from that date on is given as the solution of the continuation program for any realized set of state variables, as shown in Appendix 1, with necessary conditions for the full program given in Appendix 2. In the next section, we consider in detail the strategy to follow, according to our welfare criterion, as long as no threshold is hit, given the planner's expectations or beliefs about the initial state of the continuation program.

4. An optimal saving strategy

As shown in Appendix 2, an optimal strategy to follow at *t*, given no threshold so far, must obey the following conditions:

First, define the spot price (measured in units of utility) of consumption at some point in time *t*, conditional on a regime switch not having occurred so far, as $P(t) := \dfrac{p(t)}{1 - G(Z(t))}$ with $p(t)$ as the current shadow value of the capital stock. In Appendix 2, we have that prior-to-a-shift-consumption, at some point in time t, must obey:

$$(1) \quad (1 - G(Z^*(t)) \cdot U'(c^*(t)) = p(t) \Rightarrow U'(c^*(t)) = P(t) := \frac{p(t)}{1 - G(Z^*(t))}$$

As seen from the start of the planning period ($t = 0$), $e^{-rt} p(t)$ is the present value price of consumption, in units of utility. For a program to be optimal this present value price should be equal to the expected present value of marginal utility of consumption,

$e^{-rt}(1-G(Z^*))\cdot U'(c^*(t))$. One might regard $e^{-rt}p(t)$ as the price to be paid at $t=0$, for delivery of one unit of consumption at *t*, given the non-occurrence of a catastrophe by *t*.

On using (a-4) in Appendix 2, we can state the following modified Keynes-Ramsey result:

Proposition 1: For an optimal extraction (and emission) path and for a given output, the planner should, as long as no catastrophic event has occurred, trade off current consumption and capital accumulation according to the catastrophic risk-modified Keynes-Ramsey condition, as given by:

$$(2)\quad r+h(Z^*(t))\cdot R^*(t)+\hat{\omega}(c^*(t))\frac{\dot{c}^*(t)}{c^*(t)}=r-\frac{\dot{p}(t)}{p(t)}=F_K(K^*(t),R^*(t))-\delta+h(Z^*(t))\cdot R^*(t)\frac{W_K}{P(t)},$$

The common rate equalizing the two sides is the social rate of discount at t (with output as numeraire), denoted in the following as $\rho(t):=r-\dfrac{\dot{p}(t)}{p(t)}$.

On the LHS of (2) we have a time-dependent risk-adjusted consumption rate of interest (a required rate of return from deferring consumption at some point in time as long as we have not entered the continuation regime). The term on the RHS, on the other hand, is a risk-adjusted real rate of return on capital, with the last term, according to Dasgupta (op. cit.), being interpreted as a risk premium term.

Here $W_K:=\dfrac{\partial W(S(\tau),K(\tau))}{\partial K}$ gives the expected marginal value of capital at the outset of the continuation regime.

We observe that, compared to the risk-free optimality condition given in footnote 1, the pure rate of impatience (the utility discount rate, r), is supplemented by a hazard rate in time space due to accumulation of GHGs from extraction of fossil fuels, which, on its own, should induce less saving and more consumption. This upwards adjustment captures a pure "risk-magnified impatience effect", also found in Aronsson et al. (op. cit.). This additional term can be justified by taking into account that a positive pure rate of impatience was introduced into dynamic modelling so to capture a fear of extinction. From that point of view, adding this hazard rate, as done on the LHS of (2), due to catastrophic risk, will make sense; cf. Yaari (1965).

How capital accumulation is affected by a catastrophic risk, depends on a risk premium term, as given by the marginal rate of substitution, $\frac{W_K}{P}$, on the RHS of (2), as well as the hazard rate (in time space) which is affected by the history of resource extraction or accumulation of GHGs. The risk premium term is in some way related to what van der Ploeg and de Zeeuw (op. cit.) have called "the precautionary return on capital accumulation", which operates as a counteracting "be prepared"- motive for increased saving.

