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Abstract

We use perturbation methods to derive a rule for the optimal risk-adjusted social cost of carbon (SCC) that incorporates the effects of uncertainties associated with climate and the economy from a calibrated DSGE model. We allow for different aversions to risk and intertemporal fluctuations, convex damages, uncertainties in economic growth, atmospheric carbon, climate sensitivity and damages, their correlations, and distributions that are skewed in the longer run to capture long-run climate feedbacks. Our non-certainty-equivalent rule for the SCC incorporates precaution, risk insurance, and climate sensitivity and damage rate hedging effects to deal with future economic and climatic and damage risks.

JEL-Codes: H210, Q510, Q540.

Keywords: precaution, insurance, hedging, economic, climatic and damage uncertainties, skewness, mean reversion, correlated risks, risk aversion, intergenerational inequality aversion, convex damages.

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The social cost of carbon (SCC) is the Pigouvian tax that internalizes the expected harm of emitting one ton of carbon to the economy, i.e. the expected present discounted value of all future marginal utility losses resulting from emitting one ton of carbon today, converted from utility into dollars today. The risk-adjusted SCC incorporates uncertainties[1] associated with climate and the economy when calculating this tax. If global warming is the only market failure, it is optimal in a decentralized economy to set the price of carbon emissions (e.g. a specific carbon tax or the price in a competitive permit market) to the SCC. To evaluate the SCC, one must know how much of one ton of carbon emitted at time t is left in the atmosphere at each future time $s \geq t$, i.e. $\partial S(s)/\partial F(t)$ with $S(s)$ the stock of atmospheric carbon at time s and $F(t)$ the rate of carbon emissions at time t; the effect of the atmospheric carbon stock on temperature $\partial T(s)/\partial S(s)$; the effect of temperature on damages to aggregate output $\partial \Pi(s)/\partial T(s) < 0$; and utility of consumption $U\left(C(s)\right)$ at time s. For time-separable utility with exponential discounting, the SCC is thus defined by

$$(1) \qquad P(t) \equiv \mathrm{E}_t\left[\int_t^\infty \left(U'\left(C(s)\right)\frac{\partial \Pi(s)}{\partial T(s)}\frac{\partial T(s)}{\partial S(s)}\frac{\partial S(s)}{\partial F(t)}\right)e^{-\rho(s-t)}ds\right]\frac{1}{U'\left(C(t)\right)},$$

where $\rho \geq 0$ is the rate of pure time preference. We begin by illustrating the simplest case, in which only consumption is uncertain. We assume that atmospheric carbon decays at rate φ, so $\partial S(s)/\partial F(t) = e^{-\varphi(s-t)}$, the marginal effect of atmospheric carbon on damages is proportional to aggregate consumption, $\left(\partial \Pi(s)/\partial T(s)\right)\left(\partial T(s)/\partial S(s)\right) = \Theta C(s)$ with Θ a constant[2], consumption follows a geometric Brownian motion, $dC = gdt + \sigma C dW$ with g the drift, σ the volatility and W a unit Wiener process, and utility is of the form $U(C) = C^{1-\gamma}/(1-\gamma)$ with $\gamma > 0$ the coefficient of relative aversion to

[1] We use the terms risk and uncertainty interchangeably.
[2] This assumes that the concavity of temperature as a function of the atmospheric carbon stock is exactly offset by the convexity of damages as function of temperature, so that damages are proportional to the carbon stock and marginal damages (with respect to the carbon stock) are constant (cf. Golosov et al., 2014).

intergenerational inequality aversion and risk. We then obtain from (1) that[3]

$$(2) \qquad P(t) = \Theta C(t) / (r^* + \varphi) \quad \text{with} \quad r^* = \rho + (\gamma - 1)(g - \frac{1}{2}\gamma\sigma^2).$$

The discount rate r^* in (2) is the safe interest rate $r_{\mathrm{rf}} = \rho + \gamma g - \gamma\left(1+\gamma\right)\sigma^2 / 2$ (a Keynes-Ramsey rule with a prudent correction for growth volatility), plus the risk premium $\gamma\sigma^2$ for damages proportional to GDP, minus the growth rate g to correct for growing damages. The SCC is proportional to aggregate consumption, since marginal damages are too. Higher future affluence and less growth volatility push up the SCC if $\gamma > 1$.[4]

The literature concerned with finding the optimal risk-adjusted SCC consists of two strands. Numerically, different authors have performed numerical calculations of the optimal SCC under multiple sources of uncertainty, first with Monte-Carlo simulations (e.g., Ackerman and Stanton, 2012; Dietz and Stern, 2015) and more recently by tackling the dynamic programming problem with advanced numerical methods (e.g., Crost and Traeger, 2013; Traeger, 2014a; Jensen and Traeger, 2014; Hambel et al., 2017).[5] Analytically, the literature on discounting under uncertainty and optimal carbon prices typically deals with one uncertainty at a time (e.g., Gollier, 2012; Traeger, 2014b). At the start of this analytical literature, Golosov et al. (2014) obtained a simple rule for the optimal SCC reacting to world GDP only, making bold assumptions including logarithmic utility[6], which imply that growth uncertainty does not affect the

[3] It is easy to allow for richer dynamics in the atmospheric carbon stock. E.g., Joos et al. (2013) use a N-box linear carbon cycle, $\left(1/\Delta t\right)\partial S(s)/\partial F(t) = \mu_0 + \sum_{i=1}^{N-1}\mu_i \exp(-\varphi_i(s-t))$ with $\sum_{i=0}^{N-1}\mu_i = 1$, where φ_i is the rate of decay of the i-th transient component of atmospheric carbon and $0 < \mu_0 < 1$ the fraction of emissions that stays in the atmosphere forever. The SCC then becomes $P = [\mu_0 / r^* + \sum_{i=1}^{N}\mu_i / (r^* + \varphi_i)]\Theta C$.

[4] See also Gollier (2012) and Traeger (2014b) for the effect of growth volatility on the discount rate.

[5] Lemoine and Traeger (2014; 2016), Lontzek et al. (2015) and Cai et al. (2016) study numerically the optimal SCC in the face of climate tipping risks. Lemoine and Rudik (2017) review recursive numerical assessment and Monte Carlo evaluation of climate policy under uncertainty and discuss learning.

[6] They have a discrete-time (decadal) model, assume logarithmic utility, Cobb-Douglas production, 100% depreciation of capital each period, and total factor productivity as an exponential function of the atmospheric carbon stock.

SCC (cf. Traeger, 2017). Gerlagh and Liski (2016) also derive a simple rule[7] and examine this in the context of learning about uncertain impacts. Jensen and Traeger (2016) show how the effect of climate sensitivity on the risk premium in the SCC depends on prudence and convexity of marginal damages. Lemoine (2017) decomposed the SCC into different components due to uncertain warming, damages and economic growth. He showed that the sign of the effect of so-called climate betas, representing the normalized covariances of different climatic uncertainties with the rate of economic growth, on the SCC depends on whether relative risk aversion is greater than one or not.[8] In both the decompositions by Jensen and Traeger (2016) and Lemoine (2017) consumption is set exogenously. Very recently, two first steps have been made towards a simple rule for the risk-adjusted SCC in a general equilibrium model. Traeger (2017) transforms an integrated assessment model with a range of climate uncertainties, in which consumption is determined endogenously but with full capital depreciation after one period following Golosov et al. (2014) and the restriction that the model that is linear in states with additively separable controls. Finally, Bretschger and Vinogradova (2018) extend the endogenous growth model of Pindyck and Wang (2013) to allow for Poisson shocks in the capital stock in their analysis of carbon pricing.

Our aim here is to derive a general rule for the optimal SCC that maximizes expected welfare in a Dynamic Stochastic General Equilibrium (DSGE) model. We allow for uncertainty in projections of the carbon stock, of the impact of the atmospheric carbon stock on temperature, of temperature on damages and of the GDP growth rate, as well as their correlations, and analyse the precautionary, insurance and hedging determinants of the SCC. We allow skewness and uncertainty of the response in temperature resulting from doubling the atmospheric carbon stock, as captured by the climate sensitivity, to rise with

[7] Van den Bijgaart et al. (2016) compare such a simple rule to numerical evaluations of the SCC from a standard integrated assessment model in a deterministic setting.
[8] Alternatively, it depends on whether the risk insurance effect (what we will call the climate hedging effect) dominates the exposure effect (what we will call the offsetting effect due to damages being proportional to GDP).

time, reflecting key differences in short-term (cf. transient climate response) and long-term (cf. equilibrium climate sensitivity) uncertainty predictions. We allow risk aversion to differ from intergenerational inequality aversion (Kreps and Porteus, 1978; Epstein and Zin, 1989; Duffie and Epstein, 1992). Finally, we allow for general concave or convex relationships between the carbon stock and temperature and between temperature and damages.[9] In doing so, we generalize Golosov et al.'s (2014) rule for non-unitary coefficients of relative risk aversion and intergenerational inequality aversion, more convex damages, uncertainty in the carbon stock, climate sensitivity and damages, and skewness and mean reversion in the distributions governing these variables.

Our DSGE model adapts the endogenous growth model with investment adjustment costs of Pindyck and Wang (2013) to allow for fossil fuel use, climate change and damages. To obtain a simple result akin to (2), we solve our DSGE model using perturbation methods around a known analytical solution path, where the "small" perturbation parameter is the fraction of damages in GDP.[10] By using a power-function transformation of a normal variate displaying a variance that grows in time,[11] we capture the significant right-skew evident in the equilibrium (i.e. long run) climate sensitivity, but not in the transient climate response, whilst capturing the difference in time scales on which these apply, but avoiding the fat tails in Weitzman's (2009) 'dismal theorem'.

We derive three results. Result 1 gives our general expression for the optimal risk-adjusted SCC and can be evaluated by numerical evaluation of a multi-dimensional integral, avoiding the daunting task of numerically solving the underlying multi-state Hamilton-Jacobi-Bellman equations. For the case of damages proportional to the atmospheric carbon stock and focusing only on the leading-order effects of uncertainty, Result 2 evaluates this rule in closed form. Result 3 generalizes it for convex dependence of damages on the carbon stock,

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[9] Hence, the so-called flow damage coefficient Θ in $(\partial D(s)/\partial T(s))(\partial T(s)/\partial S(s)) = \Theta C(s)$ will no longer be constant but depend on the stochastic atmospheric carbon, temperature and damages.

[10] Using scaling, we identify the damages ratio as the only "small" non-dimensional quantity. Judd (1996, 1998), Judd and Guu (2001) and Binsbergen et al. (2012) use perturbation analysis (e.g. Bender and Orszag, 1999) in discrete time.

[11] Specifically, we use an Ornstein-Uhlenbeck process.

giving the SCC in the form of one-dimensional deterministic integrals.

Generalizing (2) to recursive preferences, we find that precaution about uncertain growth outcomes implies a lower discount rate and a higher optimal SCC, whilst a risk-insurance term increases the discount rate and curbs the SCC. If intergenerational inequality aversion exceeds one, the discount rate is adjusted downwards and the SCC upwards with riskier growth prospects. The upward correction to the SCC to allow for temperature uncertainty depends on the combination of the skewness of its equilibrium probability distribution, the convexity of damages, the (non-climatic) risk-adjusted discount rate and, crucially, on the time scale on which the equilibrium distribution is reached.

In our analysis, the three different climatic uncertainties have their own betas, representing their normalized covariances with shocks to the rate of economic growth: the carbon stock beta, the temperature beta and the damage beta with the latter two the most important. If the economy is concentrated in economic sectors that benefit from high (low) temperature, the temperature beta is positive (negative), and we show that the optimal risk-adjusted SCC is lower (higher) provided risk aversion exceeds one, as found by Lemoine (2017). If the economy is concentrated in adaptation industries (e.g. flood defences), shocks to future damages are associated with higher assets returns so the damage beta is positive. We show that, if the coefficient of relative risk aversion exceeds one, the optimal SCC is then reduced,[12] although we note that such capital allocation is rare, especially in the developing world. Finally, we calibrate our model and show how the optimal SCC is quantitatively by the different uncertainties.

Section I presents our model. Sections II derives Result 1. Section III derives Result 2 with Result 3 in Appendix A. After a discussion of our calibration in Section IV, Section V estimates the optimal SCC and the effects of the various uncertainties. Finally, Section VI concludes.

[12] This differs from the *built-in* climate beta due to damages being proportional to GDP, which also implies that uncertainty depresses the SCC. The *built-in* climate beta also has a deterministic effect, namely that trend growth increases the SCC, as is clear from (2).

