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Multigenerational Transmission of Culture

Abstract

This paper explores intergenerational transmission of culture and the consequences of a plausible assumption: that people care not only for their children's culture but also for how their grand-children are raised. This departs from the previous literature which, without exception, assumes parents either do not care about, or fail to consider, the effect their actions have on all future generations. The current paper models a sequential game where parents take actions trading off being close to their own preferences and influencing their children, and where parents take into account that the children face a similar trade-off when raising their children. Predictions regarding endogenous extremism, the effect of societal socialization, parents. discounting, social pressure and interaction between groups are derived. In equilibrium, parents behave more extremely than their own preferences and this effect is intensified the more extreme preferences the parent has. There may be perpetual extremizing whereby an arbitrarily long sequence of generations will behave more extremely than the first ancestor's preferences. Furthermore, interaction of groups implies more extreme initial behavior but also faster integration.

JEL-Codes: D900, J150, Z100.

Keywords: culture, integration, social pressure.

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1 Introduction

In the literature on intergenerational transmission of culture it is generally assumed that parents, when raising their children, do not take into account how their children in turn will raise the grandchildren (see, e.g., Bisin and Verdier, 2001; Vaughan, 2013; Buechel et al. 2015; Patacchini and Zenou, 2016; Cheung and Wu, 2017). This paper explores the consequences of an evolutionarily and empirically motivated observation: that humans care how their grandchildren are raised.[1]

There are at least two reasons why parents in reality would care about how their grandchildren are raised. 1) Parents may care directly about the preferences, values and more broadly the culture that the grandchildren adopt and similarly about the culture of the grand-grandchildren and so on. 2) Parents may care indirectly about how their grandchildren are raised since they care for the behavior of their children as parents. For instance, a parent may want her adult child to visit the temple regularly, but since visiting the temple is a family activity the adult child's decision is made with the purpose of affecting the grandchild.

The key complexity that arises in such a setting is that parents need to take into account that their children will solve a similar problem as themselves – the parent takes an action to influence the action of the child, but the action of the child is determined with the purpose of influencing the action of the grandchild which is meant to influence the action of the grand-grandchild and so on. This paper presents a simple and tractable framework for analyzing such decisions of *multigenerational* transmission of culture.

The baseline setting analyzes a parent from a minority culture and how it raises her child vis-à-vis a majority culture. The parent has a (for her exogenous) bliss point that she wants her action to be close to. This action along with the (fixed) culture of mainstream society determine the preference of the child. The child's preference in turn affects the child's action which affects the preferences and thus actions of the grandchild etc. The parent wants the actions of all future generations (her child, grandchild, grand-grandchild etc.) to be close to her own blisspoint.

It is shown in this setting that parents will extremize their behavior: they will take actions which are more extreme (in distance to mainstream society) than their own blisspoints. This extremizing is the strongest for intermediate values of parental influence: if the child's blisspoint is mainly determined by mainstream society then it is *pointless* for the

[1] For evolutionary motivations see, e.g., Hawkes (2003), Lahdenperä (2004) and Silverstein (2007). For anecdotal empirical evidence that grandparents care about how their grandchildren are raised see nearest family. For more scientifically oriented empirical support see, e.g, Kaptijn (2010).

parent to behave extremely; and if the child's blisspoint is mainly determined by the parent then it is *needless* for the parent to behave extremely. Furthermore, extremizing is stronger the more extreme preferences the parent has.

When extremizing is at its strongest, also the child's (and possibly also the grandchild's, grand-grandchild's etc.) actions are more extreme than the original parent's blisspoint. Perpetual extremizing – an arbitrary number of future generations behave more extremely than the original parent's blisspoint – arrives as a limit result as the minority group becomes sufficiently patient (caring sufficiently about the actions of future generations).

Also social sanctioning is analyzed: actions that deviate from mainstream society are sanctioned. Parents then take into account that not only they themselves are sanctioned the more their actions deviate from mainstream society, but also that this sanctioning will affect the actions of the children, grandchildren etc. Is the fact that the actions of future generations may be pulled towards mainstream a source for further extremizing of the parent? No, it is shown that such social sanctioning has an unambiguous effect of making all generations behave less extremely.

Finally, interaction between two groups with differing cultures is analyzed. Parents from each group take into account that their children are influenced by their own group's action and by the actions of the other group which in itself is a strategic choice. Parents also take into account that the actions of their children are determined by a similar strategic interaction with the children of the other group. It is shown that such interaction makes actions of the parents more extreme: parents compensate for the actions of the other group which fuels further extremizing by the other group, thus more extremizing by the parent. However, while strategic interaction fuels extremism within one generation, long-run integration between the groups (convergence of blisspoints) is still faster than if the parent would interact with a fixed (non-strategic) mainstream society.

2 Related literature

The novelty of this paper is in analyzing multigenerational transmission of culture – parents that take into account how their actions affect not only the next generation (like in for instance Bisin and Verdier, 2001; Buechel et al. 2015; Patacchini and Zenou, 2016; Cheung and Wu, 2017) but also later generations. The three most closely related papers are by Buechel et al. (2015), by Cheung and Wu (2017) and by Vaughan (2013).

Buechel et al. (2015) present a model where parents purposefully raise their children to become like them and, importantly, where the parent's choice variable and the children's culture are continuous variables.[2] The important and novel feature of their model, which is also used in the current paper, is that the action and traits are (control and state) variables along the same dimension. The interpretation is that parents are role models. For instance, how often the parent prays (action) affects how often the child wants to pray (trait). If one is interested in questions such as whether a person's behavior is more extreme than her own preferences this feature is essential. This is since in the basic binary-trait model (first developed by Bisin and Verdier, 2001, and then extended by many, for instance, Saez-Marti and Zenou, 2012, see Bisin and Verdier 2011 for an overview) the action (a continuous choice of effort in raising children) is along a different dimension than the parent's cultural trait. Hence actions and tastes cannot be compared in those models.

However, unlike the current paper, in Buechel et al. (2015) the parents care only about the cultural trait of the next generation and not about their actions or about traits of subsequent generations. Hence, their model boils down to a series of disconnected static decisions. In the current paper a parent cares about her child's action (and possibly later generations' actions too) but this action is determined strategically as the child tries to affect the grandchild. Hence, the game is truly dynamic in the sense that the parent has to take into account the (sub-)games all future generations are facing. Naturally, such a game is more complex to solve (the game is solved recursively and a Markov-perfect equilibrium is identified) hence the functional forms chosen and the network structure used are less general than in Buechel et al. (2015).[3] Nevertheless, the current framework is tractable in allowing analysis of various extensions. The current paper echoes some of the results of Buechel et al. (2015) but also adds to them, for instance, by showing when perpetual extremizing will arise and the effect of social pressure.[4]

Cheung and Wu (2017) present a framework similar to Buechel et

[2]Predecessor papers are Bisin and Topa (2003) who suggest a continuous model but do not solve it; Brueckner and Smirnov (2007, 2008) who show dynamics in a continuous-trait model but where parents are not making choices; and Pichler (2010) and Panebianco (2014) where the child's trait is determined by the own parent's effort (which is made along a different dimension than the trait) but not by other parents' actions (only their types).

[3]See also Hellman and Panebianco (2018) for a model where the parents choose the network.

[4]Social pressure cannot really be studied in the framework of Buechel et al. (2015) since there the parent only cares about her own child's type while social pressure is something that plausibly will affect the child's actions. Perpetual extremizing does not arise in their framework except for under a somewhat special case where in equilibrium the child adopts the opposite culture of the parent.

al. (2015) with the main difference being that in Cheung and Wu's (2017) model a child's trait will be precisely equal to the action of one individual from the previous generation (instead of a weighted average like in Buechel et al., 2015, and in the current paper). This departure is important as it may lead to divergence of traits in society. In that sense it is related to the current paper which shows a different mechanism for limited convergence – parents caring about more than one generation. Another main contribution of Cheung and Wu (2017) is showing the effect of preference curvatures (the cost of deviating from the blisspoint may be convex or concave) similar to Michaeli and Spiro's (2015, 2017) study of social norms. The current paper is less general in that sense. The model is solved for less general functional forms since the dynamics of the game are more complex than Cheung and Wu's (2017) model which, like all previous models, boils down to a sequence of disconnected static games.[5]

Finally, in Vaughan's (2013) model the child's preferences are decided in a conformity game she plays against other children and the parent can affect how much the child resists conforming to others. As such, the model provides an elegant micro-theoretic foundation for how transmission of preferences may actually take place. In that model, however, the child plays the game against the peers without taking into account how this will affect her own future actions. Hence, the agents in the model are myopic or naive in the sense that they make choices without taking into account how they affect their own future welfare. Vaughan's (2013) game therefore boils down to a sequence of disconnected two-period games. The current paper departs from this by specifically letting all generations take into account how their actions affect future generations.

[5]It is noted that in Cheung and Wu (2017) the parent cares about the child's action in a game the child plays later in life. However, this game does not affect the actions or traits of subsequent generations hence the parent does not need to take into account what grandchildren and grand-grandchildren will do and indeed does not directly care about their actions or traits. Hence, each parent essentially plays a static game against other parents in the same generation.

3 Basic framework and results

"Actions are what count, not merely thoughts" Rabbi Boruch Leff[6]

"Thought is the parent of the deed" Thomas Carlyle

The basic model consists of a dynasty with an infinite sequence of overlapping generations indexed by $i \in \mathbb{N}$. Each generation consists of one individual (thus indexed by i as well). This individual is "the parent" of one individual $(i+1)$ referred to as "the child". Each i in the dynasty is associated with a type $t_i \in \mathbb{R}$ and takes an action $s_i \in \mathbb{R}$. The game is sequential so that i takes her action before $i+1$ who takes her action before $i+2$ etc. Parent i has the objective function

$$\min_{s_i} L_i\left(s_i; t_i\right) = \min_{s_i} \sum_{j=0}^{\infty} \beta^j \frac{\left(t_i - s_{i+j}\right)^2}{2} \tag{1}$$

subject to the following constraints. For $j = 1, ..., \infty$

$$t_{i+j} = \alpha s_{i+j-1} + (1-\alpha)\, \bar{s}_{i+j-1} \tag{2}$$

$$s_{i+j} = \arg\min L_{i+j}\left(s_{i+j}; t_{i+j}\right). \tag{3}$$

That is, following (1), parent i gets a disutility when taking an action (s_i) that differs from her type (t_i), when her child's action (s_{i+1}) differs from i's type, when the grand-child's action (s_{i+2}) differs from i's type and so on for the actions of all future generations.[7] The parent cares more about the actions of close-by generations as represented by $\beta \in]0,1[$.