To see how manmade catastrophic risk should affect the current optimal consumption-saving decision, as long as no catastrophe yet has occurred, we can alternatively look at the contingent consumption path, as characterized by:

$$(3) \quad \hat{\omega}(c^*(t))\frac{\dot{c}^*(t)}{c^*(t)} = F_K(K^*(t), R^*(t)) - \delta - r + h(Z^*(t)) \cdot R^*(t)\left[\frac{W_K - P(t)}{P(t)}\right]$$

This condition is similar to one derived by Dasgupta (op. cit.), and by van der Ploeg and de Zeeuw (op. cit.). The last term on the RHS of (3) is the additional risk element required for taking account of the prospect of a future catastrophe. Suppose that we can keep the elasticity of marginal utility of consumption constant (say, with an absolute value around two). Then, for a given extraction path, the consumption path will be more positively sloped the higher is the last term on RHS of (3). In particular, if $W_K > P = U'(c^*)$, along with a higher value of the hazard rate $h(Z(t))\dot{Z}(t)$, the "steeper" is the consumption path, because there is an upwards adjustment of the rate of return on capital. This feature will indicate a motive for pushing down or lowering current consumption (as long as no threshold has been hit), accompanied by higher capital accumulation. In this case, we have a further precautionary saving motive.

On the other hand, if $W_K < U'(c^*)$, then the last term will make the consumption path less steep, indicating a motive for less capital accumulation. In that case, the precautionary saving motive becomes weaker (stronger impatience effect).

A special case of the model is the "Doomsday Scenario"; as stated below in Proposition 2:

Proposition 2: Under the assumption that any input is essential; with $W(0,0) = W(K,0) = W(0,S) = 0$, *for positive values of either* K *or* S, *and with*

$W_K(K,0) = W_S(0,S) = 0$*, as well, a "Doomsday"- scenario, with complete destruction of the production possibilities should a threshold be hit, the only risk correction in the consumption-saving trade-off (2), is through increasing the impatience effect ("a higher probability of extinction").*

In this special case, the saving incentive becomes significantly weaker, because we are now in an ex-ante situation where a natural question to ask is "why save if we cannot expect to reap the future or post-catastrophic benefits?" This is also in accordance with the results derived by van der Ploeg and de Zeeuw (op. cit.).

So far, our conclusions confirm the ones being derived in the literature. However, the catastrophe-modified Keynes-Ramsey result in (2) is only one element of the full set of optimality conditions. We have to consider the multi-role played by resource extraction as well; as an essential input in production, as an asset due to exhaustibility, but also as a source for the potential environmental damage or catastrophic risk. Because the future state of the world is strongly affected by accumulated emissions and the corresponding conditional probability distribution for a catastrophe, it is not possible to separate the discussion of the consumption-saving decision from resource extraction (the "Hotelling-issue"). We might conjecture that if the planner anticipates an extreme drop in production capacity should a threshold be crossed, the planner will either adopt a more capital-intensive technique or a general contraction of output with less use of both inputs to prevent the threshold to be crossed. Then we will not have the problem with "Green Paradox". Hence, there are countervailing operating forces in light of a more or less likely catastrophe. If a catastrophe is expected to have severe welfare implications – like a "Doomsday" – one obvious way is to implement a policy with less resource extraction, and hence less emissions. Let us see that this feature follows from our set-up.

First we have to consider the decision rule for current extraction, related directly to the use of fossil fuel as input; thereafter we have to consider the overall asset management decision, related to the balancing between extraction and non-extraction (resource saving), as well as capital accumulation, as a supplement to (2). These decisions are of course closely related. Below we let $Q(t)$ be the conditional shadow value of the remaining stock of resources at time t, conditional on no catastrophe prior to t.

An optimal strategy for resource extraction must obey the following condition; cf. (a-2), (a-4) and (a-5) in Appendix 2:

(4) $$P(t)F_R(K^*(t),R^*(t)) = Q(t) + \Theta(t)$$

where $\Theta(t) := \int_t^\infty e^{-r(\tau-t)} \cdot \frac{1-G(Z^*(\tau))}{1-G(Z^*(t))} \cdot h(Z^*(\tau))R^*(\tau) \cdot \frac{U(c^*(\tau)) - rW^+}{R^*(\tau)} d\tau$.

The LHS of (4) is the conditional (on not having crossed a threshold) valuation, in terms of utility, of using a marginal resource unit as input as viewed from the vantage point in time t, whereas the RHS shows the corresponding conditional expected social marginal cost, in units of utility. The first term, $Q(t)$, is the conditional expected shadow value of the remaining reserves – see below – or the conditional expected resource rent, whereas the second term, $\Theta(t)$, is the present value of the conditional expected environmental marginal cost of extraction. This term is linked to what Tsur and Zemel (op. cit.) have called "the Pigouvian hazard" tax; see also Aronsson et al. (op. cit.).