I. A DSGE Model of Global Warming and the Economy

We start from the DSGE model with endogenous AK growth of Pindyck and Wang (2013) and add fossil fuel use as a production factor. Fossil fuel use gives rise to global warming and damages to output. The coefficient of relative risk aversion, η = CRRA ≥ 0, may differ from the coefficient of relative intergenerational inequality aversion, IIA = 1/EIS = $\gamma \geq 0$, where EIS is the elasticity of intertemporal substitution. We use the continuous-time version of recursive preferences (Duffie and Epstein, 1992), where the recursive aggregator $f(C,J)$ depends on consumption C and the value function

$$(3) \quad J = E_t\left[\int_t^\infty f\left(C(s),J(s)\right)ds\right] \text{ with } f\left(C,J\right) = \frac{1}{1-\gamma}\frac{C^{1-\gamma}-\rho\left(\left(1-\eta\right)J\right)^{\frac{1-\gamma}{1-\eta}}}{\left(\left(1-\eta\right)J\right)^{\frac{1-\gamma}{1-\eta}-1}}.$$

The dynamics of the aggregate capital stock follow from

$$(4) \qquad dK = \Phi(I,K)dt + \sigma_K K dW_1 \quad \text{with} \quad \Phi(I,K) = I - \frac{1}{2}\omega\frac{I^2}{K} - \delta K,$$

where K denotes the capital stock, I investment, $\delta \geq 0$ the depreciation rate of physical capital, and $\omega > 0$ the adjustment cost parameter.[13,14] Adjustment costs are quadratic and homogenous of degree one in capital and investment. Capital is subject to continuous geometric shocks with relative volatility σ_K, and W_1 is a Wiener process, representing both economic growth and asset return uncertainty in the context of the AK-model considered. Investment is $I = Y - C - bF$, where Y is aggregate production, F fossil fuel use, and b the production cost of fossil fuel. Fossil fuel is supplied inelastically at fixed cost. The final goods production function is $Y = AK^\alpha F^{1-\alpha}$ with $0 < \alpha < 1$ and

[13] With AK growth, shocks to the capital stock and productivity are equivalent. To avoid an extra state, we introduce volatility directly in the capital dynamics (cf. Pindyck and Wang, 2013).
[14] For ease of presentation, we first introduce the separate evolution equations for the four stochastic variables before introducing the covariance matrix of these four state variables.

$A \equiv A^*(1-D)$ is total factor productivity. Damages as share of pre-damages aggregate output D increase in global mean temperature relative to preindustrial temperature *T*. We use the power-function specification

$$(5) \qquad D(T,\lambda) = T^{1+\theta_T}\lambda^{1+\theta_\lambda} \quad \text{with} \quad \theta_T \geq -1 \quad \text{and} \quad \theta_\lambda \geq -1,$$

where the (positive) stochastic variable λ captures the uncertain nature of damages for a given temperature. Convexity of damages with respect to temperature corresponds to $\theta_T \equiv TD_{TT} / D_T > 0$.[15] To allow for potential skewness in the impact of damage shocks even if λ has a symmetric distribution, we raise it to the power $1+\theta_\lambda$.

The part of atmospheric carbon, *S*, associated with man-made emissions is $E \equiv S - S_{\mathrm{PI}}$, where S_{PI} is the preindustrial carbon stock. The rate of carbon emissions is $F\exp(-gt)$, where *F* is fossil fuel use and $\exp(-gt)$ the emission intensity which declines at the endogenous economic growth rate g. A proportion $0 < \mu < 1$ of fossil fuel emissions ends up in the atmosphere. Atmospheric carbon decays at the rate $\varphi \geq 0$. The carbon stock dynamics is

$$(6) \qquad d\tilde{E} = (\mu F e^{-gt} - \varphi\tilde{E})dt + \sigma_E dW_2 \quad \text{and} \quad E = \max(0, \tilde{E}),$$

where W_2 denotes a second Wiener process, so the carbon stock is described by a (truncated) Arithmetic Brownian motion with absolute volatility $\sigma_E \geq 0$.[16] This specification ensures that the expected value of the carbon stock returns to its preindustrial value when emissions cease. We have for temperature

[15] Subscripts of functions denote partial derivatives.
[16] One can allow for a permanent and one (Golosov et al., 2014), two (Gerlagh and Liski, 2018) or three (Millar et al., 2017) temporary basins of atmospheric carbon. Appendix F.3 shows that our 1-box model reproduces historical atmospheric carbon stocks well, and section IV illustrates how it captures all the key features of future projections. Millar et al. (2017) allow the speed at which oceans absorb atmospheric carbon (akin to our φ) to fall with warming. We ignore such positive feedback effects and associated multiplicative uncertainty.

$$(7) \qquad T(E,\chi) = \chi^{1+\theta_\chi}(E/S_{\mathrm{PI}})^{1+\theta_E} \quad \text{with} \quad \theta_E \ge -1 \quad \text{and} \quad \theta_\chi \ge -1,$$

where the (positive) stochastic variable χ captures the uncertain nature of temperature for a given carbon stock. A negative value of θ_E captures the concavity of Arrhenius' law. The parameter θ_χ captures skewness in the impact of shocks even if χ has a symmetric distribution. We allow for the effect of lags via the time-varying dynamics of the stochastic process for the random variable χ.[17] The climate sensitivity is the temperature increase from doubling the carbon stock from its preindustrial level, i.e. $T_2 \equiv T(E = S_{PI}, \chi) = \chi^{1+\theta_\chi}$, and depends on the stochastic climate sensitivity parameter χ. We will show that its leading-order skewness is $\text{skew}\left[T_2\right] = 3\theta_\chi(1+\theta_\chi)^3 \mu_\chi^{3(1+\theta_\chi)}(\Sigma_\chi / \mu_\chi)^4$ and increases in the skewness parameter θ_χ and the coefficient of variation Σ_χ / μ_χ (see Appendix F.5). Combining equations (5) and (7), damages become

$$(8) \qquad D(E,\chi,\lambda) = \chi^{1+\theta_{\chi T}} \lambda^{1+\theta_\lambda} (E/S_{PI})^{1+\theta_{ET}} \quad \text{with} \quad \theta_{\chi T} \equiv \theta_\chi + \theta_T + \theta_\chi \theta_T.$$

The parameter $\theta_{ET} \equiv \theta_E + \theta_T + \theta_E \theta_T$ captures the combined effect of the concave relationship between temperature and the carbon stock ($-1 \le \theta_E < 0$) and the convex relationship between damages and temperature ($\theta_T > 0$). It is positive or negative depending on which effect dominates. We refer to $\theta_{ET} = 0$ as *proportional damages* (cf. Golosov et al., 2014 and the introductory example) and $\theta_{ET} > 0$ as *convex damages*, reflecting the dependence of damages on the carbon stock. The parameter $\theta_{\chi T}$ captures the joint effect of skewness of the climate sensitivity ($\theta_\chi > 0$) and convexity of the damage function with respect

[17] We thus include potential effects of temperature lags from ocean heating, which affect estimates of the long-run climate sensitivity (e.g., Roe and Bauman, 2011). In reality, the response to small emissions is much faster and on a decadal scale (Ricke and Caldeira, 2014) than the response to larger emissions (Zickfeld and Herrington, 2015), reflecting nonlinearity in the system, which is not captured by our Ornstein-Uhlenbeck process (10a).

to temperature ($\theta_T > 0$). From (8), total factor productivity and aggregate output fall in the carbon stock and the shocks to climate sensitivity and damages:

(9) $Y = A(E,\chi,\lambda)K^{\alpha}F^{1-\alpha}$ with $A(E,\chi,\lambda) \equiv A^{*}\left(1-(E/S_{PI})^{1+\theta_{ET}}\,\chi^{1+\theta_{\chi T}}\,\lambda^{1+\theta_{\lambda}}\right).$

Uncertainties in the climate sensitivity and the damage ratio are driven by truncated mean-reverting stochastic Ornstein-Uhlenbeck processes with means $\bar{\chi}, \bar{\lambda},$ mean reversion coefficients $\nu_{\chi}, \nu_{\lambda}$, and volatilities $\sigma_{\chi}, \sigma_{\lambda},$ [18]

(10a) $d\tilde{\chi} = \nu_{\chi}(\bar{\chi}-\tilde{\chi})dt + \sigma_{\chi}dW_3$ and $\chi = \max(0,\tilde{\chi}),$

(10b) $d\tilde{\lambda} = \nu_{\lambda}(\bar{\lambda}-\tilde{\lambda})dt + \sigma_{\lambda}dW_4$ and $\lambda = \max(0,\tilde{\lambda}),$

where W_3 and W_4 are two Wiener processes. [19] Together with $T \propto \chi^{1+\theta_{\chi}}$ in (7), the process (10a) captures two features of the climate sensitivity distribution. First, temperature uncertainty increases with time, reaching a steady state associated with the equilibrium climate sensitivity (ECS) and its variance and skewness. We calibrate the ECS to the steady-state variance of (10a), $\Sigma_{\chi}^2 \to \sigma_{\chi}^2 / 2\nu_{\chi}$ as $t \to 1/\nu_{\chi}$, so that $1/\nu_{\chi}$ is the e-folding time for reaching the steady state. [20] Second, we can use our model to fit the less wide and skew distribution of the transient climate response (TCR).

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[18] Equation (10a) has solution $\tilde{\chi}(t) = \tilde{\chi}_0 e^{-\nu_{\chi}t} + \bar{\chi}(1-e^{-\nu_{\chi}t}) + \sigma_{\chi}\int_0^t e^{-\nu_{\chi}(t-s)}dW_3(s),$ and similarly for equation (10b). The random variables $\tilde{\chi}(t)$ and $\tilde{\lambda}(t)$ are normally distributed with time-varying moments: $\tilde{\chi}(t) \sim \mathrm{N}(\mu_{\chi},\Sigma_{\chi}^2)$ and $\tilde{\lambda}(t) \sim \mathrm{N}(\mu_{\lambda},\Sigma_{\lambda}^2)$. Mean and variance of $\tilde{\chi}(t)$ are $\mu_{\chi} = \chi_0 e^{-\nu_{\chi}t} + \bar{\chi}(1-e^{-\nu_{\chi}t})$ and $\Sigma_{\chi}^2 = \sigma_{\chi}^2\left(1-\exp(-2\nu_{\chi}t)\right)/2\nu_{\chi}$ with stationary limits $\mu_{\chi} \to \bar{\chi}$ and $\Sigma_{\chi}^2 \to \sigma_{\chi}^2/2\nu_{\chi}$, respectively.

[19] Although $\tilde{E}, \tilde{\chi}$ and $\tilde{\lambda}$ can formally become negative with finite probabilities due to their Gaussian distributions, we will show in section V that these probabilities are negligibly small. To avoid a formally ill-defined problem, we use truncated distributions in (6), (10a) and (10b) and in doing so place (negligibly small) probability atoms at zero values of the states. We subsequently ignore these atoms in the derivation of the asymptotic solutions for the optimal SCC presented in Result 1, 2 and 3.

[20] How long it takes for an exponentially growing quantity to rise by a factor e = 2.72.

For all three uncertain climate processes E, χ and λ, the uncertainties are exogenously given and cannot be learned in our model. Fundamentally, both statistical (or aleatoric) uncertainty and systemic (or epistemological) uncertainty play a role but cannot always be separated.[21] For all three processes, we use in our calibration the most high-level or 'consensus' range of uncertainty estimates available, which also do not make this distinction (see section IV). For example, the 'consensus' uncertainty range for the climate sensitivity (e.g. IPCC, 2014, AR5, Chapter 12, Box 12.2) captures both statistical uncertainty in individual climate models and epistemological uncertainty arising from different climate models. The climate sensitivity uncertainty we examine is this aggregate measure of uncertainty, and similarly for the carbon stock and damage ratio.

Combining (4), (6) and (10), we have one truncated multi-variate Ornstein-Uhlenbeck process:

(11) $$d\tilde{\mathbf{x}} = \boldsymbol{\alpha} - \mathbf{v} \circ \left(\tilde{\mathbf{x}} - \boldsymbol{\mu}\right) dt + \mathbf{S} d\mathbf{W}_t,$$

where the states are $\mathbf{x} \equiv (k, \tilde{E}, \tilde{\chi}, \tilde{\lambda})^T$, $\mathbf{x} = \max(0, \mathbf{x})$ with $k \equiv \log\left(K/K_0\right)$ and $\circ$ denotes the elementwise product. The growth rates of this process are

(12) $$\boldsymbol{\alpha} \equiv \left(\frac{1}{dt} \frac{E_t\left[dK\right]}{K} - \frac{1}{2}\sigma_K^2, \mu F e^{-gt}\ , 0, 0 \right)^T.$$

The vector of mean reversion rates and the vector of means of this process are

(13) $$\mathbf{v} \equiv (0, \varphi, \nu_\chi, \nu_\lambda)^T \quad \text{and} \quad \boldsymbol{\mu} \equiv (0, 0, \bar{\chi}, \bar{\lambda})^T.$$

The covariance matrix $\mathbf{SS}^T$ of the components of this multivariate process is

[21] Statistical uncertainty describes genuinely stochastic and continuously fluctuating processes, whereas systemic uncertainty is unknown and potentially learnable. Climate sensitivity is not learnable in our model. There may be aspects of climate sensitivity that are difficult or impossible to learn (cf. Roe and Baker, 2007).

$$(14)\quad \frac{1}{dt}E_t\left[d\tilde{\mathbf{x}}d\tilde{\mathbf{x}}^T\right]=\mathbf{SS}^T=\begin{pmatrix} \sigma_K^2 & \rho_{KE}\sigma_K\sigma_E & \rho_{K\chi}\sigma_K\sigma_\chi & \rho_{K\lambda}\sigma_K\sigma_\lambda \\ \rho_{KE}\sigma_K\sigma_E & \sigma_E^2 & \rho_{E\chi}\sigma_E\sigma_\chi & \rho_{E\lambda}\sigma_E\sigma_\lambda \\ \rho_{K\chi}\sigma_K\sigma_\chi & \rho_{E\chi}\sigma_E\sigma_\chi & \sigma_\chi^2 & \rho_{\chi\lambda}\sigma_\chi\sigma_\lambda \\ \rho_{K\lambda}\sigma_K\sigma_\lambda & \rho_{E\lambda}\sigma_E\sigma_\lambda & \rho_{\chi\lambda}\sigma_\chi\sigma_\lambda & \sigma_\lambda^2 \end{pmatrix},$$

where $\rho_{ij},\, i\neq j,\, i,j=K,E,\chi,\lambda$ denote the partial correlation coefficients.