Following (2), the type of an individual (t_{i+j}) is determined by a weighted average of her parent's action (s_{i+j-1}) and some other action in society $(\bar{s}_{i+j-1})$. That is, the parent instills preferences in her child by being a role model in her actions. $\alpha \in]0,1[$ is exogenous and captures what influence the parent has on her child vis-à-vis the rest of society.[8] In

[6]Interpretation of weekly Torah portion, Matot (Numbers 30:2-32:42), http://www.aish.com/tp/i/ky/48959001.html, accessed May 18 2018.

[7]Thus, unlike most previous models, the parent cares about the *actions* of future generations (and not about their types). However, in line with previous models, the parent has "imperfect empathy" and evaluates deviations (of actions in this model and types in previous research) from her own type instead of from the point of view of future generations – the parent wants future generations to follow the customs she herself likes. Furthermore, unlike previous models, the parent cares about all future generations, not just the next.

[8]It may be noted that (as in Buechel et al., 2015) the child's preferences do not attain exactly the same value as the action of any one individual from the previous

this section, it will be assumed that $\bar{s}_{i+j-1}$ is constant in all generations and (w.l.o.g.) that it equals zero. This will be relaxed in Section 5, but is here meant to capture a situation where a small group is integrating in a much larger and stable society. For notational ease it will be further assumed that $t_0 > 0$.

As stated, the parent cares about the actions of future generations and not directly about their types. However, the only thing the parent can directly influence is her own child's type. The child's preferences in turn affect the child's actions since she solves an equivalent problem as the parent, as stated in (3), but, of course, taking her own and not the parent's preferences into account. Thus, the parent can instill preferences in the child which affect the actions of the child, hence also preferences and actions of later generations. Naturally, the parent takes into account this chain of intergenerational transmission of preferences when trying to affect future generations. This is clearly seen in the first-order condition (in case of an interior solution)

$$-\left(t_i - s_i\right) - \beta\left(t_i - s_{i+1}\right)\frac{ds_{i+1}}{ds_i} - \beta^2\left(t_i - s_{i+2}\right)\frac{ds_{i+2}}{ds_i} - \ldots = 0$$

where the transmission to future generations is captured by the derivatives.

The problem is complex and it is generally hard to show the universe of equilibria. Hence, the treatment is constrained to Markov Perfect equilibria. Denote the equilibrium action of an individual s_i^*. A Markov Perfect Nash Equilibrium (MPNE) is one where the strategy of each generation is a stationary function of its type: $s_i^* = s^*\left(t_i\right)$ for all i.

The quadratic nature of the objective function is tractable and enables guessing and verifying that an MPNE exists where the equilibrium strategy of each individual is linear in her type:

$$s_i^*\left(t\right) = Ct_i \ \forall i$$

where C is a constant. The first step in showing the existence of an equilibrium is to derive the parent's best response if later generations follow such a strategy and verify that the best response is linear. If later generations follow a linear strategy then $\frac{ds_{i+1}}{dt_{i+1}} = C$ and by (2) (recall $\bar{s}_{i+j-1} = 0$) follows that $\frac{dt_{i+1}}{ds_i} = \alpha$ with the chain of derivatives up to generation $i+j$ being $\frac{ds_{i+j}}{ds_i} = \left(C\alpha\right)^j$. Using this, the first-order condition

generation. Rather it is a weighted average. Thus, the parent's action affects how *similar* the child will be to the parent, but the type of the child is not dichotomous as in the basic Bisin-Verdier framework (1998).

for an interior solution becomes

$$(s_i - t_i) + \beta (C\alpha s_i - t_i) C\alpha + \beta^2 \left((C\alpha)^2 s_i - t_i\right) (C\alpha)^2 + ... = 0 \quad (4)$$

$$\leftrightarrow \sum_{j=0}^{\infty} \beta^j \left((C\alpha)^j s_i - t_i\right) (C\alpha)^j = 0. \quad (5)$$

Differentiating this with respect to s_i yields a second–order condition for an interior solution

$$\sum_{j=0}^{\infty} \beta^j (C\alpha)^{2j} > 0. \quad (6)$$

Since the power on the parenthesis is an even number, the SOC is satisfied globally[9] implying that, for each C used in the strategy of later generations, there exists a unique and interior solution to parent i's minimization problem – a unique best response $s_i^* (t_i)$ that solves (5).

Lemma 1 *If generations $i + j$ for $j = 1...\infty$ follow a linear strategy $s_{i+j} = Ct_{i+j}$ where $C \neq 0$, then the parent in generation i has a unique linear best response given by*

$$s_i^* = C_i t_i \text{ where}$$
$$C_i \equiv \frac{\sum_{j=0}^{\infty} \beta^j (C\alpha)^j}{\sum_{j=0}^{\infty} \beta^j (C\alpha)^{2j}} \quad (7)$$

Proof. *Follows from the SOC (6) holding and by rewriting the FOC (5) and noting that the expression for C_i only consists of constants.* ■

For the best response to be part of an MPNE (stationary strategies) it is necessary and sufficient that $C_i = C$. Denote by C^* a C that solves the implicit expression in (7). Analyzing it yields:

Lemma 2 *In any MPNE with linear strategies, $C^* \in]0, 1/\alpha[$.*
Proof. *See Appendix A.* ■

The lemma specifies bounds on the equilibrium strategies. The bounds imply that $C\alpha \in]0, 1[$ and hence that the implicit expression for C^* in (7) can be rewritten to

$$C^* = \frac{1 - \beta (C^*\alpha)^2}{1 - \beta (C^*\alpha)}. \quad (8)$$

Any $C^* \in]0, 1/\alpha[$ that solves this expression constitutes an MPNE with linear strategies. Solving it for C^* yields the following proposition.

[9] Except when $C = 0$ whereby the best response is $s_i = t_i$.

Proposition 1 *There exists a unique Markov-Perfect Equilibrium with linear equilibrium strategies* $s_i^* = C^* t_i \ \forall i$. *In this equilibrium* $C^* = \frac{1-\sqrt{1-4\beta\alpha(1-\alpha)}}{2\beta\alpha(1-\alpha)}$ *and*

1. *agents behave more extremely than their own type* ($C^* > 1$)*;*

2. *the more extreme a type is the more she extremizes* ($s_i^*(t_i) - t_i$ *is increasing in* t_i)*;*

3. *there is dynastic integration* ($t_{i+1} = \alpha C^* t_i < t_i$)*;*

4. *the most extremizing is attained for intermediate values of parental influence* (C^* *is hill-shaped in* α)*;*

5. *the more influential the parents are the slower is the integration* (αC^* *is increasing* α)*;*

6. *the more patient the agents are the more they extremize* (C^* *is increasing in* β)*.*

Proof. *See Appendix B.* ■

The proposition shows that agents extremize their behavior (point 1) in the sense they will take an action which is further away from the rest of society (normalized to zero) than their own type is. The reason for this is that, while extremizing implies the parent deviates from her blisspoint, she does so to mitigate the influence of society so that the actions of her children and grandchildren will be closer to her type. The most extreme types extremize their actions the most (point 2) since for any given level of extremizing ($s_i - t_i$) types far from the mainstream will see their children end up having types and thus also actions further away from them (in absolute level). Hence, extreme parents need to compensate for the pull of mainstream society more than more moderate parents. Despite this extremizing, the equilibrium displays integration over time (point 3) since $\lim_{i\to\infty} t_i = \lim_{i\to\infty} (\alpha C)^i t_0 = 0$. The logic behind this result is easiest understood by considering the payoffs if this result did not hold, that is, if the child would be more extreme than the parent. Then the child would spawn even more extreme grandchildren whose children would again be even more extreme. This way the actions would diverge implying that the original parent would want to mitigate it by behaving less extremely.

As is stated in point 4, extremizing is most pronounced for intermediate values of α. To understand the intuition for why this is the case, consider first the polar case of $\alpha \to 1$. Here it is *needless* for the parent to extremize since the child will anyway grow up to be like the parent – there is no influence from society to mitigate. In the other polar case ($\alpha \to 0$) society has a large impact on the child which the parent would want to mitigate, but extremizing is *pointless* for the parent since her influence is small. That is, she would need to take an action far from her blisspoint in order to have an effect on the child, but that is too painful. Roughly speaking, for intermediate values of α there is both a reason for the parent to try to mitigate the influence of society and the parent's attempts will also have an effect. Hence, extremizing will be high. The prediction is thus, for instance, that newly arrived immigrants will behave the most extremely in societies which neither force assimilation nor abdicate from influencing children. However, while the extremizing in the first generation is strongest for intermediate values of α, the cross-generational resistance to integrate, that is αC^*, is stronger the larger is α (point 5). Hence, while parents can offset the influence of society, this only happens partially and the more influential society is (the smaller is α) the faster the integration will go.

Finally, parents that care more about how much future generations' actions deviate from t_i (high β) will extremize more (point 6). At an intuitive level this is of course natural since they will put a relatively higher weight on later actions than on their own action's deviation. Also, they will put a relatively higher weight on generations very far into the future compared to closer generations, hence extremize more since the far-away generations are those who will potentially be deviating the most from the original parent's type. Now, while the comparative statics with respect to β may seem intuitive it should be noted that it does, in its details, not follow trivially from the assumption that parents care more about later generations. The reason is that a high β of the parent (which has a direct effect of extremizing) is accompanied by a high β also of its offspring implying that they will extremize more too hence (the indirect effect) that the parent does not need to extremize as much in the first place. As is shown, the direct effect is always stronger for the logical reason that the indirect effect on the parent is contingent on a direct effect of β on the child. Hence, if the direct effect is small, then the indirect effect need to be small as well.