Let us take a closer look at $\Theta(t)$. Suppose that current extraction is increased by one unit during a short interval of time $\left[t, t+dt\right]$, given that we have not yet reached the continuation regime at t, and we follow an optimal strategy thereafter. On following the elegant interpretation provided by Loury (op. cit.), we then have that the rate of increase in accumulated stock Z will be higher during this interval, implying that the critical level of accumulated emissions that will trigger a regime shift is realized sooner. If the critical level were to be reached at some $\tau > t$, when $Z = Z(\tau)$, a unit increase in previous extraction as suggested, will make that point in time to come sooner, by $\frac{1}{R(\tau)}$ time units. The rate of utility loss from hitting the threshold at τ, is $U(c^*(\tau)) - rW^+$, with a corresponding *total loss*, discounted back to t, as given by $e^{-r(\tau-t)} \frac{U(c^*(\tau)) - rW^+}{R^*(\tau)}$. However, as seen from t, the critical instant τ is stochastic with a conditional density function $\frac{G'(Z^*(\tau))R^*(\tau)}{1-G(Z^*(t))} d\tau = \frac{1-G(Z^*(\tau))}{1-G(Z^*(t))} \cdot h(Z^*(\tau))R^*(\tau) d\tau$; as the product of "the odds ratio", $\frac{1-G(Z^*(\tau))}{1-G(Z^*(t))} \leq 1$, declining in τ, and the hazard rate in time space, $h(Z^*(\tau))R^*(\tau)d\tau$. Hence the last term on the RHS of (4), $\Theta(t)$, is simply the present value of the expected welfare loss caused by the shortening the time until the threshold is hit due to a marginal increase in emissions or extraction at t.

One might interpret $\Theta(t)$ in (4) as a conditional *precautionary* tax on fossil fuels or simply a carbon tax, in units of utility, imposed at t as long as no catastrophe has occurred. The marginal tax rate internalizes, as should be the case, future expected cost of current emissions (due to extraction) as long as no climate change has yet occurred. The tax formula depends on traditional discounting, on the conditional probability beliefs of a climate change, and hence on the current stock of GHGs in the atmosphere, on the utility loss from a regime switch, and on the emission technology. Dividing through by $P(t)$, gives the tax, in units of output, per unit extraction. The tax differs from the one derived by Golosov et al. (2014), within their DSGE-framework, and also from the carbon tax derived in the non-stochastic framework studied by Acemoglu et al. (2012). The marginal tax at t is higher the higher is the welfare loss from hitting a threshold, as expected.

However, it is not obvious how the tax rate itself behaves over time and its dependence on accumulated stock of emissions: The discounted odds ratio $e^{-r(\tau-t)}\dfrac{1-G(Z^*(\tau))}{1-G(Z^*(t))}$ is declining with τ, whereas $h(Z^*(\tau))$ is increasing in τ, with a rate of welfare loss depending on the consumption path.

To complete the characterization of an optimal strategy, we should consider how the conditional resource rent $Q(t)$ moves over time. Above we have defined the conditional shadow value of the remaining resource as $Q(t) := \dfrac{q(t)}{1-G(Z(t))}$, where q is the unconditional shadow value of the remaining resource stock. Then from Appendix 2 we can derive a modified no-arbitrage condition in the pre-threshold regime, which has the flavor of a risk-modified Hotelling Rule, as given by:

$$(5) \quad rQ(t) = \dot{Q}(t) + h(Z^*(t))R^*(t)\left[W_S - Q(t)\right]$$

where $W_S := \dfrac{\partial W(S(\tau), K(\tau))}{\partial S}$.

This no-arbitrage condition has a standard interpretation. At any point in time as long as no tipping point has been hit, the marginal cost of delaying extraction (resource saving), in terms of utility, given by rQ, must be balanced against the expected marginal benefit from delaying extraction. The latter is the sum of the instantaneous "capital gain", $\dot{Q}(t)$, and the change in the expected shadow value of the resource, should a catastrophe occur in the "very near" future. We can alternatively write (5) as:

$$(5)' \quad \frac{\dot{Q}(t)}{Q(t)} = r + h(Z^*(t))R^*(t) \cdot \frac{Q(t) - W_S}{Q(t)} = r - h(Z^*(t))R^*(t)\left[\frac{W_S}{Q} - 1\right]$$