II. Asymptotic Solutions for the Optimal Risk-Adjusted SCC

The optimal solution must satisfy the Hamilton-Jacobi-Bellman equation

$$(15)\qquad \max_{C,F}\left[f\left(C,J\right)+\frac{1}{dt}E_t\left[dJ\left(t,K,\tilde{E},\tilde{\chi},\tilde{\lambda}\right)\right]\right]=0,$$

where $\left(1/dt\right)E_t\left[dJ\right]$ is Ito's differential operator applied to J. Using $I(C,F,K,E,\chi,\lambda)=A(E,\chi,\lambda)K^\alpha F^{1-\alpha}-C-bF$ and Ito's lemma gives[22]

$$(16)\quad \begin{aligned} &\max_{C,F}\left[f\left(C,J\right)+J_K\Phi\left(I(C,F,K,E,\chi,\lambda),K\right)+J_{\tilde{E}}(\mu Fe^{-gt}-\varphi\tilde{E})\right]+J_t \\ &+J_{\tilde{\chi}}\nu_\chi(\bar{\chi}-\tilde{\chi})+J_{\tilde{\lambda}}\nu_\lambda(\bar{\lambda}-\tilde{\lambda})+\frac{1}{2}J_{KK}K^2\sigma_K^2+\frac{1}{2}J_{\tilde{E}\tilde{E}}\sigma_E^2+\frac{1}{2}J_{\tilde{\chi}\tilde{\chi}}\sigma_\chi^2+\frac{1}{2}J_{\tilde{\lambda}\tilde{\lambda}}\sigma_\lambda^2 \\ &+J_{K\tilde{E}}K\rho_{KE}\sigma_K\sigma_E+J_{K\tilde{\chi}}K\rho_{K\chi}\sigma_K\sigma_\chi+J_{K\tilde{\lambda}}K\rho_{K\lambda}\sigma_K\sigma_\lambda+J_{\tilde{E}\tilde{\chi}}\rho_{E\chi}\sigma_E\sigma_\chi \\ &+J_{\tilde{E}\tilde{\lambda}}\rho_{E\lambda}\sigma_E\sigma_\lambda+J_{\tilde{\chi}\tilde{\lambda}}\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda=0. \end{aligned}$$

Differentiating (16) with respect to C and F gives the optimality conditions $f_C=C^{-\gamma}\left(\left(1-\eta\right)J\right)^{\frac{\gamma-\eta}{\eta-1}}=J_K\Phi_I(I,K)$ and $(1-\alpha)Y/F=b+Pe^{-gt}$, where the optimal SCC is defined by $P\equiv-\mu J_{\tilde{E}}/J_K\Phi_I(I,K)>0$. Our command optimum corresponds to the outcome in a decentralized market economy if emissions are priced at the SCC and no other externalities or market failures exist. Hence, we

[22] Strictly, equation (15) is not continuously differentiable, but we already ignore the (negligibly small) probability atoms at zero values of the states here (see section IV).

use the terms 'carbon price' and SCC interchangeably and denote these by P.

There is no closed-form analytical solution to the stochastic dynamic optimal control problem (15). Solving numerically by approximating the value function and its derivatives in 5-dimensional space (time and the four states) is challenging due to the curse of dimensionality and does not yield analytical insight into the stochastic drivers of the optimal SCC. Instead, we derive an approximate solution using perturbation methods. We first examine the system for small parameter(s) (see Appendix B), then perform asymptotic expansions to leading-order in the thus identified small parameter, namely the share of climate damages in total GDP,

(17) $$\epsilon \equiv D(E_0, \overline{\chi}, \overline{\lambda}) = \overline{\lambda}^{1+\theta_\lambda} \, \overline{\chi}^{1+\theta_{\chi T}} \left(E_0 / S_{PI} \right)^{1+\theta_{ET}} ,$$

where $E_0 \equiv S_0 - S_{PI}$ (with climate damage and sensitivity parameters at their equilibrium values and the atmospheric carbon stock at its initial value). It is typically only a few percentage points and lower than 10% even at high temperatures (see section IV). Our perturbation solutions consider terms up to first order in ϵ. The resulting error scales with $\epsilon^2 \cong 0.01,$ which is small even for the large value of $\epsilon \cong 0.1$. We judge this to be sufficiently accurate for estimating the optimal SCC, since Nordhaus and Moffat (2017) suggests that available empirical estimates of the damage ratio are well below 10%.

To solve our problem, we perform a perturbation expansion in the small parameter ϵ around a base solution for which $\epsilon = 0$ and the analytical solution is known. At each order *n* of the expansion, the problem is linear in the value function $J^{(n)}$, but remains fully nonlinear in the states, thus retaining risk-aversion and prudence properties without approximation. Mathematically, at each order *n*, the problem is of the form $L\left[\epsilon^n J^{(n)}\right] = \Gamma,$ where L is a linear differential operator in the states and the nonlinear forcing Γ is formed from products or derivatives of lower-order solutions (in *n*), so that the order of the forcing thus obtained (from products or derivatives) is also $O(\epsilon^n)$. We use the following truncated series solution up to n = 1 and thus restrict our attention to zeroth- and first-order terms in ϵ only, as denoted by the superscripts[23]

$$(18)\quad J(K,\tilde{E},\tilde{\chi},\tilde{\lambda},t) = J^{(0)}(K,\epsilon D(\tilde{E},\tilde{\chi},\tilde{\lambda})) + \epsilon J^{(1)}(K,\tilde{E},\tilde{\chi},\tilde{\lambda},t,\epsilon D(\tilde{E},\tilde{\chi},\tilde{\lambda})) + O(\epsilon^2),$$

and similarly for F and C. The parameter ϵ appears both as small parameter of the series solution and as the multiple-scales parameter in front of the dependence on damages. We let total factor productivity be a slowly-varying power-law function of the climate-related variables $\tilde{E}, \tilde{\chi}$ and $\tilde{\lambda}$: higher derivatives required to model strong variation are thus ignored at leading order. The zeroth-order value function in (18) inherits the properties of the production function (9). Our consistent leading-order estimate of the optimal SCC from the zeroth and first-order value function is thus

$$(19)\qquad P = -\mu\left(J_{\tilde{E}}^{(0)} + \epsilon J_{\tilde{E}}^{(1)}\right)\Big/\phi'(i^{(0)})J_K^{(0)}.$$

In the limit as $\epsilon \to 0,$ climate has no effect, and the corresponding zeroth-order solution of our model reduces to an AK model for which the closed-form solution is given by Pindyck and Wang (2013).[24] The only difference with Pindyck and Wang (2013) is that our solution depends slowly on the climate variables, as determined by the implicit equation for optimal investment (C7) and the dependence of the marginal productivity of capital on climate damages therein. Our derivation of the first-order solution is given in Appendix D with the solution for $J^{(1)}$ given by (D3.14).[25] The first-order value function captures

[23] We emphasize we do not perform a Taylor-series expansion in the state variables around their steady states, since this requires too many terms due to the large number of states and derivatives needed to capture risk aversion and prudence. At every order in n, the problem remains nonlinear in the states. To overcome this, we will choose products of power-law functions as the functional form for the dependence of J on the states.

[24] The derivation is in Appendix C with $J^{(0)}$ given by (C4) (in terms of non-dimensional variables of Appendix B).

[25] We only show the solution for $J_E^{(1)}$ as this is what is needed to evaluate the SCC in (19).

changes to the economy resulting not from climate-induced changes to the marginal productivity of capital (as captured by $J^{(0)}$'s slow dependence the climate-related states), but from direct damages to the economy arising from the three climate-related states. Combining the zeroth- and first-order solutions, we obtain the following result (corresponding to (C3.19) in Appendix C).

Result 1: The optimal risk-adjusted SCC is:

$$(20) \qquad P = \frac{\mu\ \Theta(E,\chi,\lambda)Y\big|_{P=0}}{r^*}\left(1 - \frac{\Omega}{E^{\theta_{ET}}\chi^{1+\theta_{ET}}\lambda^{1+\theta_\lambda}K^{1-\eta}}\right) + O(\epsilon^2),$$

where $\Theta \equiv D_E/(1-D)$ and $r^* \equiv r^{(0)} - g^{(0)} = \rho + (\gamma-1)(g^{(0)} - \eta\sigma_K^2/2)$.

Further, $\Omega = E_t\left[\int_t^\infty \Gamma e^{-r_\Omega(s-t)} ds\right]$ where $r_\Omega \equiv r^* - (\eta-1)\left(\phi(i^{(0)}) - \eta\sigma_K^2/2\right) + \varphi,$ $\phi \equiv \Phi/K = i - \omega i^2/2 - \delta,$ $i = I/K$ and (A.1) in Appendix A gives $\Gamma = \Gamma(E,\chi,\lambda)$. □

The term in (20) in front of the brackets is the present value of marginal damages when only economic growth/asset return uncertainty is considered (and carbon does not decay); the second term in brackets is the mark-up for carbon stock, climate sensitivity and damage ratio uncertainties (and carbon stock decay). The optimal SCC (20) is proportional to world GDP and depends directly on the stock of atmospheric carbon and the climate sensitivity and damage ratio parameters through the function $\Theta(E,\chi,\lambda)$. It depends on preferences (ρ, γ and η), geophysical parameters (μ, φ and ν_χ), and the properties of the stochastic processes driving GDP, the carbon stock, climate sensitivity and damages. The optimal SCC depends on the growth-corrected return on capital r^*, which is given to leading-order by its value when there is no climate policy ($P=0$). The expected return on investment $r^{(0)}$ is the risk-free rate, $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - (1+\gamma)\eta\sigma_K^2 / 2,$ plus the risk premium $\eta\sigma_K^2$.[26]

Result 1 indicates that the absolute error in our expression for the optimal SCC is $O(\epsilon^2)$ and that the error as fraction of the SCC (which is $O(\epsilon)$) is thus $O(\epsilon).$ Consistently, we ignore the slow dependence of the discount rate on the atmospheric carbon stock (via the marginal productivity of capital) when evaluating the discounting integral in Result 1. As $\epsilon \to 0,$ the SCC in Result 1 becomes exact. Generally, a closed-form solution to the time integral and the expectations operator over the stochastic states in Ω is unavailable, so Result 1 must be evaluated numerically.[27] However, if we consider only the leading-order effects of uncertainty, we can derive the closed-form expression in Result 2 below (and Result 3 in Appendix A) with only minimal quantitative errors.[28]

III. A Closed-Form Rule for The Optimal Risk-Adjusted SCC

To obtain a closed-form solution for the optimal SCC in Result 1, we consider only leading-order terms in the climatic and damage uncertainties $\sigma_E^2,\ \sigma_\chi^2,\ \sigma_\lambda^2$ and their covariance terms (including with the capital stock). Appendix E then shows that the five-dimensional integral in Result 1 can be explicitly evaluated except for one time integral, and we obtain Result 3 given in Appendix A. For ease of exposition, we present in Result 2 below the special case of *proportional damages* ($\theta_{ET} = 0$) also examined by Golosov et al. (2014), in which marginal damages do not depend on the carbon stock, and we further assume the temperature and damage ratio are at their steady-state values ($\chi_0 = \bar{\chi},\ \lambda_0 = \bar{\lambda}$).

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[26] The investment and growth rates of GDP are given to leading-order by their values without climate policy (cf. (C7)). Implicitly, we get from the Euler and capital accumulation equations $i^{(0)} = Y_K\big|_{P=0} - q^{(0)}\left(\rho + (\gamma-1)\left(\phi(i^{(0)}) - \eta\sigma_K^2/2\right)\right)$ with $Y_K\big|_{P=0} = \alpha A(E,\chi,\lambda)^{1/\alpha}\left((1-\alpha)/b\right)^{(1-\alpha)/\alpha}$ and $g^{(0)} = i^{(0)} - \ \omega(i^{(0)})^2/2 - \delta \ \equiv \phi(i^{(0)}).$ Tobin's q is $q(i) = 1/\phi'(i).$

[27] This requires five-dimensional numerical integration over the probability space corresponding to the four states and with respect to time. If the processes are independent, the integrals over the probability space of states can be evaluated independently.

[28] In Appendix G we examine the accuracy of Results 2 and 3 by comparing with Result 1.

Result 2: If $\chi_0 = \bar{\chi}, \lambda_0 = \bar{\lambda}$ and $\theta_{ET} = 0,$ the optimal SCC is

(21) $$P = \frac{\mu\ \Theta Y|_{P=0}}{r^* + \varphi}(1 + \Delta_\chi + \Delta_\lambda + \Delta_{\mathrm{CK}} + \Delta_{\mathrm{CC}}),$$

where $r^* = \rho + (\gamma - 1)(g^{(0)} - \frac{1}{2}\eta\sigma_K^2),$ $\Delta_\chi = \frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\frac{(\sigma_\chi / \bar{\chi})^2}{r^* + 2\nu_\chi + \varphi},$ $\Delta_\lambda = \frac{1}{2}\theta_\lambda(1+\theta_\lambda)\frac{(\sigma_\lambda / \bar{\lambda})^2}{r^* + 2\nu_\lambda + \varphi},$ $\Delta_{\mathrm{CK}} = -(\eta - 1)\sigma_K\left((1+\theta_{\chi T})\frac{\rho_{K\chi}\sigma_\chi / \bar{\chi}}{r^* + \nu_\chi + \varphi} + \frac{\rho_{K\lambda}\sigma_\lambda / \bar{\lambda}}{r^* + \nu_\lambda + \varphi}\right)$

and $\Delta_{\mathrm{CC}} = (1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda / \bar{\chi}\bar{\lambda}}{r^* + \nu_\chi + \nu_\lambda + \varphi}.$ □

Without uncertainty, $P = \mu\ \Theta Y|_{P=0} / (r^* + \varphi)$ with $r^* = \rho + (\gamma - 1)g^{(0)}.$ This expression shows the geophysical (μ and φ), economic (Y and g), damage (Θ) and ethical (ρ and γ) determinants of the optimal deterministic SCC. More patience (lower ρ) boosts the SCC.[29] Rising affluence (higher $g^{(0)}$) pushes up the discount rate, if intergenerational inequality aversion exceeds one, and thus curbs the appetite of current generations for ambitious climate policy (the $\gamma g^{(0)}$ term in r^*). Higher economic growth also implies growing damages and a lower (growth-corrected) discount rate (the $-g^{(0)}$ term in r^*), which increases the optimal SCC. Economic growth thus depresses the SCC if $\gamma > 1.$ Higher economic activity (Y) and flow-damage coefficient (Θ) also push up the SCC.