Now, while point 3 states that over time the actions will converge to the mainstream, this is only an asymptotic property. In practice integration may take a long time which is illustrated in Figure 1. The action of generation 0 relative to its type is given by C^* which we know is greater than 1. The action of the next generation is $C^{*2}\alpha$ relative to generation zero's type; the action of the generation after this is $C^{*3}\alpha^2$ and so on. The figure illustrates that, depending on parameter conditions, it

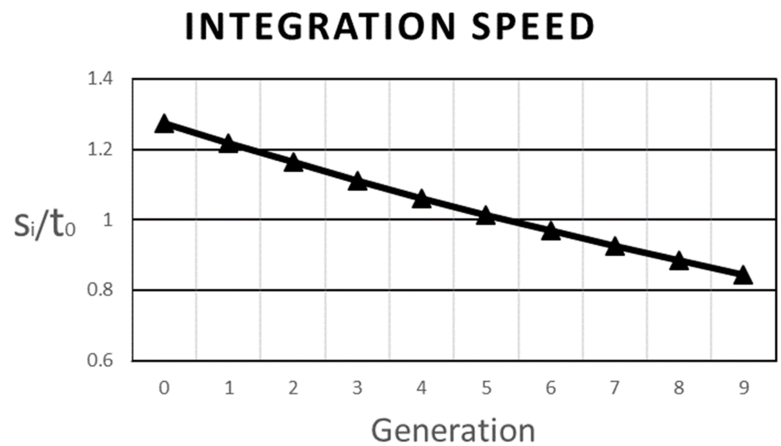

Figure 1: Action of individual of generation i relative to type of generation 0. In the illustration $\alpha = 3/4$ and $\beta = 0.9$.

may take many generations (in this case six) before the actions are less extreme than the type of generation zero. Extremizing may in fact be virtually without end.

Corollary 1 *Suppose $\alpha > 1/2$. For any finite j there exists a sufficiently large β such that $s^*_{i+j} > t_i$.*
Proof. *Follows by noting that $\lim_{\beta \to 1} \sqrt{1 - 4\beta\alpha(1-\alpha)} = |1 - 2\alpha| = 2\alpha - 1$ when $\alpha > 1/2$ implying that (from Proposition 1) $\lim_{\beta \to 1} C^* = 1/\alpha$ and $\lim_{\beta \to 1} s^*_{i+j} = \lim_{\beta \to 1} (C^*\alpha)^j C^* t_i = t_i/\alpha > t_i$ for any finite j.* ■

The corollary says that extremizing may be perpetual in the sense that it may take arbitrarily many generations before actions are closer to mainstream society than the original ancestor's preferences. This result only holds if $\alpha > 1/2$ so that parents have a larger influence on their children than society has. The intuitive reason for it *not* holding when $\alpha < 1/2$ is that, for perpetual extremizing, "all" future generations have to take an action $s_{i+j} = t_{i+j}/\alpha$ in order to mitigate the influence of society. But, when society is very influential ($\alpha < 1/2$), $s_{i+j} - t_{i+j} = t_{i+j}(1/\alpha - 1) > t_{i+j}$. That is, the action needed, by the current and future generations, to uphold perpetual extremizing is further from the parent's blisspoint than if she and her offspring fully integrated. Hence, the parent has a profitable deviation to immediate integration so that perpetual extremizing cannot be an equilibrium.[10] When, on the other

[10] The actual equilibrium when $\alpha < 1/2$ does not actually display immediate convergence. Under the knife's-edge case $\alpha = 1/2$ the intuition is somewhat different.

hand, the parent is more influential ($\alpha > 1/2$) the perpetually-extreme action ($s_{i+j} = t_{i+j}/\alpha$) is always closer to her own blisspoint than full integration is. Hence, if she cares sufficiently about the culture of future generations she will ensure that this integration takes an arbitrarily long time. Appendix F discusses the (im-)possibility of reputation equilibria using trigger strategies.

4 The effect of social pressure

I now move to analyze the effect of social sanctioning from mainstream society. To this end I incorporate the element that the individual incurs a direct loss when her behavior deviates from the majority culture ($s_i \neq \bar{s}$). This element makes it more difficult to analyze the equilibrium properties. Hence a simplification is made here that the parent only cares about the behavior of her children and not of later generations. Discounting is also abstracted from by setting $\beta = 1$. The objective function of the parent thus is

$$\min_{s_i} L_i(s_i; t_i) = \min_{s_i} \frac{(t_i - s_i)^2 + (t_i - s_{i+1})^2 + K(s_i - \bar{s}_i)^2}{2} \tag{9}$$

$$\text{s.t.} \quad t_{i+j} = \alpha s_{i+j-1} + (1-\alpha)\bar{s}_{i+j-1} \text{ and} \tag{10}$$

$$s_{i+j} = \arg\min L_{i+j}(s_{i+j}; t_{i+j}) \text{ for } j = 1...\infty \tag{11}$$

where K is the weight of social sanctioning which is a quadratic function of the deviation from mainstream society. Children are socialized in the same way as before by a convex combination of the parent's action and the fixed mainstream society (which is kept constant and normalized so that $\bar{s}_i = 0\ \forall i$). As before, the parent takes into account that her action affects the child's type and thus action where the child is in turn taking into account the type and action of the next generation. Hence, despite the parent here not caring directly about what her *grand*children and later generations do, she still has to take their actions into account since these considerations affect the child's behavior.

I again look for an MPNE guessing and verifying that the equilibrium strategy of each individual is linear in her type. Similar steps as in the previous section yield the following proposition.

Proposition 2 *There exists a unique Markov-Perfect Nash Equilibrium with a linear equilibrium strategy $s_i^* = D^* t_i\ \forall i$ where $D^* \in]0, 1/\alpha[$ is implicitly given by $D = \frac{1+\alpha D}{1+(\alpha D)^2+K}$. In this equilibrium extremizing is reduced by social sanctioning (D^* is decreasing in K).*
Proof. *See Appendix C.* ■

This proposition focuses on the effect of social sanctioning.[11] Now, given the previous finding that parental influence (α) has a non-monotonic effect on extremizing, it is ex-ante not obvious whether the effect of social sanctioning (K) will have similar or some other effect. A first direct effect of social pressure is that the parent (i) will choose a behavior more similar to mainstream society. However, since this will induce also the children to show more mainstream behavior, the indirect effect is that the parent will want to compensate for this by behaving more extremely. Is the direct or the indirect effect stronger? As the proposition expresses, the effect of social sanctioning is unambiguous – it reduces extremizing – hence the direct effect is stronger. The intuition for this is similar to why β has an unambiguous total effect despite there being a direct and indirect effect (see previous section). The existence of the indirect effect is contingent on the existence of the direct effect hence can never surpass it.

A brief discussion about the generality of this result may be in place. In this model it has been assumed that the parent cares about the behavior of her child as manifested when the child is affecting the grandchild. Hence, this is essentially a model of parents that care about the behavior of their children when the children become parents themselves. As shown, the effect of social pressure is then to unambiguously reduce extreme behavior. An alternative model choice could, however, have been made so that parents care about the actions of their children when they are still young and where these actions would not have a direct effect on the grandchildren's type (similar to Vaughan, 2013). In such a model there would have been a disconnect between actions that agents take as children and the actions they take as adults and the effect of social pressure could well have been reversed so that peer pressure makes parents more extreme.

5 Two interacting groups

I now move to analyzing interaction between groups. To this end, the simpler framework of the previous section will be used as a base but including two groups (A and B) influencing each other. W.l.o.g., the types in the first generation ($i = 0$) are normalized so that $t_{0,B} = -t_{0,A}$ and $t_{0,A} \geq 0$. Each group consists of one parent. An A-parent in generation i cares about the behavior of the next A-generation only, without discounting and without social pressure. That is, the A-parent does not directly care about B-children or later generations of A-children. The

[11] The other properties of the equilibrium are the same as in Proposition 1, short of those regarding β which does not exist here.

A-parent thus solves

$$\min_{s_{i,A}} L_{i,A}\left(s_{i,A}; t_{i,A}, s_{i,B}\right) = \min_{s_{i,A}} \frac{\left(t_{i,A} - s_{i,A}\right)^2 + \left(t_{i,A} - s_{i+1,A}\right)^2}{2}$$

subject to the following constraints. For $j = 1...\infty$

$$t_{i+j,A} = \alpha s_{i+j-1,A} + (1-\alpha)\, s_{i+j-1,B}, \tag{12}$$
$$s_{i+j,A} = \arg\min L_{i+j,A}\left(s_{i+j,A}; t_{i+j,A}, s_{i+j,B}\right) \tag{13}$$
$$t_{i+j,B} = \alpha s_{i+j-1,B} + (1-\alpha)\, s_{i+j-1,A}, \tag{14}$$
$$s_{i+j,B} = \arg\min L_{i+j,B}\left(s_{i+j,B}; t_{i+j,B}, s_{i+j,A}\right). \tag{15}$$

By (12), the child's type ($t_{A,i+1}$) is influenced both by her own parent's behavior (with weight α) but also by the behavior of the parent in the other group (with weight $1-\alpha$). I will restrict $\alpha > 1/2$ so that the own parent influences the child more than the other group's parent. Furthermore, by (13), the parent (like before) takes into account that the child's behavior is determined by an equivalent tradeoff as the parent's behavior. Finally, (14) and (15) capture that the A-parent takes into account the influence on the B-child which will affect the behavior of the A-child in (13). By (15) the A-parent is aware, and takes into account in her own actions, that the B-children are solving a similar problem. (12) and (14) jointly contain the assumption that the A-parent has en equal effect on the B-child as the B-parent has on the A-child. The A-parent takes her action simultaneously with the B-parent (thus takes the action of the B-parent $s_{i,B}$ as given).