How this shadow value will move over time, will depend on whether the future expected shadow value exceeds or fall below the current conditional one. The term $\frac{W_S}{Q}$ is a marginal rate of substitution between the two states, or a risk premium term. It shows how many units of the resource one is willing to save at some point in time *t* as long as no threshold has been in order to have one more unit at the start of a continuation regime, should that occur in the "very near future". If this rate of substitution is less than one (i.e., with $Q > W_S$) – or that the expected rate of return from resource saving, net of capital gain, $\frac{W_S}{Q} - 1$, is negative – then there is a weak incentive for saving, with a "low" value of the current (conditional) net price to the resource owners. In this case the resource rent will increase at a rate above the utility discount rate, with a downwards adjustment of the resource rent. *Without* a global principal and no carbon tax imposed, but with a competitive resource sector having rational expectations, then we would have too high extraction and too high emissions now, and an increased likelihood for hitting a catastrophe. Hence, in that case, we will identify a "Green Paradox". If, on the other hand, $W_S > Q$, there is in itself a motive for further resource saving. Even without a global carbon tax as suggested above, this will reduce the "Green Paradox" feature.

In the full, regulated optimum, with an optimal precautionary tax on emissions, resource extraction is determined from the consumer price, $PF_R = Q + \Theta$. For instance, if expectations are extremely pessimistic, as in Proposition 2, the resource rent will be low, but to counteract the private incentive to increase extraction, the carbon tax is high, with both terms affected by the rate of extraction itself. Hence, there is not obvious that the model features the problem with "Green Paradox". In fact, the global principal's objective is expected to prevent a catastrophe through lowering extraction and hence emissions.

5. The risk-modified Ramsey-Hotelling condition

The shadow price Q defined above is in units of utility. To reconcile the dynamics of the resource rent with the Ramsey-condition in (3), we have to measure the resource rent in units of output. Defined in this way we get $\pi(t) := \dfrac{Q(t)}{P(t)} = F_R - \Lambda(t)$,

where $\Lambda(t) := \dfrac{\Theta(t)}{P(t)}$ is the expected marginal environmental cost of resource extraction ("carbon tax") in units of output.

From (*a-3*) and (*a-4*) in Appendix 2, we then have $\dfrac{\dot{\pi}(t)}{\pi(t)} = \dfrac{\dot{q}(t)}{q(t)} - \dfrac{\dot{p}(t)}{p(t)}$, while, on using (2) above, we get

$$(6) \quad \frac{\dot{\pi}(t)}{\pi(t)} = \rho(t) - h(Z^*(t))R^*(t)\frac{W_S}{Q(t)}$$

From the resource owners' point of view, what matters, as long as no catastrophe has occurred, is how the newly defined conditional rent or producer price, π, will move over time. As noted above, a high post-catastrophic shadow value W_S, relative to the conditional current one, Q, gives an incentive to delay current extraction; but if W_S is small relative to the current resource rent, then there is the opposite incentive. We can therefore state the following proposition:

Proposition 3: Let $\rho(t)$ be the social rate of discount (with output as numeraire), and $\pi(t)$ being the expected resource rent in units of output. Then, when the global planner faces a stochastic catastrophe induced by extraction of fossil fuels, the Hotelling-rule or, the conditional no-arbitrage condition for managing the resource as an asset, will be given by

$$(6)' \quad \rho(t)\pi(t) = \dot{\pi}(t) + h(Z^*(t))R^*(t)\frac{W_S}{P(t)}$$

We observe that the higher is the last term on the RHS of (6)', the higher is, cet. par., the expected benefit from delaying extraction. Hence, the incentive to delay extraction, is stronger the higher is the hazard rate, for a given expected shadow value, and/or, the higher is the post-threshold shadow value, $\frac{W_S}{P}$, for a giver hazard rate. Because all terms on both sides of (6)' are in fact determined simultaneously, we must be careful when interpreting the condition.

In the Doomsday scenario, with $W_S = 0$, the incentive to delay extraction becomes weaker, as seen from (6)', where the last term now will vanish. From before we have, however, that pre-catastrophic extraction should obey $F_R(K^*(t), R^*(t)) = \pi(t) + \Lambda(t)$, with $\Lambda(t)$ as a marginal precautionary tax on fossil fuel in units of output. Hence the dynamics of the resource rent, as seen from (6), shows how the difference between the consumer (or market) price of energy and the tax (both in units of output), given by the resource rent, $F_R - \Lambda := \pi$, should behave over time, i.e., according to $\frac{\dot{\pi}(t)}{\pi(t)} = \frac{d}{dt}\ln[F_R - \Lambda]$.