[29] In contrast to exogenous Ramsey growth models such as Golosov et al. (2014) and Nordhaus (2017), our rate of economic growth $g^{(0)}$ is endogenous (see footnote 26). Hence, there are indirect effects on the optimal SCC via the growth rate $g^{(0)}$. For example, the direct effect of a higher rate of pure time preference ρ is to lower the SCC and the indirect effect is to raise the SCC as economic growth is lowered (for $\gamma \geq 1$). Together, the effect of a higher rate of pure time preference on the discount rate is always positive $\partial r^*/\partial\rho = 1 + (\gamma - 1)\partial g/\partial\rho = 1/\gamma$ with $\partial g/\partial\rho = -1/\gamma$ (and thus always negative on the SCC). Although the optimal SCC does not depend directly on the share of fossil fuel in value added, the cost of fossil fuel, adjustment costs or the depreciation rate of physical capital, it does depend on adjustment costs and the depreciation rate via their effect on the endogenous rate of economic growth, which we treat as fixed in the analysis below. Furthermore, Ramsey growth models with an exogenous long-run growth rate include a second time scale associated with economic convergence, which will typically be faster than the climatic time scales. We conjecture that our formula for the optimal SCC derived in an AK growth model will therefore be a good approximation to the optimal SCC for a Ramsey growth model.

A small fraction of emissions that stays forever in the atmosphere (μ) and fast decay of atmospheric carbon (higher φ) curb the SCC.

A. Economic growth uncertainty and the climate beta

Including economic, but not climatic uncertainty, Result 2 gives $P = \mu\, \Theta Y\big|_{P=0} / (r^* + \varphi)$ with $r^* = \rho + (\gamma - 1)(g^{(0)} - \eta\sigma_K^2 / 2)$. The estimate of future economic growth is thus cut to take account of its uncertain nature, especially if risk aversion η is high. When $\gamma > 1$ and rising affluence dominates the effect of growing damages, growth uncertainty cuts the discount rate and pushes up the risk-adjusted SCC. We rewrite the risk-adjusted discount rate as

$$(22) \qquad r^* = \underset{\text{impatience}}{\rho} + \underset{\text{rising affluence}}{\gamma g^{(0)}} - \underset{\text{growing damages}}{g^{(0)}} \underbrace{-\frac{1}{2}(1+\gamma)\eta\sigma_K^2}_{\text{prudence}} + \underset{\text{insurance}}{\eta\sigma_K^2},$$

where we recover the first three terms of the introductory example. The prudence term depresses the discount rate and pushes up the SCC (cf. Leland, 1968; Kimball, 1990). This effect increases in the coefficient of relative prudence $1+\gamma$, risk aversion η, and economic growth uncertainty. The insurance term stems from the perfect correlation between damages and GDP (damages are proportional to GDP). The insurance term acts to increase the optimal discount and reduce the optimal SCC, reflecting that positive shocks to damages are associated with positive shocks to GDP and are thus less harmful to welfare. This corresponds to a *built-in* climate beta of one.[30] For $\gamma > 1$ the prudence term dominates the insurance term, so growth uncertainty curbs the discount rate and boosts the optimal SCC, and vice versa for $\gamma < 1$. If utility is logarithmic,

[30] Dietz et al. (2018) use Monte Carlo simulations of DICE (Nordhaus, 2008) and find that, with emissions-neutral technical change, future states with rapid technical progress imply more emissions, more warming and a greater benefit from curbing emissions. The positive correlation between consumption and the benefits of mitigation implies a positive climate beta. This beta is close to one if damages are proportional to GDP, but closer to zero if damages are additive. Section III.C analyses correlated risks and climate betas more generally.

$\gamma = \eta = 1$ and $r^* = \rho,$ so economic growth (or asset return) uncertainty does not affect the optimal SCC, as in Golosov et al. (2014).[31]

It is instructive to consider the risk-adjusted discount rate when damages are not proportional to GDP:

$$(22') \qquad r^* = \rho + \gamma g^{(0)} - \underbrace{\theta_D \left(g^{(0)} - \frac{1}{2}(1-\theta_D)\sigma_K{}^2 \right)}_{\text{growing damages}} - \underbrace{\frac{1}{2}(1+\gamma)\gamma\sigma_K{}^2}_{\text{prudence}} + \underbrace{\theta_D \gamma \sigma_K{}^2}_{\text{self-insurance}},$$

where $0 \le \theta_D \le 1$ is the elasticity of damages with respect to GDP and we only consider the case $\eta = \gamma.$[32] The growing damages term indicates that due to the direct effect of a lower θ_D and the further stochastic reduction to the expected growth rate of damages (compared with $\theta_D = 1$), the discount rate is higher and the SCC smaller for a given $g^{(0)} > 0$. The prudence term is unaffected. The self-insurance effect now depends on the *built-in* beta of $\theta_D > 0.$ Focussing on uncertain growth, a smaller elasticity of damages with respect to GDP (lower θ_D) pushes up the SCC, since there is now less self-insurance due to the reduced *built-in* climate beta of θ_D.[33] If damages are additive and do not depend on GDP, the risk-insurance term drops out.

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[31] With logarithmic preferences and proportional damages, $\Delta_{\mathrm{CK}} = 0$ and (21) becomes $P = \mu \Theta Y|_{P=0} (1 + \Delta_\chi + \Delta_\lambda + \Delta_{CC}) / (\rho + \varphi).$ Economic growth uncertainty and the temperature and damage betas do not affect the optimal SCC, but climate sensitivity and damage rate uncertainty and its correlation do. The simple rule put forward by Golosov et al. (2014) does not consider these uncertainties and reduces to $P = \mu \Theta Y|_{P=0} / (\rho + \varphi).$ As Golosov et al. (2014) have a 2-box carbon cycle with a permanent as well as a temporary reservoir, we obtain more specifically $P = \Theta Y|_{P=0} \left[(1-\mu)/\rho + \mu/(\rho+\varphi) \right],$ where μ is the fraction of emissions that goes into the temporary reservoir.

[32] In our model, we only consider the case of proportional damages. We have derived (22′) in a similar, ad-hoc, fashion to the introductory example by assuming $\left(\partial \Pi(s)/\partial T(s)\right)\left(\partial T(s)/\partial S(s)\right) = \Theta C^{\theta_D}$. A similar expression is derived by Svenssen and Traeger (2014) and Dietz et al. (2018). Rewriting (22′), the risk-adjusted discount rate becomes $r^* = r_{\mathrm{rf}}^{(0)} - \theta_D (g - \sigma_K^2/2 - \gamma\sigma_K^2) - \theta_D^2 \sigma_K^2/2$ with $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - \gamma(1+\gamma)\sigma_K^2/2$ the risk-free interest rate, corresponding to Proposition 1 in Dietz et al. (2018).

[33] This follows from $\partial^2 r^* / \partial \sigma_K{}^2 \partial \theta_D = \gamma - \theta_D + 1/2 > 0$ for $\gamma > \theta_D - 1/2$, which is generally true.

B. Climate and damage uncertainties

The term $(1/2)\theta_{\chi T}(1+\theta_{\chi T})(\sigma_\chi/\bar{\chi})^2/(r^*+2\nu_\chi+\varphi)$ in (21) is the *climate sensitivity risk correction* and depends on $\theta_{\chi T}\equiv\theta_\chi+\theta_T+\theta_\chi\theta_T$ which combines positive skewness of the (equilibrium) climate sensitivity distribution ($\theta_\chi>0$) and convex dependence of damages on temperature ($\theta_T>0$). The climate sensitivity uncertainty correction is positive and larger for a more convex damage function, a more skewed climate sensitivity distribution with high uncertainty (σ_χ), smaller discount rate (r^*), and faster carbon decay rate (φ). The *damage rate risk correction* $(1/2)\theta_\lambda(1+\theta_\lambda)(\sigma_\lambda/\bar{\lambda})^2/(r^*+2\nu_\lambda+\varphi)$ in (21) is zero if the distribution of the damage ratio is not skewed ($\theta_\lambda=0$). A right-skewed distribution requires an upward-correction of the SCC, more so if damages are more uncertain. In both cases, when keeping the steady-state uncertainties $\Sigma_\chi^\infty\equiv\sigma_\chi/\sqrt{2\nu_\chi}$ and $\Sigma_\lambda^\infty\equiv\sigma_\lambda/\sqrt{2\nu_\lambda}$ fixed, increasing the rates of mean reversion ν_χ and ν_λ increases the risk corrections, as the near future becomes more uncertain.

C. Hedging: temperature beta and damage beta

We rewrite the term in Result 2 that corrects for correlations between climate and damage risks, on the one hand, and economic risks, on the other hand, as

$$(23)\qquad \Delta_{CK}=-(\eta-1)\sigma_K^2\left(\left(1+\theta_{\chi T}\right)\frac{\beta_{K\chi}}{r^*+\nu_\chi+\varphi}+\left(1+\theta_\lambda\right)\frac{\beta_{K\lambda}}{r^*+\nu_\lambda+\varphi}\right),$$

where $\beta_{K\chi}\equiv\rho_{K\chi}\sigma_\chi/\bar{\chi}\sigma_K$ and $\beta_{K\lambda}\equiv\rho_{K\lambda}\sigma_\lambda/\bar{\lambda}\sigma_K$ denote the temperature beta and damage beta, respectively. These betas measure the normalized covariance with shocks to the rate of economic growth analogously to the beta in asset pricing theory (e.g. Lucas, 1978; Breeden, 1979).[34] The sign of (23) depends on whether relative risk aversion η exceeds one or not, i.e. on whether the climate hedging effect dominates the offsetting effect due to growing damages.[35] We will first discuss the *hedging* effects, corresponding to the terms $-\eta\sigma_K^2\left((1+\theta_{\chi T})\beta_{K\chi}/(r^*+\nu_\chi+\varphi)+(1+\theta_\lambda)\beta_{K\lambda}/(r^*+\nu_\lambda+\varphi)\right)$ in (23).

A negative *temperature beta* $\beta_{K\chi}$ implies that asset returns in industries producing, for example, agricultural products, heating systems or winter garments are low in future states of nature in which temperature is high. It is then optimal to hedge these investments more by raising the SCC. If the economy is dominated by industries whose returns benefit from higher temperature (e.g. air conditioning), the temperature beta is positive, and it becomes optimal to have a lower SCC. The adjustment is large if risk aversion is high, climate sensitivity is more uncertain and skew, damages are more convex, and the climate sensitivity beta is large (high $\eta, \sigma_\chi, \theta_\chi, \theta_T,\ \beta_{K\chi}$) and non-zero even for a symmetric climate sensitivity distribution and a linear dependence of the damage ratio on temperature (i.e. $\theta_{\chi T}=0$).

A negative *damage beta* $\beta_{K\lambda}$ implies that asset returns in industries will be low in future states of nature in which damages are high, over and above the effect of the *built-in* climate beta. It is likely to be negative, especially in vulnerable areas (e.g. investments in flood prone regions), justifying a higher SCC. Economies dominated by industries that make money from climate damage (e.g. water engineering) have a positive damage beta ($\beta_{K\lambda}>0$) and should price carbon less vigorously. The adjustment is large if risk aversion is high, the damage ratio has high uncertainty and skewness (high $\eta, \sigma_\lambda, \theta_\lambda$) and is non-zero even for a symmetric damage ratio distribution (i.e. $\theta_\lambda=0$).

[34] Consistent with our perturbation scheme, the volatility of GDP is given to leading order by the volatility of the capital stock neglecting the effect of climate damages and thus the carbon stock, climate sensitivity and damage uncertainties.
[35] Lemoine (2017) calls these the risk insurance and risk exposure effects, respectively.

The *offsetting* effects in (23), $\sigma_K^2\left((1+\theta_{\chi T})\beta_{K\chi}/(r^*+\nu_\chi+\varphi)\right.$ $\left.+(1+\theta_\lambda)\beta_{K\lambda}/(r^*+\nu_\lambda+\varphi)\right)$ occur because future states of nature that are associated with high asset returns are associated with high growth in damages (as damages are proportional to GDP). E.g. if $\beta_{K\chi}<0$, future states of nature with negative GDP shocks are associated with lower damages, which requires a lower SCC. *Hedging* effects dominate *offsetting* effects if $\eta>1$.[36]

D. Correlation between temperature and damage ratio risks

The term $\Delta_{\mathrm{CC}}=(1+\theta_{\chi T})\,\rho_{\chi\lambda}(\sigma_\chi/\bar{\chi})(\sigma_\lambda/\bar{\lambda})\big/(r^*+\nu_\chi+\nu_\lambda+\varphi)$ in Result 2 captures the effect of correlation between temperature and damage ratio uncertainty on the SCC. This is positive if high temperature is associated with disproportionally high damages (e.g. extreme events such as hurricanes and fires as far as they are not captured by the convex dependence of damages on temperature), in which case the optimal SCC is higher. Risk aversion η plays no role, since there is no possibility of hedging the returns on assets.