The B-parent is solving an equivalent problem:

$$\min_{s_{i,B}} L_{i,B}\left(s_{i,B}; t_{i,B}, s_{i,B}\right) = \min_{s_{i,B}} \frac{\left(t_{i,B} - s_{i,B}\right)^2 + \left(t_{i,B} - s_{i+1,B}\right)^2}{2}$$
$$\text{s.t. (12)-(15).}$$

This way there is direct strategic interaction between the groups. Moreover, since the A-parent takes into account that there will be equivalent strategic interaction also between her child and the B-child, the A-parent also needs to take into account that her own actions influence the type of the B-child through expressions (14)-(15).

An implicit assumption underlying this structure is worth commenting. The fact that each group only consists of one parent can be interpreted in two main ways. Firstly, that the groups are small so that each parent in each group is a sizeable player. Secondly, and perhaps more interestingly, it can be interpreted as if the actions represent the leadership in each group, for instance, religious clergy in a society consisting of two religions.

In this model structure an MPNE is an equilibrium where the strategy of each generation in group A is a stationary function of its own type and the type of group B

$$s_{i,A}^{*} = s_{A}^{*}\left(t_{i,A}, t_{i,B}\right) \ \forall i;$$

and where the strategy of each generation in group B is a stationary function of its own type and the type of group A

$$s_{i,B}^{*} = s_{B}^{*}\left(t_{i,B}, t_{i,A}\right) \ \forall i.$$

The first- and second-order conditions for interior solutions for the A-parent are

$$-\left(t_{i,A} - s_{i,A}\right) - \left(t_{i,A} - s_{i+1,A}\right)\frac{ds_{i+1,A}}{ds_{i,A}} = 0 \text{ and} \tag{16}$$

$$1 - \left(t_{i,A} - s_{i+1,A}\right)\frac{d^2 s_{i+1,A}}{ds_{i,A}^2} + \frac{ds_{i+1,A}}{ds_{i,A}}\frac{ds_{i+1,A}}{ds_{i,A}} > 0. \tag{17}$$

The last element in the first-order condition (16) represents that the parent takes into account the effect her behavior has on the child's behavior. I guess (and later verify) that A-parents of all generations use a strategy

$$s_{i,A} = t_{i,A} + E\left(t_{i,A} - t_{i,B}\right) \ \forall i,$$

that is, that each parent will use her own type as baseline (first term on the right-hand side) and then extremize her actions by an amount proportional to the distance between herself and the B-group type of the same generation. An additional guess is that the equilibrium strategies for group A and group B are symmetric so that

$$s_{i,B} = t_{i,B} + E\left(t_{i,B} - t_{i,A}\right) \ \forall i. \tag{18}$$

To verify that such strategies are part of an MPNE one has to show that, if the B parent, B-child and A-child use such a strategy then it is a best response for an A-parent to use this strategy.

Under the supposed strategy, the action of the A-child is

$$\begin{aligned} s_{i+1,A} &= t_{i+1,A} + E\left(t_{i+1,A} - t_{i+1,B}\right) \\ &= \alpha s_{i,A} + (1-\alpha)\, s_{i,B} + E\left(\alpha s_{i,A} + (1-\alpha)\, s_{i,B} - \alpha s_{i,B} - (1-\alpha)\, s_{i,A}\right). \end{aligned} \tag{19}$$

The second line utilizes the A- and B-parents' influence on both child types (equations (12) and (14)). Using (19) yields

$$\frac{ds_{i+1,A}}{ds_{i,A}} = \alpha + (2\alpha - 1) E$$
$$\frac{d^2 s_{i+1,A}}{ds_{i,A}^2} = 0$$

which verifies that the second-order condition (17) is fulfilled globally. Hence, given that future A- and B-generations and the B-parent use a linear strategy the A-parent has a unique best response which is given by the solution to the first-order condition. Using the guess of the linear strategies in the first-order condition (16) and rewriting gives the best-response function of the A-parent:

$$s_{i,A}^* = \frac{t_{i,A}\left[1 + (\alpha + (2\alpha - 1) E)\right] - s_{i,B}\left[(1-\alpha) + E(1 - 2\alpha)\right](\alpha + (2\alpha - 1) E)}{1 + \alpha(\alpha + (2\alpha - 1) E) + (2\alpha - 1) E(\alpha + (2\alpha - 1) E)}. \tag{20}$$

Inserting the guessed strategy for what the B-parent does in equilibrium (18) into $A's$ best response (20) and rearranging yields

$$\begin{aligned} s_{i,A}^*\left[1 + (\alpha + (2\alpha - 1) E)^2\right] &= t_{i,A}\left[1 + (\alpha + (2\alpha - 1) E)^2\right] \\ + (t_{i,A} - t_{i,B})(1 + E)&\left[(\alpha + (2\alpha - 1) E) - (\alpha + (2\alpha - 1) E)^2\right]. \end{aligned} \tag{21}$$

It can be noted that the square brackets on the left-hand side is the same as the square brackets multiplying $t_{i,A}$ on the right of the first line. Furthermore, the second square brackets on the right consist of only parameters and multiply $(t_{i,A} - t_{i,B})$. From this follows:

Lemma 3 *If generations $i + j$ for $j = 1...\infty$ of the A-group follow a linear strategy $s_{i+j,A} = t_{i+j,A} + E(t_{i+j,A} - t_{i+j,B})$ and generations $i + j$ for $j = 0...\infty$ of the B-group follow a linear strategy $s_{i+j,B} = t_{i+j,B} + E(t_{i+j,B} - t_{i+j,A})$, then the A-parent in generation i has a unique linear best response given by*

$$s_{i,A} = t_{i,A} + E_i(t_{i,A} - t_{i,B})$$

where

$$E_i \equiv \frac{(1 + E)\left[(\alpha + (2\alpha - 1) E) - (\alpha + (2\alpha - 1) E)^2\right]}{1 + (\alpha + (2\alpha - 1) E)^2}. \tag{22}$$

Proof. *Follows from the previous text and by rearranging (21).* ■

For the best response to be part of an MPNE (stationary and symmetric strategies) it is necessary and sufficient that $E_i = E$. That is, any E that solves the equation is part of an MPNE with symmetric strategies which are linear in the distance between the groups' types. Let E^* denote such a solution to (22). By the properties of (22) a proposition follows.

Proposition 3 *Consider the model with interaction between two groups. There exists a unique Markov-Perfect Nash Equilibrium with linear and symmetric equilibrium strategies given by* $s_{i,A} = t_{i,A} + E^* (t_{i,A} - t_{i,B})$ *and* $s_{i,B} = t_{i,B} + E^* (t_{i,B} - t_{i,A})$ $\forall i$, *where* $E^* > 0$ *is given by the unique solution to (22). In this equilibrium:*

1. *agents behave more extremely than their own type* ($E \in]0, 1/4[$);
2. *the more influential the other group is, the more extremely parents behave* ($dE/d\alpha < 0$);
3. *there is dynastic integration* ($0 < t_{i+1,A} < t_{i,A}$).

Proof. *See appendix.* ∎

The proposition establishes the existence of the equilibrium as stipulated above. It also establishes that the groups extremize their behavior (point 1). This extremizing takes place to mitigate the actions of the other group as can be interpreted by point 2. Nevertheless the two groups will converge over time (point 3). The equilibrium has two additional properties which are particularly interesting but are difficult to show analytically. They are shown numerically in Appendix E hence should, strictly speaking, be seen as conjectures.

Conjecture 1 *i) Integration is faster the more influential the groups are on each other (for any* $t_{i,A}$, $t_{i+1,A}$ *is increasing in* α*). ii) The model of interaction between groups displays more extremizing but faster integration than the simple model without interaction from Section 4.*

These properties are shown to hold for a tight grid of $\alpha \in]1/2, 1[$. To illustrate the first part consider $\alpha \to 1/2$. When this is the case, extremizing is at its strongest since each group needs to mitigate the other group the most. Yet, in this case integration is very fast since the equal influence of the groups on the children implies that in the next generation the children of both groups will have fully converged. Hence, somewhat counterintuitively, the interaction model predicts a correlation in equilibrium between extreme behavior on the part of the parents and fast integration between the groups.

Point (ii) of the conjecture suggests that strategic interaction between groups has the effect of making both groups more extreme (compared to if a small group would face a large mainstream society). The reason is that an A-parent, in order to mitigate the effect from group B, behaves more extremely. But since the action taken by the A-parent affects the B-child as well, the B-parent counters this by behaving more extremely in her own direction which in turn induces more extreme behavior by the A-parent and so on – a vicious circle of extreme behavior. However, as the ending of point (ii) suggests, despite the strong extremizing, integration is faster than if there was only interaction with mainstream society. The intuitive reason for this is twofold. First, even though the A-parent is extremizing, so is the B-parent and this pulls the A-child towards the center. Second, the B-child is moving towards the center too implying integration from both sides (unlike in the mainstream-society model where one side – mainstream society – is fixed).

6 Concluding remarks

This paper has presented a tractable model of transmission of culture where parents take into account how their actions influence the culture of all future generations, not just the next generation as in the previous literature. This leads to a dynamic game between subsequent generations and predictions are derived regarding endogenous and perpetual extremism, the effect of societal socialization, parent's discounting, social pressure and interaction between groups. While this paper has considered a number of questions and extensions, the tractable nature of the model allows for more extensions and questions to be asked. In particular, it would be interesting to analyze a richer network structure, groups consisting of many parents and other functional forms.

References

[1] Büchel, B., Hellmann, T. and Pichler, M.M., (2014). "The dynamics of continuous cultural traits in social networks". *Journal of Economic Theory*, 154, pp.274-309.

[2] Bisin, A. and Topa, G., (2003). "Empirical models of cultural transmission". *Journal of the European Economic Association*, 1(2-3), pp.363-375.

[3] Bisin, A. and Verdier, T., (2001). "The economics of cultural transmission and the dynamics of preferences". *Journal of Economic theory*, 97(2), pp.298-319.

[4] Bisin, A. and Verdier, T., (2011). "The economics of cultural transmission and socialization". In *Handbook of social economics* (Vol. 1, pp. 339-416). North-Holland.