In one special case, with no prospect of any regime switch, the resource rent as measured above is simply given by $F_R(K, R)$, which along an intertemporal optimal plan should increase at a rate equal to the social rate of discount; cf. footnote 1. In the opposite case, if the catastrophe should turn out to be of the Doomsday-type, with $W_S = 0$, the resource rent, $\pi = F_R - \Lambda$, should increase at a rate equal to the social rate of discount, but with a net price $\pi(t)$ adjusted downwards. In this case, a high precautionary tax should be imposed in order to induce less resource extraction, and hence lower the likelihood for a catastrophic outcome.

By putting all these elements together, we can characterize the catastrophe-modified (Ramsey-Hotelling) optimality conditions for the global economy by the risk-modified social rate of discount at some point in time as long as no catastrophe has occurred; cf. the conditions as given in footnote 1:

$$(7)\begin{cases} \rho(t) = r + h(Z^*(t))R^*(t) + \widehat{\omega}(c^*(t))\dfrac{\dot{c}^*(t)}{c^*(t)} \\ = \dfrac{d}{dt}\ln[F_R - \Lambda] + h(Z^*(t)) \cdot R^*(t)\dfrac{W_S}{Q(t)} \\ = F_K(K^*(t), R^*(t)) - \delta + h(Z^*(t)) \cdot R^*(t)\dfrac{W_K}{P(t)} \end{cases}$$

In the first line, we have the risk-modified social rate of discount or risk-modified consumption interest rate (or required rate of return on saving). Along an optimal contingency plan, this social rate of discount should be equal to the risk-modified rate of return on both assets. In the third line we have the one related to capital equipment; in the second line we have the risk-modified rate of return on resource saving. On each asset, we have to add a risk premium term to the "ordinary" rate of return, which will capture a "be-prepared-motive" for saving. The modified Hotelling rule in the second line will also take into account the hazards of resource extraction through an optimal carbon tax, affecting the net price to resource owners. This adjustment captures the hazardous features of resource extraction or a precautionary motive for resource saving to prevent a climate change. The strength of this precautionary motive depends, as we have seen, on the expected consequences of a regime switch. In the Doomsday-scenario, with $W_S = W_K = 0 = W^+$, only pure discounting (impatience) will be affected, but of course, the carbon tax will be high as well, so as to lower extraction and emissions, and by so postpone Doomsday to come.

At last let us consider the dynamics of the carbon tax or social cost of carbon in units of output, $\Lambda(t) = \dfrac{\Theta(t)}{P(t)}$. On using our previous findings, we get, as long as $\Lambda > 0$, that:

$$(8) \quad \dot{\Lambda}(t) = \rho(t)\Lambda(t) - h(Z^*(t))R^*(t)\frac{\dfrac{U(c^*(t)) - rW^+}{R^*(t)}}{P(t)}$$

The rate at which this carbon tax changes over time, will capture the correct rate of discount, $\rho(t)$, as given in (7), minus the expected welfare loss from a regime shift. Again, there are strong similarities between the properties of the dynamics of the social cost of carbon as derived by van der Ploeg and de Zeeuw (op. cit.) and our carbon tax.

The higher is the expected welfare loss from hitting a threshold, the lower is the rate of change in the carbon tax, but the higher is the tax itself. To prevent a catastrophe, emissions have to go down, which should, in our model, be implemented by having a high tax "early".

6. Conclusion

The contribution of the present paper has been to derive a modification of the conditions for optimal wealth management in a global economy with several assets, when one of the assets (natural fossil resources) has a double-edged character. Fossil fuel from extracting a non-renewable resource provides services as input in current production, along with reproducible capital equipment. However, fossil fuel is also the source for accumulation of GHGs in the atmosphere. This stock of GHGs affects the likelihood for a future catastrophe or regime switch. Should a catastrophe occur, when a tipping point of uncertain location is hit, the overall production capacity of the economy is stochastically downgraded. Therefore, to reduce the future costs of a catastrophe, precautionary or mitigating actions should be taken today through increasing current saving. The implied saving behavior will depend on current beliefs, as given by the anticipated consequences of hitting a threshold. In the extreme pessimistic case, a Doomsday scenario, the precautionary actions taken by the global planner (the "climate protocol") should be to impose a sufficiently high tax on fossil fuel to reduce the consumption of the resources as well as emissions of GHGs. An upward shift in the hazard rate of passing a threshold for a catastrophic event, given that no such event has yet occurred, will shift the tax upwards. This shift will induce less extraction of fossil fuels and may imply less capital investments as well, dependent on the substitution possibilities between fossil fuels and capital in the production of goods and services. The consumption path will be affected making the path steeper, which means lower consumption level "now".