E. Result 3

To derive Result 2, we have made two important simplifying assumptions: damages are proportional to the carbon stock ($\theta_{ET}=0$) and the mean of the climate sensitivity parameter (a proxy for temperature) is at its equilibrium

[36] Two further climate-beta effects have been suggested in the literature. First, Sandsmark and Vennemo (2007) only have one stochastic parameter, i.e. the loss of GDP for a given temperature, and additive damages (not proportional to GDP, so $\theta_D=0$). In this setup high future damages are associated with low levels of future aggregate consumption, and a large benefit from mitigating future climate change. The corresponding beta is thus negative. It relies on the product of the change in marginal utility due to damages and marginal damages themselves, is thus $O(\epsilon^2)$ in our perturbation scheme and too small to be included. Second, Nordhaus (2011) argues on basis of simulations with the RICE-11 integrated assessment model that "those states in which the global temperature increase is particularly high are also ones in which we are on average richer *in the future*", suggesting a positive beta. In the asymptotic approach framework of the paper,it does not feature in our correction factors, since it requires the integration of a Geometric Brownian Motion (for *K*), when solving the differential equation for the carbon stock, which cannot conveniently be done in closed form. Crucially, if $\theta_{ET}=0$, this effect is zero as marginal damages are no longer proportional to the carbon stock *E* and enhanced uncertainty of this term due to uncertain new emissions does not contribute to the optimal SCC. For the case $\theta_{ET}>0$, we examine this effect by numerically solving the stochastic differential equations and the integral in Result 1 and find it to be small (see Appendix G).

value from the outset ($\chi_0 = \bar{\chi}$), which are relaxed in Result 3 (see Appendix A). As a result of the first, the adjustment to the SCC for carbon stock uncertainty (A4) is zero in Result 2. Generally, this adjustment is negative as marginal damages $D_E \propto E^{\theta_{ET}}$ typically remain concave even for convex damages $D \propto E^{1+\theta_{ET}}$ (i.e., $\theta_{ET} < -1$, see section IV). Furthermore, when damages are convex ($\theta_{ET} > 0$), marginal damages will not be constant but increase with future emissions, resulting in a higher SCC, as captured by multiplicative correction factors in the form of single-variable deterministic integrals in Result 3. As a result of the second assumption, the mean temperature response at initial times and thus the SCC is overestimated by Result 2, but, due to the multiplicative correction factors, this is not the case for Result 3.

IV. Calibration

Table 1 summarizes our calibration starting from base year 2015 with further details in Appendix F. To calibrate the non-climatic part of our model to match historical asset returns, we follow Pindyck and Wang (2013) but abstract from catastrophic shocks to economic growth (see Appendices F.1 and F.2). This gives a coefficient of relative risk aversion of $\eta = 4.3$, intergenerational inequality aversion of $\gamma = 1.5$, pure time preference of $\rho = 5.8\%$ per year, trend growth of $g^{(0)} = 2.0\%$ per year, annual volatility of asset returns of $\sigma_K = 12\%$ and a risk premium of $\eta\sigma_K^2 = 6.4\%$ per year. In line with the specification in equation (6), we assume the global ratio of CO_2 emissions to GDP declines at a rate of 2.0% per year, which matches recent data.[37] Following Nordhaus (2017), we use world GDP at PPP of 116 trillion US dollars in 2015. Table 1 gives details for investment, depreciation and the cost of fossil fuel.

[37] The global ratio of CO_2 emissions to GDP ratio declined at 2.1% per year during 2000-15 versus a decline of 0.8% per year in the decade before. Nordhaus (2017) uses a decline of 1.5% per year.

TABLE 1 – SUMMARY OF BASE CASE CALIBRATION

Impatience and aversion to intergenerational inequality and risk	ρ = 5.8%/year, IIA = 1/EIS = γ = 1.5, RRA = η = 4.3
World economy	A^* = 0.113 /year, GDP PPP= 116\$T/year, $g^{(0)}$ = 2.0 %/year
Investment, depreciation and adjustment cost	$i^{(0)}$ = 2.8%/year, δ = 0.33%/year, ω = 12.5 year
Asset volatility and returns	σ_K = 12%/year$^{1/2}$, $r^{(0)}$ = 7.2%/year, $r_{\mathrm{rf}}^{(0)}$ = 0.80%/year, $r^{(0)} - r_{\mathrm{rf}}^{(0)}$ = $\eta\sigma_K^2$ = 6.4%/year
Share of fossil fuel and production cost	$1-\alpha$ = 4.3%, b = \$5.4×10^2 /tC
Preindustrial and 2015 (t = 0) carbon stocks Concavity of Arrhenius'law & stochastic carbon stock dynamics	S_{PI} = 596 GtC, S_0 = 854 GtC, E_0 = 258 GtC, θ_E = –0.36, μ = 0.65, φ = 0.35%/year, σ_E = 13 ppmv/year$^{1/2}$
Distribution of the climate sensitivity	χ_0 = 1.1109, $\bar{\chi}$ = 1.2619, σ_χ = 0.020%/year$^{1/2}$ ν_χ = 0.0086%/year, θ_χ = 3.0
Distribution of the damage ratio	θ_T = 0.56 ($\theta_{ET} = 0$), $\bar{\lambda}$ = 0.21, σ_λ = 2.3%/year$^{1/2}$, θ_λ = 2.7, ν_λ = 0.20/year
Flow impact of global warming damages	Θ_0 = 2.07% GDP/TtC
Conversion factors	1 ppmv CO_2 = 2.13 GtC, 1 tC = 3.664 tCO_2

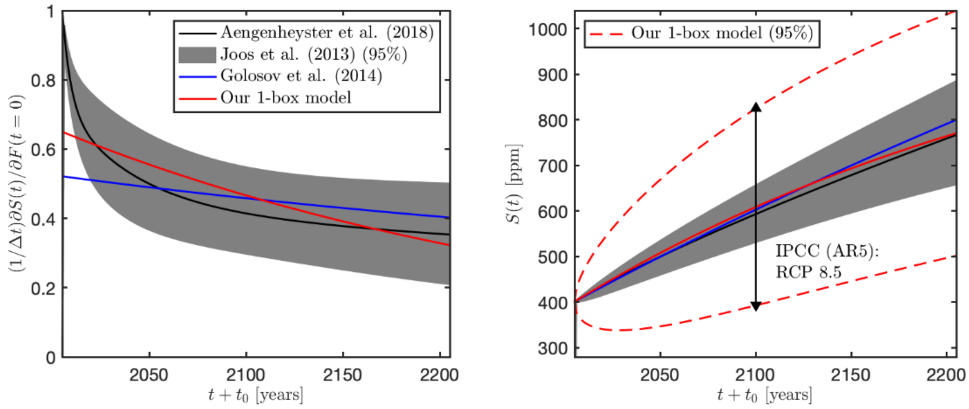
A. Carbon stock uncertainty

To calibrate our 1-box model for carbon stock dynamics (6), we use the 17 linear impulse response functions from the survey in Joos et al. (2013) and find $\mu =$ 0.65 and $\varphi = 0.35$%/year.[38] We use the 90% confidence range 794-1149 ppmv in 2100 predicted by simulations for the high temperature scenario RCP 8.5 (Chapter 12.4.8.1, IPCC, 2014 AR5) to calibrate $\sigma_E = 13$ ppmv/year$^{1/2}$. Fig. 1a shows the impulse response function for our 1-box model and Fig. 1b shows the stock of atmospheric carbon, including 95%-confidence bounds.[39] Fig. 1 shows that our simple 1-box model compares well with the 4-box model fitted to the same data by Aengenheyster et al. (2018) and the 2-box model of Golosov et al.

[38] It is possible to estimate these values from historical data too (see Appendix F.3).

[39] From $\Sigma_E(t) = $ (1149-794)/3.29 = 108 ppmv, $\sigma_E = \Sigma_E(t)\sqrt{2\varphi/(1-\exp(-2\varphi t))}$ = 13 ppmv CO_2 /year$^{1/2}$ with t = 2100 − 2005 = 95 years and using $\varphi = 0.35$%/year, which corresponds to a steady-state uncertainty of $\Sigma_E^\infty = \sigma_E/\sqrt{2\varphi} = 155$ ppmv CO_2. The confidence band from IPCC (2014, AR5) is shown centred around the mean of our prediction and translated in time to 2110 to reflect different initial times. The probability of a value of $\tilde{E} \le 0$ is indeed negligibly small, as previously assumed, and we formally have a negligibly small atom at $E = 0$.

(2014).[40,41,42] Our confidence bands are much wider than those obtained from Joos et al. (2013)[43] and still much wider than the uncertainty range obtained from historical data,[44] suggesting that model uncertainty far exceeds any inherent variability. Nevertheless, we will show in section V that even with our high value of σ_E, the correction to the optimal SCC is small for $\theta_{ET} \neq 0$.[45]



(a) Impulse response function (b) Stock of atmospheric carbon

FIGURE 1. ATMOSPERHIC CARBON CYCLE AND UNCERTAINTY

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[40] For a linear N-box carbon cycle $\tilde{S} = \sum_{i=0}^{N} \tilde{S}_i$ by $d\tilde{S}_i/dt = \mu_i F e^{-gt} - \varphi_i \tilde{S}_i$, Aengenheyster et al. (2018) obtain $\mu =$ {0.2173, 0.2240, 0.2824, 0.2763}, $\varphi =$ {0, 0.25, 2.74, 23.23}%/year with $\tilde{S}(t=0) =$ {328, 40, 27, 5}ppmv scaled so $\tilde{S}(t=0) = 401$ ppm. We adapt Golosov et al. (2014) to continuous time and get $\mu =$ {0.2, 0.3215}, $\varphi =$ {0, 0.23}%/year and $\tilde{S}(t=0) =$ {0.85, 0.15} $\times$ 401 ppm, ignoring its third box for carbon that decays within the first decadal period.

[41] We set the initial atmospheric carbon concentration to $S_0 = 401$ ppm of CO_2 (May 2015), corresponding to 0.854 TtC or 3.13 $TtCO_2$, and the preindustrial atmospheric carbon concentration to 280 ppm CO_2, 0.596 TtC or 2.19 $TtCO_2$, so that $E_0 = 121$ ppm CO_2, 0.258 TtC or 0.94 $TtCO_2$. Updated and historical values can be found online at http://www.esrl.noaa.gov/gmd/ccgg/trends/global.html.

[42] Although the impulse response function is less well captured by our 1-box model, this must be time integrated (after discounting) to evaluate the SCC. Agreement of the time path of the atmospheric stock (Figure 1b) is thus more important, especially if $\theta_{ET} \neq 0$ and the dependence on the stock is nonlinear.

[43] Using the distribution at t = 95 years and $\varphi = 0.35$%/year, we get $\sigma_E = 3.7$ ppm/year$^{1/2}$, which is much higher than the value of $\sigma_E = 0.65$ ppm/year$^{1/2}$ obtained by Aengenheyster et al. (2018) based on Joos et al. (2013).

[44] Based on the historical Law Dome Ice Core 2000-year dataset for emissions and concentrations, we estimate $\sigma_E =$ 0.1-0.15 ppmv CO_2/year$^{1/2}$ (see Appendix F.3). Using the same dataset but fitting a Geometric Brownian Motion, Hambel et al. (2017) find a much larger volatility of 0.78 %/year$^{1/2}$. Estimating this volatility, we find 1.4, 0.5 and 0.2 %/year$^{1/2}$ for the periods 1800-2004, 1900-2004 and 1959-2004. This large variation of volatility with time suggest that historical volatility in the atmospheric carbon concentrations is better described by an Arithmetic Brownian Motion, as in (6).

[45] The adjustment to the SCC is potentially larger than we calculate here, since there is a risk that as global warming continues (sudden) releases of greenhouse gases (e.g. from thawing permafrost) and reductions in the capacity of oceans to absorb CO_2 cause additional global warming. The existing modelling of such positive feedbacks "do not yield coherent results beyond the fact that present-day permafrost might become a net emitter of carbon during the 21st century under plausible future warming scenarios (low confidence)" (IPCC, 2014, AR5, Chapter 12.4.8.1) and we thus exclude it here.

B. Climate sensitivity uncertainty

We calibrate our temperature model (7) and (10a) to capture the key features of *both* the transient climate response (TCR) and the equilibrium climate response (ECS).[46] The ECS is the equilibrium or long-term change in annual mean global temperature following a gradual doubling of the atmospheric carbon stock relative to pre-industrial levels. The TCR is the change in temperature following an increase of 1% in the atmospheric stock of carbon each year at the time of doubling (i.e. 70 years). The distributions of the ECS and the TCR are in our view the best characterized measures of the uncertainty associated with predicted temperature increase in the climate science literature.

Figure 2 shows the range of probability density functions proposed for the TCR and ECS in IPCC (2014, AR5).[47] We take the mean of these distributions and fit our model to the first two moments of the TCR (mean and variance) and the first three moments of the ECS (mean, variance and skewness), as well as an initial temperature of $T_0 = 0.89$°C above preindustrial.[48,49] Table 2 shows that we match these moments well, and Table 3 shows good agreement with the consensus likelihood ranges in IPCC (2014, AR5). For comparison Fig. 2b also shows the thin-tailed Gamma distribution fitted by Pindyck (2012).