[5] Brueckner, J.K. and Smirnov, O., (2007). "Workings of the melting pot: Social networks and the evolution of population attributes". *Journal of Regional Science*, 47(2), pp.209-228.

[6] Brueckner, J.K. and Smirnov, O., (2008). "Social networks and the convergence of population attributes: A generalization". *Journal of Regional Science*, 48(2), pp.359-365.

[7] Cheung, M.W. and Wu, J., (2018). "On the probabilistic transmission of continuous cultural traits". *Journal of Economic Theory.*

[8] Hawkes, K., (2003). "Grandmothers and the evolution of human longevity". *American Journal of Human Biology*, 15(3), pp.380-400.

[9] Hellmann, T. and Panebianco, F., (2018). "The transmission of continuous cultural traits in endogenous social networks". *Economics Letters*, 167, pp.51-55.

[10] Kaptijn, R., Thomese, F., Van Tilburg, T.G. and Liefbroer, A.C., (2010). "How grandparents matter". *Human Nature*, 21(4), pp.393-405.

[11] Lahdenperä, M., Lummaa, V., Helle, S., Tremblay, M. and Russell, A.F., (2004). "Fitness benefits of prolonged post-reproductive lifespan in women". *Nature*, 428(6979), p.178.

[12] Michaeli, M. & Spiro, D., (2015), "Norm conformity across societies," *J. of Public Economics*, Vol. 132, pp. 51-65.

[13] Michaeli, M. and Spiro, D. (2017). "From Peer Pressure to Biased Norms" *American Economic Journal: Microeconomics*, 9(1): 152-216.

[14] Panebianco, F., (2014). "Socialization networks and the transmission of interethnic attitudes". *Journal of Economic Theory*, 150, pp.583-610.

[15] Patacchini, E. and Zenou, Y., (2016). "Social networks and parental behavior in the intergenerational transmission of religion". *Quantitative Economics*, 7(3), pp.969-995.

[16] Pichler, M.M., (2010). "The economics of cultural formation of preferences". Working papers No. 431, Institute of Mathematical Economics.

[17] Sáez-Martı, M. and Zenou, Y., (2012). "Cultural transmission and discrimination". *Journal of Urban Economics*, 72(2-3), pp.137-146.

[18] Silverstein, M., (2007). "Benefits of grandparents raising grandchildren". *Journal of Intergenerational Relationships*, 5(3), pp.131-134.

[19] Vaughan, D., (2013). "Nurture vs. nurture: Endogenous parental and peer effects and the transmission of culture" (No. 2013-04).

Working Papers, Banco de México.

A Proof of Lemma 2

We need to show that the best response $s_i = C_i t_i$ that solves the first-order condition (5) and meets the requirement $C_i = C$ in (7) has no solution outside the range $]0, 1/\alpha[$. If $C \geq 1/\alpha$ (also implying $C > 1 \rightarrow s_i > t_i$), then $s_i \sum_{j=0}^{\infty} \beta^j (C\alpha)^{2j} > t_i \sum_{j=0}^{\infty} \beta^j (C\alpha)^j$ violating the FOC. If $C \leq -1/\alpha < -1$ then $(C\alpha)^{2j} s_i - (C\alpha)^j t_i = t_i \left((C\alpha)^{2j} C - (C\alpha)^j \right) < 0$ since either both terms are negative or the second term is smaller in absolute value than the first which is always negative. This violates the FOC. If $C = 0$ then all terms containing C in (4) disappear hence for (4) to hold $s_i = t_i$ implying $C = 1 \neq 0$. Suppose $C \in \,]-1/\alpha, 0[$. Then adding terms from the sum for pairs of j in (5) we get for $j = 0$ and 1:

$$\begin{aligned} & \beta^j \left[(C\alpha)^{2j} s_i - (C\alpha)^j t_i \right] + \beta^{j+1} \left[(C\alpha)^{2(j+1)} s_i - (C\alpha)^{j+1} t_i \right] \\ = & \, t_i \left[C - 1 \right] + \beta t_i \left[(C\alpha)^2 C - (C\alpha) \right] = t_i \left[C - 1 + \beta (C\alpha)^2 C - \beta C\alpha \right] < 0 \end{aligned}$$

since all but the last term are negative (with $C < 0$) and $-1 < \beta C\alpha$. For $j = 2$ and 3

$$\begin{aligned} & \beta^j \left[(C\alpha)^{2j} s_i - (C\alpha)^j t_i \right] + \beta^{j+1} \left[(C\alpha)^{2(j+1)} s_i - (C\alpha)^{j+1} t_i \right] \\ = & \, \beta^2 \left[(C\alpha)^4 C t_i - (C\alpha)^2 t_i \right] + \beta^3 \left[(C\alpha)^6 C t_i - (C\alpha)^3 t_i \right] \\ = & \, t_i (C\alpha)^2 \beta^2 \left[(C\alpha)^2 C - 1 + \beta (C\alpha)^4 C - \beta C\alpha \right] < 0 \end{aligned}$$

since again all terms but the last are negative and $-1 < \beta C\alpha$. This pattern of negative additions continues for higher values of j. Hence $\sum_{j=0}^{\infty} \beta^j \left[(C\alpha)^{2j} s_i - (C\alpha)^j t_i \right] < 0$ which violates the FOC. In total this implies that no MPNE with linear strategies exists where $C \notin]0, 1/\alpha[$.

B Proof of Proposition 1

We first prove the statement on existence of precisely one equilibrium with a linear strategy and then Points 1-6.

The existence, uniqueness and linearity of the best response (given that later generations play linear strategies $s_{i+j} = C t_{i+j} \forall j > 0$) was established in Lemma (1). The best response constitutes an MPNE if and only if (8) has a solution and, by Lemma 2, $C^* \in]0, 1/\alpha[$. If this solution is unique then the linear MPNE is unique. We now investigate the properties of the left-hand side (LHS) and right-hand side (RHS) of

(8).

$$\lim_{C\to 0} LHS = 0 < \lim_{C\to 0} RHS = 1$$
$$\lim_{C\to 1/\alpha} LHS = 1/\alpha > \lim_{C\to 1/\alpha} RHS = 1$$

This implies at least one intersection (or an odd number of intersections). This establishes existence of an MPNE with linear strategies. The first intersection is one where the RHS intersects the LHS from above. Now rewrite (8):

$$1-\beta\,(C\alpha)^2 = C-\beta C^2\alpha \text{ which has the solution}$$
$$C = \frac{1\pm\sqrt{1-4\beta\alpha\,(1-\alpha)}}{2\beta\alpha\,(1-\alpha)} \tag{23}$$

where we note that $1-4\beta\alpha\,(1-\alpha)>0$ since $\max_\alpha 4\beta\alpha\,(1-\alpha)=\beta<1$ (for $\alpha=1/2$). We know there is an odd number of solutions $C^* \in]0,1/\alpha[$ to (8) hence (since (23) has two roots) C^* is unique. This establishes that there exists precisely one MPNE with linear strategies. We now check which of the two roots in (23) is permissible ($C\in[0,1/\alpha[$).[12] The larger root is positive hence permissible iff $\frac{1+\sqrt{1-4\beta\alpha(1-\alpha)}}{2\beta\alpha(1-\alpha)} < 1/\alpha \leftrightarrow$

$$\sqrt{1-4\beta\alpha\,(1-\alpha)} < 2\beta\,(1-\alpha)-1$$

which is violated if $2\beta\,(1-\alpha)-1$ is negative, hence is true only if

$$1-4\beta\alpha\,(1-\alpha) < (2\beta\,(1-\alpha)-1)^2 \leftrightarrow \ldots \leftrightarrow 0<(1-\alpha)\,(\beta-1)$$

which is not true, hence this root is not permissible. We now verify that the smaller root $C=\frac{1-\sqrt{1-4\beta\alpha(1-\alpha)}}{2\beta\alpha(1-\alpha)}$ is permissible, i.e., is in the range $]0,1/\alpha[$. We have already established that $1-4\beta\alpha\,(1-\alpha)>0$ and it is immediate that it therefore also is smaller than 1. Hence the C-root is positive hence permissible iff $\frac{1-\sqrt{1-4\beta\alpha(1-\alpha)}}{2\beta\alpha(1-\alpha)} < 1/\alpha \leftrightarrow 1-2\beta\,(1-\alpha) < \sqrt{1-4\beta\alpha\,(1-\alpha)}$. If the LHS of this inequality is negative then the inequality holds. If the LHS is positive then the inequality holds iff

$$1-\sqrt{1-4\beta\alpha\,(1-\alpha)} < 2\beta\,(1-\alpha) \leftrightarrow \ldots \leftrightarrow 0<(1-\beta)\,(1-\alpha)$$

which is true. Hence this root is permissible and unique. We get that the unique MPNE with linear strategies has

$$C^* = \frac{1-\sqrt{1-4\beta\alpha\,(1-\alpha)}}{2\beta\alpha\,(1-\alpha)}.$$

[12] Recall that the step from (7) to (8) hinges on $C^*\in[0,1/\alpha]$. Hence any solution to (8) outside this range violates the first-order condition.

Point 1: Use the implicit expression for C^* in (8) where we note that $C\alpha < 1$ (by Lemma 2) hence the RHS numerator is larger than the denominator implying that the C solving this equation must be greater than 1.

Point 2: In the equilibrium just proven $s_i^*(t_i) - t_i = Ct_i - t_i$ which is increasing in t_i since C was shown to be greater than 1.

Point 3: From Lemma 2 we know $\alpha C < 1$.