An important result is that we derive an overall investment criterion, with a risk-adjusted social rate of discount. However, the estimation of this social rate of discount will require information that may be hard to acquire. We have a highly aggregated global model, with only one (ideally) overall planner or global principal, operating in an economy with a single production activity without any substitute for fossil fuel, facing at most one catastrophe, and with no pollution abatement. However, despite the crudeness of the model, the conclusion of the paper is clear: To avoid a catastrophe in the future from hitting a threshold of unknown position, precautionary actions seem highly necessary, and – not surprisingly – to reduce current emissions of GHGs worldwide. The paper has shown how this target may be reached, by imposing a tax on fossil fuel.

Appendix 1: The continuation payoff

For some given point in time $\tau > 0$, when the threshold is hit, with a new set of initial state variables $(K(\tau^+) = k = aK(\tau^-), S(\tau^+) = s = aS(\tau^-)$, with a realization of the random fraction $a \in [0,1]$ of the two assets from the pre-threshold regime being left over to the continuation regime, the planner's problem is (when we assume that the integral will converge and that a solution exists):

$$w(k,s) = Max_{(c,R)} \int_{\tau}^{\infty} e^{-r(t-\tau)} U(c(t)) dt$$
$$s.t. \begin{cases} K(\tau) = k, \lim_{t\to\infty} K(t) \geq 0, \dot{K}(t) = F(K(t), R(t)) - c(t) - \delta K(t) \\ S(\tau) = s, \lim_{t\to\infty} S(t) \geq 0, \dot{S}(t) = -R(t) \end{cases}$$

Because all uncertainty is resolved, this is a standard optimal control problem with a solution obeying the standard Pontryagin-conditions, with a current value Hamiltonian, where $\tilde{\lambda}$ and $\tilde{\mu}$ are non-negative costate variables, as given by:

$$H(c,R,K,S,\tilde{\lambda},\tilde{\mu}) = U(c) + \tilde{\lambda}\left[F(K,R) - c - \delta K\right] - \tilde{\mu}R.$$

Necessary conditions for an optimal solution are:

$U'(c) - \tilde{\lambda} \leq 0$, and when strict inequality implies $c = 0$

$\tilde{\lambda}F_R - \tilde{\mu} \leq 0$, and when strict inequality implies $R = 0$

$\dot{\tilde{\lambda}} = r\tilde{\lambda} - \tilde{\lambda}\left[F_K - \delta\right]$, $\lim_{t\to\infty} e^{-rt}\tilde{\lambda}(t) \geq 0$ and $\lim_{t\to\infty} e^{-rt}\tilde{\lambda}(t)K(t) = 0$

$\dot{\tilde{\mu}} = r\tilde{\mu}$, $\lim_{t\to\infty} e^{-rt}\tilde{\mu}(t) \geq 0$ and $\lim_{t\to\infty} e^{-rt}\tilde{\mu}(t)S(t) = 0$

According to our assumptions $(U' > 0, U'' < 0, \lim_{c\to 0} U'(c) = \infty$ and $\lim_{c\to\infty} U'(c) = 0)$ and the production function (essential inputs and the Inada conditions satisfied), we will have positive and finite values of the two control variables, as well as positive costate variables, so that the transversality conditions are satisfied.

Given our assumptions, it is shown, cf. Weitzman (2003); pp. 215), that $w(k,s)$, which is a time-independent value function for some given downgrading, is increasing and concave in (k,s), with $\frac{\partial w}{\partial k} = \tilde{\lambda}(\tau)$ and $\frac{\partial w}{\partial s} = \tilde{\mu}(\tau)$. For any given pair of the co-state variables, the maximized Hamiltonian $\hat{H}(K,S,\tilde{\lambda},\tilde{\mu}) = \max_{(c,R)} H(c,R,K,S,\tilde{\lambda},\tilde{\mu})$ is concave in the state variables. Hence, Arrow's sufficiency theorem holds and we have found an optimal program. But then we have as well that $\frac{\partial^2 w}{\partial k^2} = \frac{\partial \tilde{\lambda}}{\partial k} \leq 0$ and $\frac{\partial^2 w}{\partial s^2} = \frac{\partial \tilde{\mu}}{\partial s} \leq 0$.