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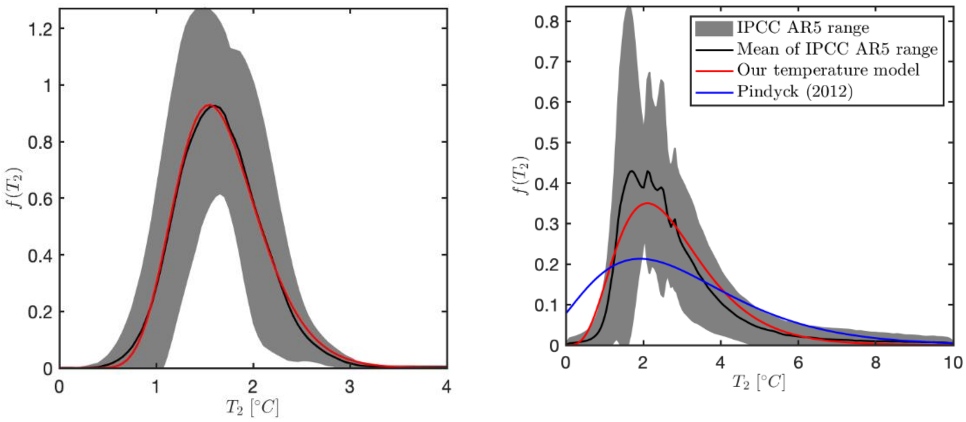
[46] From (7), $T_2 \equiv T(E = E_{PI}, \chi) = \chi^{1+\theta_\chi}$ $T_2 \equiv T(E = E_{PI}, \chi) = \chi^{1+\theta_\chi}$ with χ normally distributed with time-varying mean $\mu_\chi = \chi_0 \exp(-\nu_\chi t) \; + \bar{\chi}\left(1 - \exp(-\nu_\chi t)\right)$ and standard deviation $\Sigma_\chi = \sigma_\chi \sqrt{\left(1 - \exp(-2\nu_\chi t)\right) / 2\nu_\chi}$, and its skewness is given to leading-order by $\text{skew}\left[T_2\right] \equiv E\left[\left(T_2 - E[T_2]\right)^3\right] = 3\theta_\chi (1 + \theta_\chi)^3 \mu_\chi^{3(1+\theta_\chi)} (\Sigma_\chi / \mu_\chi)^4 + O(\Sigma_\chi^6)$.

[47] We take the 7 distributions for the TCR and the 13 distributions for the ECS from Figure 10.20 in Chapter 10.8 of IPCC (2014, AR5). The grey area in Figure 2 corresponds to one standard deviation either side of the mean of these different distributions (negative values not shown).

[48] To capture these and initial temperature, we match the TCR at $t = \ln(2S_{\mathrm{PI}} / S_0) / 0.02 =$ 17 years from 2015 (instead of 70 years from preindustrial). We thus deviate slightly from the formal definition of TCR, but argue this is justified as the high-level uncertainties in TCR and ECS are by far the best characterized of all summary statistics. This gives $\chi_0 = 1.11$, $\bar{\chi} = 1.26$, $T_0 = 0.89$°C, $\sigma_\chi = 0.020\%/\text{year}^{1/2}$, $\theta_\chi = 3.0$ and $\nu_\chi =$ 0.0086% per year corresponding to an e-folding scale of $1/2\nu_\chi = 58$ years. Climate sensitivity (as a proxy for temperature) is initially below its long-run value ($\chi_0 \le \bar{\chi}$).

[49] The probability of $\tilde{\chi} \le 0$ is indeed negligibly small, as previously assumed. Since $\chi \equiv \max[\tilde{\chi}, 0]$ cannot take negative values but $\tilde{\chi}$ can, we formally have a negligibly small atom at $\chi = 0$ (and $T_2 = 0$).



(a) Transient climate response (b) Equilibrium climate sensitivity

FIGURE 2. CLIMATE SENSITIVITY (PROBABILITY DENSITY FUNCTION)

TABLE 2. CLIMATE SENSITIVITY UNCERTAINTY

	TCR		ECS	
	IPCC (2014, AR5)	Our calibration	IPCC (2014, AR5)	Our calibration
E[T_2]	1.7°C	1.7°C	2.8°C	2.8°C
var[T_2]	0.19°C^2	0.20°C^2	1.5°C^2	1.7°C^2
skew[T_2]	0.16°C^3	0.054°C^3	2.4°C^3	2.5°C^3

TABLE 3. CLIMATE SENSITIVITY LIKELIHOOD

		IPCC (2014, AR5)	Our calibration
TCR	1-2.5°C	'very likely' (90-100%)	91%
	> 3°C	'extremely unlikely' (0-5%)	0.72%
ECS	1.5-4.5°C	'likely' (66-100%)	75%
	< 1°C	'extremely unlikely' (0-5%)	4.2%
	> 6°C	'very unlikely' (0-10%)	2.3%

C. Damage ratio uncertainty

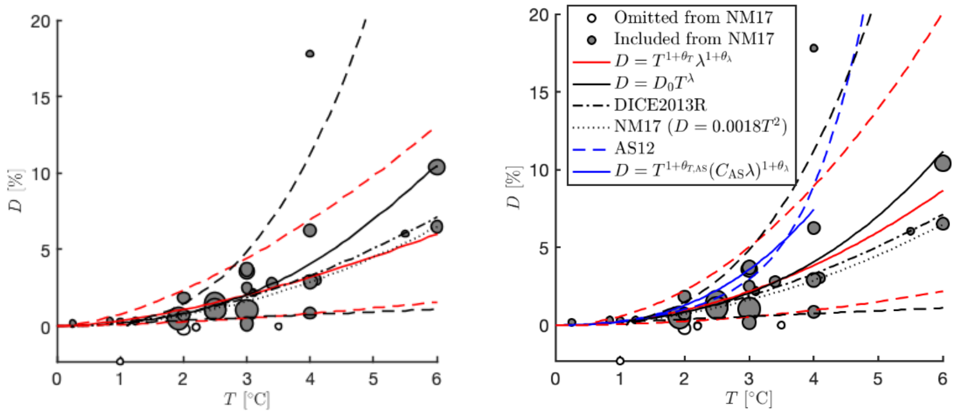
To calibrate the damage ratio and its uncertainty, we use the survey by Nordhaus and Moffat (2017) (henceforth NM17) including their subjective weights to reflect the reliability of different estimate shown in Fig. 3.[50] From these data, we estimate a mean $\lambda_0 = \bar{\lambda} = 0.21$, standard deviation $\Sigma_\lambda^\infty = 0.036$, damage convexity θ_T = 0.56 and skewness parameter θ_λ = 2.7 of the damage ratio,[51] which we take to correspond to the steady-state, setting the mean-

[50] Since our formulation does not allow for negative damages, we omit these estimates, which were given low weights of 0.1 by NM17. Fig. 3a shows omitted estimates in open circles and included estimates in closed circles. Since $\lambda \equiv \max[\tilde{\lambda}, 0]$ cannot take negative values, but $\tilde{\lambda}$ can, there is a negligibly small atom at λ (and $D = 0$).

[51] Ackerman and Stanton (2012) and Weitzman (2012) used a damage function that becomes even more convex at high temperatures. NM17 examined the possibly of thresholds or large convexities in the damage function but found no evidence for this in existing studies.

reversion coefficient to a large value of 20%/year (so $\sigma_\lambda = \Sigma_\lambda^\infty \sqrt{2\nu_\lambda}$ = 2.3%/year$^{1/2}$). The distribution has a positive standardized skewness $\text{skew}^*\left[D\,|\,T\right] = 3\theta_\lambda \Sigma_\lambda^\infty / \bar{\lambda}$ = 0.29.

The continuous lines in Fig. 3a denote expected damages with dashed lines denoting 5% and 95% confidence levels. Fig. 3a also shows the DICE2013R damage specification and NM17's preferred regression ($D = 0.0018T^2$). These fall within the confidence bounds of our estimates. Finally, following Nordhaus and Sztorc (2013) and NM17, we adjust damages shown in Fig. 3a upwards by 25% at all temperatures to reflect damages not included in current estimates. Combined with our calibrated value of $\theta_E = -0.36$ (see Appendix F.4), we obtain *proportional* damages (i.e. $\theta_{ET} = 0$).



(a) Proportional damages ($\theta_T = 0.56, \theta_{ET} = 0$) (b) Convex damages ($\theta_T = 1, \theta_{ET} = 0.28$)

FIGURE 3. DAMAGE RATIO UNCERTAINTY

Fig. 3b illustrates an alternative calibration in which damages are constrained to be quadratic in temperature, which implies *convex* damages ($\theta_T = 1$, $\theta_{ET} = 0.28$).[52] Our estimates imply an initial flow damage coefficient $\Theta(E_0, \chi_0, \bar{\lambda})$ of 2.1% and 1.8% of GDP per trillion ton of carbon for *proportional* and *convex* damages, respectively. This coefficient and the optimal

[52] Setting $\theta_T = 1$, we obtain $\theta_\lambda = 6.3$, $\lambda_0 = \bar{\lambda} = 0.43$, $\Sigma_\lambda^\infty = 0.039$, $\sigma_\lambda = 2.5\%/\text{year}^{1/2}$, $\nu_\lambda = 0.20$/year and $\theta_{ET} = 0.28$. This corresponds to a standardizes skewness $\text{skew}^*\left[D\,|\,T\right] = 0.27$ (similar to the unconstrained case). See Fig. 3b.

SCC rise as the atmospheric carbon stock rises with continued emissions (for $\theta_{ET} > 0$) and the climate sensitivity rises to equilibrium. For comparison, Golosov et al. (2014, p. 67-68) have a constant value of Θ = 2.4% GDP/TtC, which includes an upward adjustment for tipping risk.

V. Estimates of the Optimal Risk-Adjusted SCC

A. Market- versus. ethics-based calibration

Using Result 3 and the calibration in Table 1, Table 4 reports estimates of the optimal SCC derived from the *market-based* calibration (base case, with *proportional* damages), where all mark-ups in this and the other tables below are a percentage of the deterministic SCC.[53] The table shows the important role of the initial value of the climate sensitivity parameter χ_0: if it is mistakenly set to its higher steady state value $\bar{\chi}$, the optimal SCC roughly doubles. Similarly, if one does not allow for the lags in reaching the ECS (by setting $\nu_\chi \to \infty$), the optimal risk-adjusted SCC is considerably increased (cf. column 3), as the large uncertainties associated with the ECS are then experienced instantly. The optimal SCC of \$6.6/tCO_2 is low, since it is based on market rates of return.

TABLE 4. ESTIMATES OF THE SCC: MARKET- VS. ETHICS-BASED

	Market-based calibration			Ethics-based calibration		
	base case	$\chi_0 = \bar{\chi}$	$\nu_\chi = \infty$	base case	$\chi_0 = \bar{\chi}$	$\nu_\chi = \infty$
Deterministic SCC (\$/tCO_2)	4.1	8.4	8.4	11.5	20.8	20.8
due to economic uncertainty (\$/tCO_2)	1.3	2.4	2.4	18.7	26.2	26.2
due to carbon stock uncertainty	0	0	0	0	0	0
due to climate sensitivity uncertainty	0.4	0.6	2.6	4.7	6.4	11.2
due to damage ratio uncertainty	0.8	1.4	1.7	4.9	7.5	8.1
Risk-adjusted SCC (\$/tCO_2)	**6.6**	**12.8**	**15.0**	**39.8**	**61.0**	**66.3**
Economic risk mark-up	32%	29%	29%	163%	126%	126%
Climate sensitivity risk mark-up	9%	7%	31%	41%	31%	54%
Damage ratio risk mark-up	18%	17%	20%	43%	36%	39%
Total risk mark-up	59%	53%	80%	247%	193%	219%
Discount rate $r^{(0)}$ (per year)	7.2%	7.2%	7.2%	2.9%	2.9%	2.9%

Estimates in this table are for *proportional* damages ($\theta_{ET} = 0$), asset return volatility (σ_K = 12%/year$^{1/2}$), and ρ = 5.8%/year (*market-based* calibration) or ρ = 1.5%/year (*ethics-based* calibration).

[53] To assess the accuracy of the approximations made in Result 2 and 3 used in Tables 4-8, we evaluate Result 1 numerically and show that the error is small (see Appendix G for details).

In our *ethics-based* calibration, also shown in Table 4, we lower pure time preference from $\rho = 5.8\%$ to 1.5% per year, the risk-adjusted (not growth-corrected) discount rate $r^{(0)} = r^* + g^{(0)}$ falls from 7.2% to 2.9% per year.[54] This pushes up the deterministic SCC to \$11.5/tCO_2 and the risk-adjusted SCC to \$39.8/tCO_2. Using a market-based volatility, the mark-up for asset price risk is 163%, which exceeds those for climate sensitivity (41%) and damage ratio risk (43%). Ignoring deterministic temperature lags ($\chi_0 = \bar{\chi}$) boosts the deterministic SCC considerably as before but lowers all risk mark-ups. Ignoring stochastic temperature lags (setting $\nu_\chi \to \infty$), the climate sensitivity risk mark-up rises to an upper limit. In our calibration, the large uncertainty and skewness of the ECS (vs. TCR) only arises in the relatively long run (with an e-folding time of 58 years). From comparing the *market*- and *ethics-based* calibrations, we find that the ECS plays a more significant role for lower *ethics-based* discount rates, as is clear from the case of instantaneous ECS ($\nu_\chi \to \infty$). With high discount rates the TCR may be a better guide to climate policy than the ECS, but this is not so if a low ethics-based discount rate is used.[55]

B. Volatility from asset returns vs. GDP

The most important drawback of our *AK* model is that asset returns (capital) and GDP growth have the same volatility (see also the discussion in Pindyck and Wang (2013)), while the former is empirically much greater.[56] Table 5 shows that the mark-up for economic risk drops dramatically if volatility of

[54] E.g. Gollier (2018) relies on ethical arguments to use a zero or much lower discount rates than derived from asset market returns. To analyse this problem, the government should maximize expected welfare using low ethically-motivated discount rates subject to the constraints of the decentralized market economy with a lower discount rate. The optimal carbon price will then typically fall short of the social cost of carbon (Belfiori, 2017; Barrage, 2018).