Point 4: In (8) the fact that $RHS(0) = 1 > LHS(0)$ implies the RHS intersects the LHS from above (slope smaller than 1) at C^*. Using the implicit function theorem applied to (8)

$$\frac{dC^*}{d\alpha} = -\frac{\frac{\partial RHS}{\partial \alpha} - \frac{\partial RHS}{\partial C}}{\frac{\partial RHS}{\partial C} - \frac{\partial LHS}{\partial C}} = -\frac{\beta C \frac{\left(\beta C^2\alpha^2 - 2C\alpha + 1\right)}{(1-\beta C\alpha)^2}}{\frac{\partial RHS}{\partial C} - \frac{\partial LHS}{\partial C}}.$$

We know $\frac{\partial RHS}{\partial C} - \frac{\partial LHS}{\partial C} < 0$ at C^* since RHS intersects LHS from above at C^*. Hence the denominator is negative, hence $\frac{dC^*}{d\alpha} > 0$ iff

$$\alpha\beta \frac{(\beta C^2\alpha^2 - 2C\alpha + 1)}{(1-\beta C\alpha)^2} > 0 \ \leftrightarrow$$
$$F(\alpha) \equiv \left(\beta C^2\alpha^2 - 2C\alpha + 1\right) > 0$$

which is a U-shaped function of α with $F(0) > 0$, $F(\alpha = 1/C) < 0, F'(0) = -2C < 0$ hence there is exactly one intersection with the zero-line within the relevant range. This means $\frac{dC^*}{d\alpha}$ is positive for small α and negative for larger α within the range. Hence $C^*(\alpha)$ is a first increasing then decreasing function of α. Hence we get that C^* is a hill-shaped function of α. Note also that the implicit function for C^* gives $\lim_{\alpha\to 0} C^*(\alpha) = 1$, $\lim_{\alpha\to 1} C^*(1) = 1$.

Point 5: In the equilibrium $t_{i+A}/t_i = \alpha C^* = \alpha\frac{1-\sqrt{1-4\beta\alpha(1-\alpha)}}{2\beta\alpha(1-\alpha)}$. Differentiate

$$\frac{d(\alpha C^*)}{d\alpha} = \frac{\left(2\beta - 2\alpha\beta + \sqrt{4\beta\alpha^2 - 4\beta\alpha + 1} - 1\right)}{2\beta(\alpha-1)^2\sqrt{4\beta\alpha^2 - 4\beta\alpha + 1}}.$$

We need to show that this expression is positive. Since the denominator is positive (established earlier) the whole expression is positive iff the numerator is positive, that is, iff

$$\sqrt{4\beta\alpha^2 - 4\beta\alpha + 1} > -2\beta + 2\alpha\beta + 1 \ .$$

The inequality holds if the right-hand side is negative (as established earlier the left-hand side root is positive). If the right-hand side is positive then the inequality holds if

$$4\beta\alpha^2 - 4\beta\alpha + 1 > (-2\beta + 2\alpha\beta + 1)^2 \leftrightarrow ... \leftrightarrow 1 > \beta$$

which is true. Hence $\frac{d\alpha C^*}{d\alpha} > 0$ and Point 6 follows.

Point 6: In (8) the fact that $RHS(\alpha = 0) = 1 > LHS(\alpha = 0)$ implies the RHS intersects the LHS from above (slope smaller than 1) at C^*. Using the implicit function theorem applied to (8) to analyze the effect of β we get

$$\frac{dC^*}{d\beta} = -\frac{\frac{\partial RHS}{\partial \beta}}{\frac{\partial RHS}{\partial C} - \frac{\partial LHS}{\partial C}} = -\frac{\frac{-(C\alpha)^2(1-\beta C\alpha)+\left(1-\beta(C\alpha)^2\right)C\alpha}{(1-\beta C\alpha)^2}}{\frac{\partial RHS}{\partial C} - \frac{\partial LHS}{\partial C}} = -\frac{\frac{-(C\alpha)^2+C\alpha}{(1-\beta C\alpha)^2}}{\frac{\partial RHS}{\partial C} - \frac{\partial LHS}{\partial C}}.$$

We know $\frac{\partial RHS}{\partial C} - \frac{\partial LHS}{\partial C} < 0$ at C^* since RHS intersects LHS from above at C^*. Hence the denominator is negative. The numerator is positive (since $C\alpha < 1$). Thus, $\frac{dC^*}{d\beta} > 0$.■

C Proof of Proposition 2

The proof will be performed for $t_0 \geq 0$. Equivalent steps would follow for $t_0 < 0$. The first- and second-order conditions for an interior solution for the problem in (9)-(11) are

$$-\left(t_i - s_i\right) - \left(t_i - s_{i+1}\right)\frac{ds_{i+1}}{ds_i} + Ks_i = 0 \tag{24}$$

$$1 - \left(t_i - s_{i+1}\right)\frac{d^2 s_{i+1}}{ds_i^2} + \left(\frac{ds_{i+1}}{ds_i}\right)^2 + K > 0 \tag{25}$$

We guess and verify that there exists an MPNE with linear strategy $s_i = Dt_i \; \forall i$ which, using (10) implies

$$s_{i+1} = Dt_{i+1} = D\alpha s_i$$

which further implies

$$\frac{ds_{i+1}}{ds_i} = D\alpha$$
$$\frac{d^2 s_{i+1}}{ds_i^2} = 0.$$

Using these in (25) shows that the second-order condition is fulfilled globally for any $D \neq 0$ hence there exists a unique interior best response for parent i given that later generations have a linear strategy. If $D = 0$ then the best response is $t_i = s_i$ which implies $D = 0$ is not part of an MPNE.Using the derivatives in the FOC (24) yields

$$-\left(t_i - s_i\right) - \left(t_i - D\alpha s_i\right)\alpha D + Ks_i = 0 \leftrightarrow \tag{26}$$
$$s_i = t_i \frac{(1 + \alpha D)}{\left(1 + (\alpha D)^2 + K\right)}$$

which shows that parent i's best response is linear (the right-hand side of the last expression contains one t and constants) given by $s_i = D_i t_i$ where $D_i = \frac{(1+\alpha D)}{\left(1+(\alpha D)^2+K\right)}$. For these linear strategies to constitute an MPNE (stationary strategies) they have to be the same, that is, there has to exist a $D_i = D$ that solves

$$D_i = \frac{1+\alpha D}{1+(\alpha D)^2+K}. \tag{27}$$

and this D has to fulfill the first-order condition in (24). Denote it by D^*. Note that the solution to (27) implies $D^* < 1/\alpha$. This is since (by construction) $\alpha < 1$ which implies that when $D^* \geq 1/\alpha$ then the $LHS \geq 1$ while the $RHS < 1$. Note also that $D^*\alpha \notin \left]-1, 0\right]$ since then in (27) the $LHS \leq 0$ while $RHS > 0$. Finally note that $\alpha D^* \nleq -1$ since then the left-hand side of (26) is

$$\begin{aligned} &-t_i(1-D^*) - t_i(1-D^*\alpha D^*)\alpha D^* + KD^*t_i \\ &= t_i\left[-(1-D^*) - (1-D^*\alpha D^*)\alpha D^* + KD^*\right] < 0 \end{aligned}$$

since $-(1-D^*) \leq 0$, $-(1-D^*\alpha D^*)\alpha D^* < 0$ (since $D^*\alpha D^* > 1$) and $KD^* < 0$. This violates the first-order condition. Hence any D^* that solves (27) and meets the first-order condition has $D^* \in \left]0, 1/\alpha\right[$.

We now analyze the right-hand side (RHS) and left-hand side (LHS) of (27)

$$\begin{aligned} \frac{dRHS}{dD} &= \alpha\frac{1+K-2D\alpha-(\alpha D)^2}{\left(1+(\alpha D)^2+K\right)^2} \\ \frac{d^2RHS}{dD^2} &= \alpha\frac{(-2\alpha-2D\alpha^2)\left(1+(\alpha D)^2+K\right)^2-\left(1+K-2D\alpha-(\alpha D)^2\right)2\alpha^2D\left(1+(\alpha D)^2+K\right)}{\left(1+(\alpha D)^2+K\right)^4} \end{aligned}$$

The second derivative is positive only if $\left(1+K-2\alpha D-(\alpha D)^2\right) < 0$ which happens iff the first derivative is negative. That is, when the first derivative is positive then the second derivative is negative. Note further that $\left(1+K-2\alpha D-(\alpha D)^2\right)$ is strictly decreasing in $D \in \left]0, 1/\alpha\right[$ and positive when $D \to 0$ hence the first derivative has only one switch from positive to negative. Put together this implies the RHS is first concavely rising then possibly falling. Finally note that $RHS(0) = \frac{1}{1+K} > 0$ while $LHS(0) = 0$. This with the concave form of the RHS implies a unique intersection with the LHS of (27) (furthermore, at the intersection the RHS intersects the LHS from above). This implies that there exists a unique MPNE with linear strategies where $D^* \in \left]0, 1/\alpha\right[$ is given by (27). This proves the first statement of the proposition.

To prove the second statement of the proposition we analyze the properties of D^*. Applying the implicit function theorem to (27) yields

$$\frac{dD^*}{dK} = -\frac{\frac{\partial RHS}{\partial K} - \frac{\partial LHS}{\partial K}}{\frac{\partial RHS}{\partial D} - \frac{\partial LHS}{\partial D}} = -\frac{\frac{\partial RHS}{\partial K}}{\frac{\partial RHS}{\partial D} - \frac{\partial LHS}{\partial D}}.$$

Since we know that there is a unique intersection of the RHS and LHS where the RHS intersects the 45-degree line from above it must be that $\frac{\partial RHS}{\partial D} < \frac{\partial LHS}{\partial D}$ at that point. Hence the denominator is negative. It is furthermore immediate from (27) that $\frac{\partial RHS}{\partial K} < 0$ in total implying $\frac{dD^*}{dK} < 0$ which proves the second statement of the proposition.■

D Proof of Proposition 3

That the best response of parent A (given that the B-parent and later A- and B-generations use linear strategies of the form stipulated) is unique and has the stipulated linear properties was established in Lemma 3. To show existence of an MPNE it is sufficient to show that (22) has a solution. Rewriting (22)

$$\begin{aligned} E = \frac{(1+E)\left[(\alpha + (2\alpha - 1)E) - (\alpha + (2\alpha - 1)E)^2\right]}{\left[1 + (\alpha + (2\alpha - 1)E)^2\right]} &\leftrightarrow ... \leftrightarrow \\ 0 = \left(8\alpha^2 - 8\alpha + 2\right)E^3 + \left(12\alpha^2 - 10\alpha + 2\right)E^2 & \qquad (28) \\ + \left(6\alpha^2 - 5\alpha + 2\right)E - (1-\alpha)\alpha \equiv F & \end{aligned}$$

Note that $(8\alpha^2 - 8\alpha + 2) > 0$ for all $\alpha > 1/2$ (it is increasing in α when $\alpha > 1/2$, and it equals zero when $\alpha = 1/2$) likewise for $(12\alpha^2 - 10\alpha + 2)$ and $(6\alpha^2 - 5\alpha + 2)$. Hence this expression is increasing convexly in E when $E \geq 0$ hence (since it is negative when $E = 0$) in the range where $E \geq 0$ there is a unique intersection with the zero line. To prove the first statement (a unique MPNE) it is thus sufficient to rule out that (28) has a solution where $E < 0$.