At the point in time when the threshold is hit the assets are downgraded by a factor that is random as seen from ex ante. Let the downgrading factor be given by the random variable A (independent of time itself, by assumption), distributed on $\left[0,1\right]$, according to a differentiable and strictly increasing probability distribution $G(a) = \Pr(A \leq a)$, with density $dG(a) = g(a)da$, and with an expected downgrading EA.

Since there is one value function for each possible downgrading factor, $w(aK(\tau^-), aS(\tau^-))$, we can define the expected continuation payoff as $W(K,S) := \int_0^1 w(aK(\tau^-), aS(\tau^-))g(a)da$, and with a corresponding pair of expected shadow values or costate variables for the capital stock, and remaining reserves of the exhaustible resource, respectively as given by $\lambda(\tau) = \frac{\partial W}{\partial K} := W_K$ and $\mu(\tau) = \frac{\partial W}{\partial S} := W_S$.

Appendix 2: Optimality of the Full Program

Integrating the objective function by parts, the current value Hamiltonian can be written as:

$$H = (1 - G(Z)) \cdot U(c) + G'(Z) \cdot R \cdot W(S,K) + p\left[F(K,R) - c - \delta K\right] - qR + mR$$

where (p,q,m) are current shadow prices for the state variables (K,S,Z), while (c,R) is the pair of control variables. Let $(c^*, R^*, K^*, S^*, Z^*)$ be the vector characterizing the solution to the full program, and let W^+ be a short-hand expression for the expected continuation payoff. It is straightforward to show that an interior solution must obey:

$$(a-1) \quad U'(c^*(t)) = \frac{p(t)}{1 - G(Z^*(t))} := P(t)$$

$$(a-2) \quad p(t)F_R(K^*(t), R^*(t)) = q(t) - [m(t) + G'(Z^*(t)) \cdot W^+]$$

$$(a-3) \quad rq(t) = \dot{q}(t) + \mu(t^+) \cdot G'(Z^*(t)) \cdot R^*(t)$$

$$(a-4) \quad rp(t) = \dot{p}(t) + p(t)\left[F_K(K^*(t), R^*(t)) - \delta\right] + \lambda(t^+) \cdot G'(Z^*(t)) \cdot R^*(t)$$

$$(a-5) \quad rm(t) = \dot{m}(t) - U(c^*(t)) \cdot G'(Z^*(t)) + G''(Z^*(t))R^*(t) \cdot W^+$$

With no end-point constraint imposed on the stock pollutant, one transversality condition is $\lim_{t \to \infty} e^{-rt} m(t) = 0$. Using this in (a-5), we can solve for the conditional shadow cost of the stock pollutant to get:

$$(a-6) \quad -\frac{m(t)}{1-G(Z^*(t))} - h(Z^*(t)) \cdot W^+ = \int_t^\infty e^{-r(\tau - t)} \frac{G'(Z^*(\tau))R^*(\tau)}{1-G(Z^*(t))} \frac{U(c^*(\tau)) - rW^+}{R^*(\tau)} d\tau$$

where we have used the hazard rate for the threshold to be hit during a short period of time, $\left[t, t+dt\right]$, with a stock of emissions Z, as given by $\frac{\Omega'(t)dt}{1-\Omega(t)} = \frac{G'(Z(t))\dot{Z}(t)}{1-G(Z(t))} := h(Z(t))R(t)$. In addition we have a set of transversality conditions:

$\lim_{t \to \infty} K(t) \geq 0, \lim_{t \to \infty} e^{-rt}(1-G(Z(t)))P(t) \geq 0, \lim_{t \to \infty} e^{-rt}(1-G(Z(t)))P(t)K(t) = 0$, and $\lim_{t \to \infty} S(t) \geq 0, \lim_{t \to \infty} e^{-rt}(1-G(Z(t)))Q(t) \geq 0, \lim_{t \to \infty} e^{-rt}(1-G(Z(t)))Q(t)S(t) = 0$.

Funding: This study was funded by the Ragnar Frisch Centre of Economic Research, Oslo, Norway, and by the Department of Economics, University of Oslo.