[55] Kelly and Tan (2015) find that the mass of tail uncertainty in the climate sensitivity is curbed quickly even though overall learning is slow, because observations near the mean are evidence against fall tails. Bayesian learning curbs emissions by 50% instead of 38% without. Once the mass of the tail diminishes, remaining uncertainty is largely irrelevant for optimal emissions policy. Our formula for the optimal SCC shows that, once learning has removed tail uncertainty and skewness, in the distribution of the climate sensitivity, the SCC is increased by less.

[56] Ramsey- or Solow-type models in which consumption is a concave function of capital display a smaller relative volatility of consumption than of capital. Furthermore, a Ramsey-type model would introduce a second timescale to the problem (of economic convergence), which is likely to be fast compared to climatic timescales and, for purposes of calculating the optimal SCC, can probably be ignored.

GDP growth rates instead of asset returns is used.[57] Due to the higher risk-adjusted discount rate $r^{(0)}$, the mark-ups for climate sensitivity and damage ratio uncertainty and the risk-adjusted SCC are considerably reduced.

TABLE 5. ESTIMATES OF THE SCC: ASSET RETURN VS. GDP VOLATILITY

	Asset return volatility (σ_K = 12%/ year$^{1/2}$)			GDP growth volatility (σ_K = 1.5%/ year$^{1/2}$)			
	base case	$\eta = 6.0$	$\gamma = 2.0$	base case	$\eta = 6.0$	$\gamma = 2.0$	ρ = 0.1%/year
Deterministic SCC ($/t$CO_2$)	11.5	11.5	8.1	11.5	11.5	8.1	25.5
Risk-adjusted SCC ($/t$CO_2$)	**39.8**	**92.2**	**87.2**	**14.6**	**14.6**	**10.2**	**34.1**
Economic risk mark-up	163%	492%	691%	1%	1%	1%	2%
Climate sensitivity risk mark-up	41%	112%	149%	11%	11%	9%	15%
Damage ratio risk mark-up	43%	101%	134%	15%	15%	15%	16%
Total risk mark-up	247%	705%	974%	27%	28%	25%	34%
Discount rate $r^{(0)}$ (per year)	2.9%	2.3%	2.3%	4.5%	4.5%	5.5%	3.1%

Estimates in this table are for *proportional* damages ($\theta_{ET} = 0$) and ρ = 1.5%/year (*ethics-based* calibration), except for the last column, which considers a lower rate of impatience.

With asset return volatility, an increase in RRA[58] from 4.3 to 6.0 depresses the discount rate $r^{(0)}$ from 2.9% to 2.3% per year and pushes up the risk-adjusted SCC to \$92.2/tCO_2, corresponding to a total risk mark-up of 705%, whereas with GDP volatility this effect is negligibly small. With asset return volatility, an increase in IIA from 1.5 to 2.0 also pushes down the discount rate $r^{(0)}$ to 2.3% per year and the risk-adjusted SCC up to \$87.2/tCO_2. With GDP volatility, a similar increase in IIA instead increases the discount rate $r^{(0)}$ (from 4.5% to 5.5% per year), pushes down the deterministic SCC from \$11.5 to \$8.1/tCO_2 and the risk-adjusted SCC from \$14.6 to \$10.2/tCO_2.

Summarizing, the effect of RRA on the risk-adjusted SCC depends crucially on the magnitude of economic volatility and is very substantial for asset return volatility but negligibly small for GDP growth volatility. More IIA substantially boosts the risk-adjusted SCC for asset return volatility,[59] but decreases for GDP growth volatility. This accords with Crost and Traeger (2013), Ackerman et al.

[57] Historical data for the growth rate of world GDP for 1961-2015 imply σ_K = 1.5 %/year$^{1/2}$, which we use here.

[58] In this section, we will use the short-hands RRA and IIA to denote relative risk aversion ($\mathrm{RRA} = \eta$) and intergenerational inequality aversion($\mathrm{IIA} = \gamma$), respectively.

[59] As $g - \eta\sigma_K^2 < 0$ (cf. (22), when written as $r^* = \rho + (\gamma - 1)(g^{(0)} - \eta\sigma_K^2 / 2)$).

(2013) and Hambel et al. (2017), who all use uncertainty based on GDP.[60]

C. Convexity of the damage function

Table 6 considers the effect of our *convex* damage function ($\theta_{ET} = 0.28$) on the SCC. Generally, the SCC is larger due to larger damages for higher temperatures (cf. Fig. 3b), which is felt more strongly for lower discount rates.[61] A small mark-up for carbon stock uncertainty is now required, which is negative due to the concavity of marginal damages for $\theta_{ET} = 0.28$ (cf. (A4), Result 3).

TABLE 6. ESTIMATES OF THE SCC: CONVEXITY OF THE DAMAGE FUNCTION

	Proportional damages ($\theta_{ET} = 0$)	Convex damages ($\theta_{ET} = 0.28$)	Highly convex damages (AS12, $\theta_{ET} = 0.63$)
Deterministic SCC (\$/tCO_2)	25.5	26.8	77.2
Risk-adjusted SCC (\$/tCO_2)	**34.1**	**41.9**	**140.8**
Economic risk mark-up	2%	1%	-1%
Carbon stock risk mark-up	0%	-1%	-1%
Climate sensitivity risk mark-up	15%	30%	61%
Damage ratio risk mark-up	16%	26%	15%
Total risk mark-up	34%	56%	82%
Discount rate $r^{(0)}$ (per year)	3.1%	3.1%	3.1%
Estimates in this table are for ρ = 0.1%/year (*ethics-based* calibration) and GDP growth volatility (σ_K = 1.5%/ year$^{1/2}$).			

The climate sensitivity risk mark-up increases considerably due to the more convex damages-temperature relationship ($\theta_T = 1.0$ vs. 0.56). If we consider the highly convex damage function of Ackerman and Stanton (2012) (henceforth AS12), also shown in Fig. 3b with damages rapidly increasing above 1°C, we obtain an even larger deterministic SCC of \$77.2/tCO_2, a climate sensitivity risk mark-up of 61% and a total risk-adjusted SCC of \$140.8/tCO_2.[62,63]

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[60] With GDP growth volatility, it is possible to use an even lower *ethics-based* value of impatience of $\rho = 0.1\%$/year without negative discount rates and unbounded value of the SCC, which we will use below.

[61] This effect more than compensates the higher effective discount rate due to atmospheric decay of carbon in the case of *convex* damages (cf. $r^\star \equiv r^* + (1+\theta_{ET})\varphi$ in (A2), Result 3).

[62] The damage function of AS12 is $D = 1 - (1 + 0.00245T^2 + 5.021 \times 10^{-6} T^{6.76})^{-1}$. As our formulation has power-law damage functions, we fit $D = T^{1+\theta_{T,\mathrm{AS}}} (C_{\mathrm{AS}}\lambda)^{1+\theta_\lambda}$ to the AS12 damage function over the range 0-4.0°C to obtain $\theta_{T,\mathrm{AS}} = 0.54$ and $C_{\mathrm{AS}} = 0.90$, as illustrated in Fig. 3b. We retain the distribution for λ and the value of θ_λ for *convex* damages given in Table 1.

[63] As an alternative to our multiplicative uncertainty, Crost and Traeger (2014) have argued that the power-coefficient in the relationship between damages and temperature should be uncertain. To illustrate this, we calibrate $D = D_0 T^\lambda$

D. *Correlated risk and climate betas*

Table 7 examines the effect of the different climate betas. If the elasticity of damages with respect to world GDP (the *built-in* climate beta) is reduced from 1 to $\theta_D = 0.8$, two effects take hold: damage shocks are no longer fully insured, depressing the risk-adjusted discount rate (self-insurance term in (22′)) and pushing up the SCC, and damages now grow less rapidly than GDP, pushing up the discount rate (growing damages term in (22′)) and depressing the SCC. Table 7 shows that the former effect dominates when economic volatility is based on asset returns, and the latter when it is based on GDP growth.

TABLE 7. ESTIMATES OF THE SCC: CORRELATED RISK

	Asset return volatility		GDP growth volatility (σ_K = 1.5%/ year$^{1/2}$)							
	base	θ_D	base	θ_D	$\rho_{K\chi}$		$\rho_{K\lambda}$		$\rho_{\chi\lambda}$	
		0.8		0.8	-1	1	-1	1	-1	1
Deterministic SCC ($/t$CO_2$)	11.5	9.9	25.5	19.0	25.5	25.5	25.5	25.5	25.5	25.5
Risk-adjusted SCC ($/t$CO_2$)	**39.8**	**122.9**	**34.1**	**25.3**	**40.1**	**28.1**	**36.5**	**31.7**	**29.2**	**39.0**
Economic risk mark-up	163%	811%	2%	3%	2%	2%	2%	2%	2%	2%
Climate sensitivity risk mark-up	41%	181%	15%	14%	15%	15%	15%	15%	15%	15%
Damage ratio risk mark-up	43%	156%	16%	16%	40%	-7%	26%	7%	-3%	36%
Total risk mark-up	247%	1147%	34%	33%	57%	10%	43%	24%	15%	53%
Discount rate $r^{(0)}$ (per year)	2.9%	2.2%	3.1%	3.5%	3.1%	3.1%	3.1%	3.1%	3.1%	3.1%

Estimates in this table are for *proportional* damages ($\theta_{ET} = 0$), for $\rho = 1.5\%$/year in the case of asset return volatility (σ_K = 12%/ year$^{1/2}$), and for $\rho = 0.1\%$/year in the case of GDP growth volatility (σ_K = 1.5%/ year$^{1/2}$).

Taking economic volatility based on GDP growth, the SCC drops from \$40.1 to \$28.1/tCO_2 as the *temperature beta* $\beta_{K\chi}$, which measures correlation between temperature and GDP, is increased from its minimum to its maximum value ($\rho_{K\chi}$ from -1 to 1). Similarly, the SCC drops from \$36.5 to \$31.7/tCO_2 as the *damage beta* $\beta_{K\lambda}$, which measures correlation between damages and GDP, is

with $\lambda \sim N(\mu_\lambda, \Sigma_\lambda^2)$, to obtain $D_0 = 0.20$, $\mu_\lambda = 1.1$ and $\Sigma_\lambda = 0.59$, as shown in Fig. 3. Since damages cannot be stochastic at 1.0°C, we only use damage estimates for which temperature exceeds 1.1°C. From a leading-order expansion in λ, we obtain a standardized skewness which rises with temperature, i.e. $\text{skew}^*(D \mid T) = 3\Sigma_\lambda \log(T)$ (e.g. 2.45 at 4°C), which is much higher than our (constant) value of 0.29, especially at higher temperatures. Figure 3a indicates that this alternative gives a damage ratio distribution that is also more uncertain (wider confidence bands) at temperatures higher than 3°C or 4°C compared to *proportional* damages. Both the higher skewness and higher uncertainty push up the optimal SCC for low discount rates, but this effect is like our case of *convex* damages (Fig. 3b).

increased from its minimum to its maximum value ($\rho_{K\lambda}$ from -1 to 1). Finally, if we vary $\rho_{\chi\lambda}$ from -1 to 1, the SCC increases from \$29.2 to \$39.0/tCO_2, with the largest value corresponding to the case when future climate sensitivity shocks are perfectly (positively) correlated with future damage ratio shocks.[64]

E. Comparison with other calibrations

In Table 8, we evaluate the optimal risk-adjusted SCC for different calibrations in the literature. Golosov et al. (2014) uses proportional damages, logarithmic utility (IIA = RRA = 1), and ρ = 1.5% per year, which gives a risk-adjusted discount rate $r^{(0)}$ of 3.5% per year. With logarithmic utility, neither the expected rate of growth nor uncertainty about the future rate of growth influences the optimal SCC. Gollier (2012) uses RRA = IIA = 2 and ρ = 0, giving a risk-adjusted discount rate $r^{(0)}$ of 2.5% or 4.0% per year and a risk-adjusted SCC is \$62.6 or \$18.5/tCO_2 for economic volatility based on asset markets and GDP growth, respectively.[65] The discount rate is only substantially lowered for asset return uncertainty; asset return uncertainty depresses the discount rates and pushes up the risk-adjusted SCC as IIA exceeds one.

TABLE 8. ESTIMATES OF THE SCC: COMPARISON WITH OTHER CALIBRATIONS

Model	Base	Golosov et al. (2014)	Gollier (2012)		Stern (2007) +AS12
Volatility based on	asset returns	-	asset returns	GDP	GDP
Deterministic SCC (\$/tCO_2)	11.5	19.0	14.4	14.4	86.9
Risk-adjusted SCC (\$/tCO_2)	**39.8**	**24.6**	**62.6**	**18.5**	**165.2**
Economic risk mark-up	163%	0%	225%	1%	0%
Carbon stock risk mark-up	0%	0%	0%	0%	-1%
Climate sensitivity risk mark-up	41%	13%	57%	12%	65%
Damage ratio mark-up	43%	16%	54%	16%	26%
Total risk mark-up	247%	29%	336%	29%	90%
Discount rate $r^{(0)}$ (per year)	2.9%	3.5%	2.5%	4.0%	2.5%

Estimates in this table are for *proportional* damages ($\theta_{ET} = 0$), except for the final column, which assumes highly convex AS12 damages. The base case is for ρ = 1.5%/year (*ethics-based* calibration).