Since $-(1-\alpha)\alpha < 0$, a necessary condition for an intersection when $E < 0$ is $(8\alpha^2 - 8\alpha + 2)E^3 + (12\alpha^2 - 10\alpha + 2)E^2 + (6\alpha^2 - 5\alpha + 2)E > 0 \leftrightarrow$

$$\left(8\alpha^2 - 8\alpha + 2\right)E^2 + \left(12\alpha^2 - 10\alpha + 2\right)E + \left(6\alpha^2 - 5\alpha + 2\right) < 0.$$

As established earlier, all parenthesis in this expression are positive. Hence, the left-hand side of this expression is U-shaped in E when $E < 0$ and it is positive as $E \to -\infty$ and as $E \to 0$. For the inequality to hold it is therefore necessary that the min point (which is the unique inner extreme point) of the left-hand side is negative.

$$\frac{d(\cdot)}{dE} = 2\left(8\alpha^2 - 8\alpha + 2\right) E + \left(12\alpha^2 - 10\alpha + 2\right) = 0 \rightarrow$$
$$E_{\min} = -\frac{1}{2}\frac{(12\alpha^2 - 10\alpha + 2)}{(8\alpha^2 - 8\alpha + 2)}.$$

Plugging this back gives

$$LHS\left(E_{\min}\right) = \left(8\alpha^2 - 8\alpha + 2\right)\left(\frac{1}{2}\frac{(12\alpha^2 - 10\alpha + 2)}{(8\alpha^2 - 8\alpha + 2)}\right)^2$$
$$-\frac{1}{2}\left(12\alpha^2 - 10\alpha + 2\right)\frac{(12\alpha^2 - 10\alpha + 2)}{(8\alpha^2 - 8\alpha + 2)} + \left(6\alpha^2 - 5\alpha + 2\right)$$
$$= ... = -\frac{1}{4}\frac{\left(12\alpha^2 - 10\alpha + 2\right)^2}{(8\alpha^2 - 8\alpha + 2)} + \left(6\alpha^2 - 5\alpha + 2\right),$$

which is negative iff

$$4\left(6\alpha^2 - 5\alpha + 2\right)\left(8\alpha^2 - 8\alpha + 2\right) < \left(12\alpha^2 - 10\alpha + 2\right)^2 \leftrightarrow$$
$$... \leftrightarrow 4\left(2\alpha - 1\right)^2\left(3\alpha^2 - 4\alpha + 3\right) < 0 \leftrightarrow 3\alpha^2 - 4\alpha + 3 < 0.$$

The left-hand side of this final expression is U-shaped in $\alpha > 0$ and takes on a min point when $\alpha = 4/6$ with value $\left(3\left(4/6\right)^2 - 4\left(4/6\right) + 3\right) = \frac{5}{3} > 0$. Hence, there is no $E < 0$ that solves

$$\left(8\alpha^2 - 8\alpha + 2\right)E^3 + \left(12\alpha^2 - 10\alpha + 2\right)E^2 + \left(6\alpha^2 - 5\alpha + 2\right)E - \left(1 - \alpha\right)\alpha = 0$$

implying there is a unique $E > 0$ that solves (22). This concludes the first statement of the proposition.

We now move to proving the numbered statements of the proposition. Note first from (28) that $F(0) < 1$ hence F intersects the zero line from below at E^* implying $dF/dE > 0$ at that point. Using the implicit function theorem

$$\frac{dE}{d\alpha} = -\frac{\partial F/\partial\alpha}{\partial F/\partial E} = -\frac{(2E+1)(2\alpha - 3E + 8E\alpha + 8E^2\alpha - 4E^2 - 1)}{\partial F/\partial E}$$

where the denominator is positive due to the previous argument. $\partial F/\partial\alpha > 0$ since it is increasing in α (note that since we are here analyzing the partial derivative $\partial F/\partial\alpha$ we do not need to consider the indirect effect of α on E) and since, evaluated at $\alpha = 1/2$, $\partial F/\partial\alpha = \left(1 - 3E^* + 4E^* + 4\left(E^*\right)^2 - 4\left(E^*\right)^2 - 1\right) = \left(-3E + 4E\right) > 0$. Therefore, $\frac{dE}{d\alpha} < 0$ which proves **Point 2**.

Evaluating F at $\alpha \rightarrow 1/2$ yields

$$\lim_{\alpha\rightarrow 1/2} F(\alpha) = \left(8\left(1/2\right)^2 - 8\left(1/2\right) + 2\right)E^3 + \left(12\left(1/2\right)^2 - 10\left(1/2\right) + 2\right)E^2$$
$$+\left(6\left(1/2\right)^2 - 5\left(1/2\right) + 2\right)E - \left(1 - \left(1/2\right)\right)\left(1/2\right) = 0 \leftrightarrow ... \leftrightarrow E = \frac{1}{4}$$

and at $\alpha \to 1$ yields

$$\lim_{\alpha \to 1} F(\alpha) = (8 - 8 + 2) E^3 + (12 - 10 + 2) E^2 + (6 - 5 + 2) E$$
$$= 2E^3 + 4E^2 + 3E = 0 \leftrightarrow E = 0.$$

Hence, since $\frac{dE}{d\alpha} < 0$, $E^* \in]0, 1/4[$ which proves **Point 1**.

For dynastic integration use the equilibrium strategies $s_{i,A} = t_{i,A} + E^* (t_{i,A} - t_{i,B})$ and $s_{i,B} = t_{i,B} + E^* (t_{i,B} - t_{i,A})$ in the transmission equation (12) and symmetry of the types $t_{i,B} = -t_{i,A}$.

$$\begin{aligned} t_{i+1,A} &= \alpha s_{i,A} + (1-\alpha) s_{i,B} \\ &= (2E^* + 1)(2\alpha - 1) t_{i,A} \end{aligned} \tag{29}$$

Since $E^* > 0$ and $\alpha > 1/2$ $t_{i+1,A} > 0$ for any $t_{i,A} > 0$. Now, suppose dynastic integration does not hold, then

$$\begin{aligned} (2E+1)(2\alpha-1) &\geq 1 \leftrightarrow ... \leftrightarrow \\ E &\geq \frac{1-\alpha}{2\alpha - 1}. \end{aligned}$$

Now note that the coefficients multiplying E in (28) are positive implying F is increasing in $E \geq 0$. Since $F = 0$ in equilibrium and no dynastic integration happens iff $E^* \geq \frac{1-\alpha}{2\alpha-1}$ it follows that no dynastic integration happens iff $F\left(\frac{1-\alpha}{2\alpha-1}\right) \leq 0$.

$$F\left(\frac{1-\alpha}{2\alpha-1}\right) = \left(8\alpha^2 - 8\alpha + 2\right)\left(\frac{1-\alpha}{2\alpha-1}\right)^3$$
$$+ \left(12\alpha^2 - 10\alpha + 2\right)\left(\frac{1-\alpha}{2\alpha-1}\right)^2 + \left(6\alpha^2 - 5\alpha + 2\right)\frac{1-\alpha}{2\alpha-1} - (1-\alpha)\alpha \leq 0$$
$$\leftrightarrow -2\frac{\alpha-1}{2\alpha-1} \leq 0$$

which is not true since $\alpha \in]1/2, 1[$. Hence the supposition is inconsistent implying $t_{i+1,A} > t_{i,A}$. This proves **Point 3**.■

E Numerical results

This appendix corroborates numerically the claims in Conjecture 1.[13] The first claim is that, in the model with interaction between groups, integration is faster the more influential the groups are on each other,

[13] The numerical code is available on request. It essentially involves solving the implicit functions for D^* and E^* for a given α and repeating this for many values of $\alpha \in]1/2, 1[$. The numerical analysis does so for 50001 values of α within the range.

that is, for any $t_{i,A}$, $t_{i+1,A}$ is increasing in α. Recall from (29) that in equilibrium

$$t_{i+1,A} = t_{i,A}\left(2\alpha - 1\right)\left(1 + 2E^{*}\right). \tag{30}$$

To verify the claim we thus need to show that $\left(2\alpha - 1\right)\left(2E^{*} + 1\right)$ is increasing in α where E^{*}, given by (22), is an implicit function of α. Showing this analytically is hard, but illustrating it numerically is trivial and is shown in Figure 2, upper left panel (solid line).