Conflict of Interest: Steinar Strøm has received research grants from the Ragnar Frisch Centre of Economic Research, Oslo, Norway. Jon Vislie has received research grants from the Department of Economics, University of Oslo. The authors declare that they have no conflict of interest.

References:

Acemoglu, D., P. Aghion, L. Bursztyn and D. Hemous, The Environment and Directed Technical Change, *American Economic Review* 102, 131 – 166.

Aronsson, T., K. Backlund and K-G. Löfgren, 1998, Nuclear Power, Externalities and Non-standard Pigouvian Taxes, *Environmental and Resource Economics* 11, 177-195.

Clarke, H.R., and W.J. Reed, 1994, Consumption/pollution tradeoffs in an environment vulnerable to pollution-related catastrophic collapse, *Journal of Economic Dynamics & Control* 18, 991-1010.

Cropper, M.L., 1976, Regulating activities with catastrophic environmental effects, *Journal of Environmental Economics and Management* 3, 1-15.

Dasgupta, P., 1982, Resource depletion, research and development, and the social rate of discount, chapter 8 (273-314) in *Discounting for Time and Risk in Energy Markets,* ed. R. Lind, RFF press.

Dasgupta, P., and G.M. Heal, 1974, The Optimal Depletion of Exhaustible Resources, *Review of Economic Studies, Symposium,* 1-28.

Dasgupta, P. and G.M. Heal (1979), *Economic Theory and Exhaustible Resources*, Cambridge University Press.

Davison, R., 1978, Optimal Depletion of an Exhaustible Resource with Research and Development towards an Alternative Technology, *Review of Economic Studies* 45, 355-367.

Dorfman, R., P.A. Samuelson and R. Solow, 1958, *Linear Programming and Economic Analysis*, New York, McGraw-Hill.

Gjerde, J., S. Grepperud and S. Kverndokk, 1999, Optimal climate policy under the possibility of a catastrophe, *Resource and Energy Economics* 21, 289-317.

Golosov, M., J. Hassler, P. Krusell and A. Tsyvinski, 2014, Optimal taxes on Fossil Fuel in General Equilibrium, *Econometrica* 82, 41-88.

Hoel, M., and B. Holtsmark, 2012, Haavelmo on the climate issue, paper presented at the Trygve Haavelmo Centennial Symposium in Oslo, December 13-14, 2011.

Hoel, M., and S. Kverndokk, 1996, Depletion of fossil fuels and the impacts of global warming, *Resource and Energy Economics* 18. 115-136.

Kamien, M.I., and N.L. Schwartz, 1971, Optimal Maintenance and Sale Age for a Machine subject to Failure, *Management Science* 17, B495-B504.

Kamien, M.I., and N.L. Schwartz, 1978, Optimal Exhaustible Resource Depletion with Endogenous Technical Change, *Review of Economic Studies* 45, 179-196.

Keeler, E., M. Spence and R. Zeckhauser, 1971, The Optimal Control of Pollution, *Journal of Economic Theory* 4, 19-34.

Lemoine, D., and C.Traeger, 2014, Watch your step: Optimal Policy in a Tipping Climate, *American Economic Journal: Economic Policy* 6, 137-165.

Loury, G.C., 1978, The Optimal Exploitation of an Unknown reserve, *Review of Economic Studies* 45, 621-636.

Ploeg, F. van der, and A. de Zeeuw, 2018, Climate Tipping and Economic Growth: Precautionary Capital and the Price of Carbon, *Journal of European Economic Association* 16, 1577-1617.

Reed, W.J., and H.E. Heras, 1992, The conservation and exploitation of vulnerable resources, *Bulletin of Mathematical Biology* 54, 185-207.

Samuelson, P.A., and R.M. Solow, 1956, A Complete Capital Model involving Heterogeneous Capital Goods, *Quarterly Journal of Economics* 70, 537-562.

Stern, N., 2007, The economics of climate change: The Stern review, *Cambridge University Press*.

Tsur, Y., and A. Zemel, 2008, Regulating environmental threats, *Environmental and Resource Economics* 39, 297-310.

Weitzman, M.L., 2003, *Income, Wealth, and the Maximum Principle*, Harvard University Press, Cambridge, Mass.

Yaari, M., 1965, Uncertain Lifetime, Insurance and the Theory of Consumer, *Review of Economic Studies* 32, 137-150.