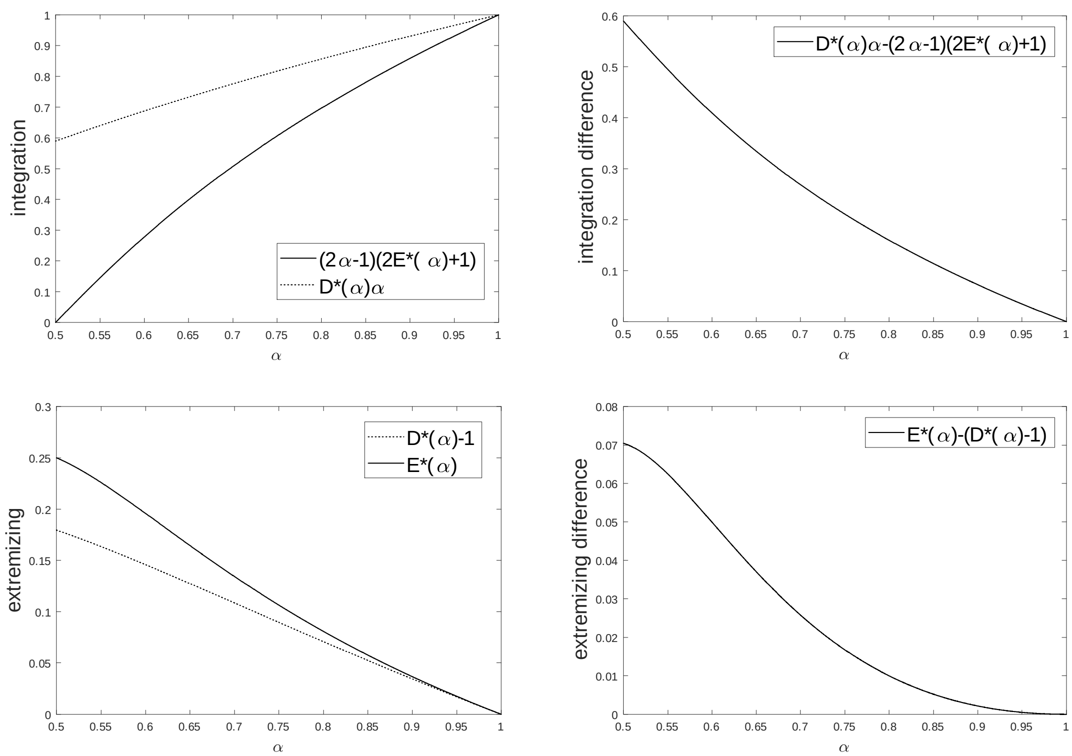

Figure 2: Numerical simulation. Upper panels: The functions $\left(2\alpha - 1\right)\left(2E^{*}\left(\alpha\right) + 1\right)$ and $D^{*}\left(\alpha\right)\alpha$ (left) and their difference (right) for different values of α. Lower panels: the functions $D^{*}\left(\alpha\right) - 1$ and $E^{*}\left(\alpha\right)$ (left)and their difference (right).

The second claim is that the model of interaction between groups displays more extremizing yet faster integration than the simple model without interaction. The model without interaction that is comparable to the one with interaction is the one presented in Section 4 when letting $K = 0$.[14]

[14] The initial model (from Section 3) is not comparable since the parent there directly cares about the actions of many generations while in the interaction model the parent cares only about the actions of the next generation.

For the part of the statement about extremizing we want to know the percentage extremizing, that is, $s-t$ divided by the distance to the other group. In the interaction model $s-t = E^* (t_A - t_B) = 2E^* t_A$ while the distance to the other group is $t_A - t_B = 2t_A$ hence the percentage extremizing is simply E which is implicitly given by (22).

In the model without interaction $s - t = D^* t - t$ and the distance to the other group (in this case mainstream society) is simply t, hence the percentage extremizing is $D^* - 1$ where D^* is given by the expression in Proposition 2. In the lower panel of Figure 2 E^* and $D^* - 1$ are compared numerically. On the lower left they are plotted separately. On the lower right the difference $(E^* - (D^* - 1))$ is plotted. As can be seen, E^* is larger than $D^* - 1$ for all values of α, but when $\alpha \to 1$ they both equal zero.

For the statement about the integration speed in Point (ii) note that, in the interaction model, integration speed is given by $\frac{t_{i+1,A} - t_{i+1,B}}{t_{i,A} - t_{i,B}}$. In equilibrium $t_{i+1,B} = -t_{i+1,A}$ hence

$$\frac{t_{i+1,A} - t_{i+1,B}}{t_{i,A} - t_{i,B}} = \frac{t_{i+1,A}}{t_{i,A}} = (2\alpha - 1)(2E^* + 1)$$

where the last step follows from (30). For the model with mainstream society the integration speed is

$$\frac{t_{i+1}}{t_i} = \frac{\alpha s_i}{t_i} = \alpha D^*$$

where the first step follows from (10) and the second step follows from Proposition 2 where the D is given in that same proposition.

For the second part of Point (ii) to be true it thus must hold that $(2\alpha - 1)(2E^* + 1) \leq \alpha D^*$ for all $\alpha \geq 1/2$. As can be seen from Figure 2 (upper left and right panel) it is indeed the case.

F Reputational equilibria

One can also consider an equilibrium with sustained extremism ($s^* = t/\alpha$ for all generations) based on a threat of a trigger strategy of gradual convergence (a reputational equilibrium). That is, if a parent in one generation does not uphold the extremeness then the next generations will gradually converge. Formally, consider the following strategy: generation 0 plays $s_0 = t_0/\alpha$; later generations i play $s_i = t_i/\alpha$ if all previous generations $k < i$ played $s_k = t_k/\alpha$, otherwise play $s_i = C^* t_i$ as given by Proposition 1. As it turns out, such a strategy is *not* an equilibrium for any α and any $\beta < 1$. To see why it is not an equilibrium when $\alpha \leq 1/2$ note the following. In the infinitely extreme subgame (superscript "inf") $s_{i+j}^{\text{inf}} = t_{i+j}/\alpha = t_i/\alpha$ for all $j \geq 0$. Furthermore, by Lemma

$2\,C^* \in]0, 1/\alpha[$. Hence in the punishment (superscript "pun") subgame $s_{i+j}^{pun} < t_i/\alpha$ for all $j \geq 0$ so that $\left|s_{i+j}^{pun} - t_i\right| < |t_i/\alpha - t_i| = \left|s_{i+j}^{\inf} - t_i\right|$ when $\alpha \leq 1/2$. The intuitive reason is that, for $\alpha \leq 1/2$, to uphold the preferences over time, each generation would need to distance itself from its own preferences more than what the stances of gradual convergence would imply even in the long run when actions have converged to mainstream. Hence, a gradual convergence is preferred.

For $\alpha > 1/2$ the intuition is more complex. It is based on the standard reasoning that, to sustain an equilibrium with trigger strategies, each agent needs to care sufficiently about the long-run outcome compared to the short-run temptation. In the setting here this means that the prospects of convergence ($s \to 0$) over the long run must be sufficiently intimidating compared to the benefit of having s close to t for a few generations. That is, the parent needs to care sufficiently about the long run to prefer that all generations behave very extremely instead of far-away generations converging. But, as was illustrated in Corollary 1, when β grows then the gradual convergence slows down so that the "threat" becomes less intimidating too. In particular, as $\beta \to 1$, the parent is indifferent between the trigger punishment and remaining in the extreme. This is since in effect also the trigger punishment implies remaining in the extreme when $\beta \to 1$. To see this more formally note that

$$
\begin{aligned}
L^{pun} &= \sum_{j=0}^{\infty} \beta^j \frac{\left(t_i - s_{i+j}\right)^2}{2} \; \{\text{using } s_{i+j}^{pun} = C^* \left(C^*\alpha\right)^j\,\} \\
&= ... = \frac{t_i^2}{2}\left[\frac{1}{1-\beta} - \frac{2C^*}{1-\beta C^*\alpha} + \frac{C^{*2}}{1-\beta\left(C^*\alpha\right)^2}\right] \\
&= \{\text{using } (8) \leftrightarrow \;\; \frac{C^*}{1-\beta\left(C^*\alpha\right)^2} = \frac{1}{1-\beta C^*\alpha}\} \\
&= \frac{t_i^2}{2}\left[\frac{1}{1-\beta} - \frac{C^*}{1-\beta C^*\alpha}\right].
\end{aligned}
$$

Use explicit $C^* = \frac{1-\sqrt{1-4\beta\alpha(1-\alpha)}}{2\beta\alpha(1-\alpha)}$

$$
\begin{aligned}
L^{pun} &= \frac{t_i^2}{2}\left[\frac{1}{1-\beta} - \frac{\frac{1-\sqrt{1-4\beta\alpha(1-\alpha)}}{2\beta\alpha(1-\alpha)}}{1-\beta\frac{1-\sqrt{1-4\beta\alpha(1-\alpha)}}{2\beta\alpha(1-\alpha)}\alpha}\right] = ...\{\text{simplify}\}... \\
&= \frac{t_i^2}{2}\frac{1}{1-\beta}\frac{1}{2\beta\alpha^2}\left(2\beta\alpha^2 - 2\alpha\beta - \sqrt{1-4\beta\alpha\left(1-\alpha\right)} + 1\right).
\end{aligned}
$$

Meanwhile

$$
\begin{aligned}
L^{\inf} &= \sum_{j=0}^{\infty} \beta^j \frac{(t_i - s_{i+j})^2}{2} = \sum_{j=0}^{\infty} \beta^j \frac{(t_i - t_i/\alpha)^2}{2} \\
&= \frac{t_i^2}{2} \frac{(\alpha - 1)^2}{\alpha^2} \frac{1}{1-\beta}.
\end{aligned}
$$

Comparing the two we get that the punishment phase is more intimidating than staying in the extreme iff

$$
\begin{aligned}
L^{\inf} < L^{pun} &\leftrightarrow (\alpha - 1)^2 < \frac{1}{2\beta}\left(2\beta\alpha^2 - 2\alpha\beta - \sqrt{1 - 4\beta\alpha(1-\alpha)} + 1\right) \\
&\leftrightarrow ...1 - 2\beta(1-\alpha) > \sqrt{1 - 4\beta\alpha(1-\alpha)} \\
&\leftrightarrow \{\text{since } \alpha > 1/2\} ... \leftrightarrow \beta > 1
\end{aligned}
$$

which is not true. Hence, the punishment phase is always preferred than staying in the extreme. This shows that, at least for this punishment scheme, a reputational equilibrium with constant extremism cannot be upheld. Whether there is some more severe punishment that could uphold it is hard to know. This is since it would require first finding another class of equilibria than the one established in the main text as otherwise the punishment phase would not be subgame perfect.