[64] The effects of ρ_{EK}, $\rho_{E\chi}$ and $\rho_{E\lambda}$ on the risk-adjusted SCC are very small, and we do not consider these here.

[65] The integrated assessment model of Nordhaus (2008) has IIA = RRA = 1.45 and ρ = 1.5%/year with a higher risk adjusted discount rate $r^{(0)}$ of 3.9% or 4.4%/year for economic volatility based on asset markets and GDP growth, respectively. Correspondingly, we obtain a lower deterministic SCC of \$11.9/tCO_2 and a lower risk-adjusted SCC of \$19.1 or \$15.1/tCO_2 corresponding to lower total risk mark-ups of 60% or 27% for economic volatility based on asset markets and GDP growth, respectively.

Finally, the last column of Table 8 uses IIA = RRA = 1.45 and a very low rate of time preference of ρ = 0.1%/year corresponding to a discount rate $r^{(0)}$ of 2.5% per year (for GDP-based economic volatility) and uses AS12 damages, which reflects the choice of low discount rate and convexity of damages used by Stern (2007). This gives very high values for the deterministic SCC of $87 and the risk-adjusted SCC of $165.2 per tCO_2.

VI. Concluding Remarks

We have derived a tractable rule for the optimal risk-adjusted SCC under climatic and damage uncertainties allowing for skewed distributions and the time scales on which they arise, as well as for uncertainty about economic growth or asset returns. Our rule is accurate if damages are a small fraction of world GDP (say, less than 10%), which is so for all estimates of damages in the literature. Our rule offers new analytical insights and complements insights from numerical solutions of stochastic, dynamic, nonlinear systems. We have calibrated our uncertainties based on what we think are the best high-level surveys (IPCC (2014, AR5) for atmospheric carbon stock and climate sensitivity uncertainties and Nordhaus and Moffat (2017) for damage ratio uncertainty)

The optimal SCC decreases in intergenerational inequality aversion if trend economic growth corrected for its uncertainty is positive but increases in risk aversion if economic growth (or asset returns) are volatile. If damages are proportional to GDP, there is a *built-in* climate beta of one. This self-insurance effect depresses the optimal SCC. If the elasticity of damages with respect to GDP is less than one, there is less potential for self-insurance, which pushes up the SCC, and damages grow less rapidly, which pushes down the SCC. The first effect dominates if economic volatility is derived from asset returns, but the second effect dominates if volatility is derived from GDP growth.

Uncertainty in atmospheric carbon stock dynamics only requires adjustments to the SCC if damages are convex, but these effects are negligible if based on historical uncertainty and negative and small if based on future projections. Uncertain climate sensitivity increases the SCC significantly, especially due to the skewness of the equilibrium climate sensitivity distribution, further enhanced by the convex dependence of damages on temperature. The magnitude of this mark-up depends crucially on the time scale on which it arises, and the much larger and more skew equilibrium climate sensitivity only plays a role for lower *ethics-based* discount rates. There is some evidence that the distribution damage ratio is right-skewed with an increase in the optimal SCC as a result.

Our rule for the optimal SCC also allows for correlated risks. If relative risk aversion exceeds one, what we call the *hedging effects* dominate the *offsetting effects* resulting from damages being proportional to GDP. It is then optimal to hedge and raise the SCC if the *temperature beta* is negative. This occurs when asset returns are high in future states of nature in which temperature is low (e.g. industries producing agricultural products, heating systems or winter garments). If risk aversion exceeds one, we also show that the optimal SCC is higher if the *damage ratio beta* is negative. This occurs when asset returns are high in future states of nature in which the damage ratio is lower than expected, which is typical, except for in adaptation industries (e.g. industries building flood defences). If risk aversion equals one, correlated risks do not affect the SCC, except for through the correlation between temperature and damage ratio risk.

We have found that the role of climate sensitivity uncertainty relies crucially on the time scale on which the large uncertainty and skewness associated with the ECS arise, a time scale that is not well understood. Instead of the TCR and the ECS, the so-called transient climate response to cumulative emissions (TCRE, e.g. Matthews et al. (2009)) is gaining traction. Although its uncertainty has not yet been as thoroughly studied as the TCR and the ECS, the absence of inherent time scales makes the TCRE useful for calculating the SCC needed to keep temperature below a cap.

Future research should be aimed at models that can have ethics-based discounting for policy makers but market-based discounting for the private sector and that are general enough to distinguish volatility of equity returns and GDP growth. We have abstracted from long-run risk in economic growth (Bansal and Yaron, 2004) and a downward-sloping term structure resulting from mean reversion in economic growth (Gollier and Mahul, 2017).[66] Models that include these three aspects should give more robust estimates of *temperature* and *damage betas*. Other challenges are to allow for compound Poisson shocks to temperature and damages (cf. Hambel et al., 2018; Bretschger and Vinogradova, 2018; Bansal et al., 2016), positive feedbacks such as the CO_2 absorption capacity of the oceans declining with temperature (Millar et al., 2016), the timing of climatic uncertainty, the risk of tipping points (e.g., Lemoine and Traeger, 2014, 2016; Lontzek et al., 2016; Cai et al., 2016; van der Ploeg and de Zeeuw, 2018), which may further increase the optimal SCC.

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Appendix A: More General Closed-form Solutions for the Optimal SCC

Result 1: The term Γ in Result 1 is given by (D3.17). Dimensionally,

$$
\begin{aligned}
\Gamma \equiv \Big(& (1+\theta_{ET})\varphi \mathrm{X}\Lambda - \nu_\chi (\bar{\chi}-\chi)\mathrm{X}_\chi \Lambda - \nu_\lambda (\bar{\lambda}-\lambda)\mathrm{X}\Lambda_\lambda \\
& -\frac{1}{2}\sigma_\chi^2 \mathrm{X}_{\chi\chi}\Lambda - \frac{1}{2}\mathrm{X}\Lambda_{\lambda\lambda}\sigma_\lambda^2 - (1-\eta)\mathrm{X}_\chi \Lambda \rho_{K\chi}\sigma_K \sigma_\chi \\
& -(1-\eta)\mathrm{X}\Lambda_\lambda \rho_{K\lambda}\sigma_K \sigma_\lambda - \mathrm{X}_\chi \Lambda_\lambda \rho_{\chi\lambda}\sigma_\chi \sigma_\lambda \Big) K^{1-\eta} E^{\theta_{ET}} \\
& -\theta_{ET}\mu A^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{b}\right)^{\frac{1}{\alpha}} \mathrm{X}\Lambda K^{2-\eta} E^{\theta_{ET}-1} e^{-g^{(0)}t} - \frac{1}{2}\theta_{ET}(\theta_{ET}-1)\sigma_E^2 \mathrm{X}\Lambda K^{1-\eta}E^{\theta_{ET}-2} \\
& -\big((1-\eta)\theta_{ET}\mathrm{X}\Lambda \rho_{KE}\sigma_K \sigma_E + \mathrm{X}_\chi \Lambda \rho_{E\chi}\sigma_E \sigma_\chi + \mathrm{X}\Lambda_\lambda \rho_{E\lambda}\sigma_E \sigma_\lambda \big) K^{1-\eta} E^{\theta_{ET}-1},
\end{aligned}
\tag{A1}
$$

where $\mathrm{X} \equiv \chi^{1+\theta_{ET}}$ and $\Lambda \equiv \lambda^{1+\theta_\lambda}$.

We can generalize Result 2 to include convex reduced-form damages ($\theta_{ET} \neq 0$) and deterministic temperature lags ($\chi_0 \neq \bar{\chi}$) (see Appendix E for the derivation). The resulting Result 3 includes additional correction factors, which can be evaluated as simple, one-dimensional integrals. The only additional assumption is that the future atmospheric carbon stock does not inherit any of the uncertainty from new emissions through their dependence on the stochastic capital stock (cf. (E2.3)), which is associated with only a very small error, as discussed in Appendix G.

Result 3: The leading-order optimal SCC is:

$$
\begin{aligned}
P \;=\; & \frac{\mu\, \Theta(E) Y|_{P=0}}{r^\star}\Bigg(1+\theta_{ET}\frac{\mu F^{(0)}}{E}\frac{\Upsilon_{\theta_{ET}\neq 0}}{r^{\star\star}} + (1+\theta_{\chi T})\frac{\nu_\chi}{r^\star}\frac{\bar{\chi}-\chi}{\chi}\Upsilon_{\chi_0\neq\bar{\chi}} \\
& +\Delta_{EE}+\Delta_{\chi\chi}+\Delta_{\lambda\lambda}+\Delta_{\chi\times\lambda}+\Delta_{CK}+\Delta_{CC}\Bigg),
\end{aligned}
\tag{A2}
$$

where $r^\star \equiv r^* + (1+\theta_{ET})\varphi$, $r^{\star\star} \equiv r^\star + (\eta-1)\sigma_K^2 - \varphi$ and $F^{(0)}$ is shorthand for optimal fossil fuel use without climate policy, $F^{(0)} = \big((1-\alpha)/b\big)^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}} K$, to the zeroth order of approximation. We refer to the Δ-terms in (A2) as uncertainty adjustments. We distinguish two types of correction factors, for $\theta_{ET} \neq 0$ and for $\chi_0 \neq \bar{\chi}$, which can be linearly combined, for example: $\Upsilon_{\chi\chi} \equiv \Upsilon_{\chi\chi,\theta_{ET}\neq 0} + \Upsilon_{\chi\chi,\chi_0\neq\bar{\chi}}$. We will discuss the uncertainty adjustments below. The correction factors are given in (E3.4)-(E3.5) in Appendix E.

The adjustments for uncertainty in the carbon stock, climate sensitivity, the damage ratio and the interaction between the two, which are now multiplied by their respective correction factors, are given by

(A4) $$\Delta_{EE} = \frac{1}{2}\theta_{ET}(1-\theta_{ET})\left(\frac{\sigma_E}{E}\right)^2 \frac{1}{r^\star - 2\varphi}\Upsilon_{EE},$$

(A5) $$\Delta_{\chi\chi} = \frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\frac{\left(\sigma_\chi/\chi_0\right)^2}{r^\star + 2\nu_\chi}\Upsilon_{\chi\chi}, \quad \Delta_{\lambda\lambda} = \frac{1}{2}\theta_\lambda(1+\theta_\lambda)\frac{\left(\sigma_\lambda/\bar{\lambda}\right)^2}{r^\star + 2\nu_\lambda}\Upsilon_{\lambda\lambda},$$

(A6) $$\Delta_{\chi\times\lambda} = \frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\sigma_\lambda^2\sigma_\chi^2}{2\nu_\chi 2\nu_\lambda}\Upsilon_{\chi\times\lambda}.$$

The adjustments for correlated climate and economic risk is

(A7) $$\Delta_{\mathrm{CK}} = -(\eta-1)\sigma_K\left(\theta_{ET}\frac{\rho_{KE}\dfrac{\sigma_E}{E}}{(r^\star-\varphi)}\Upsilon_{KE} + (1+\theta_{\chi T})\frac{\rho_{K\chi}\dfrac{\sigma_\chi}{\chi_0}}{r^\star+\nu_\chi}\Upsilon_{K\chi} + (1+\theta_\lambda)\frac{\rho_{K\lambda}\dfrac{\sigma_\lambda}{\bar{\lambda}}}{r^\star+\nu_\lambda}\Upsilon_{K\lambda}\right).$$

The correction for correlated climate sensitivity and damage ratio risk is

(A8) $$\begin{aligned}\Delta_{\mathrm{CC}} &= \theta_{ET}(1+\theta_{\chi T})\frac{\rho_{E\chi}\sigma_E\sigma_\chi/\chi_0}{(r^\star+\nu_\chi)E}\frac{r^\star}{r^\star-\varphi}\Upsilon_{E\chi} \\ &+\left(1+\theta_\lambda\right)\left(\theta_{ET}\frac{\rho_{E\lambda}\sigma_E\sigma_\lambda/\bar{\lambda}}{(r^\star+\nu_\lambda)E}\frac{r^\star}{r^\star-\varphi}\Upsilon_{E\lambda} + (1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda/\chi_0\bar{\lambda}}{r^\star+\nu_\chi+\nu_\lambda}\Upsilon_{\chi\lambda}\right).\end{aligned}$$

□

Convexity of reduced-form damages ($\theta_{ET} > 0$) causes Result 3 to be different from Result 2 in five ways. First, it changes the flow damage coefficient $\Theta(E)$. From (8), we obtain $\Theta_E(E) = (1/S_{PI})^2(1+\theta_{ET})\theta_{ET}(E/S_{PI})^{\theta_{ET}-1}$ to leading-order in our small parameter. With convex damages ($\theta_{ET} > 0$), the flow damage coefficient thus rises with the stock of atmospheric carbon. The time path for the carbon price is then steeper than that of world GDP. Its effect on the deterministic SCC is captured in (A2) through the correction factor $\Upsilon_{\theta_{ET} \neq 0} > 0$, reflecting the more harmful effect of future emissions (when the stock is higher). Second, it boosts the effective discount rate $r^{\star} = \rho + (\gamma - 1)g^{(0)} + (1+\theta_{ET})\varphi$, as the marginal damage of a unit of CO_2 decays more quickly than the unit itself. Combined, the net effect on the SCC is positive for small decay rates of atmospheric carbon. Third, a new adjustment (A4) needs to be made for carbon stock uncertainty. For reduced-form damages that are not too convex ($0 < \theta_{ET} < 1$), this adjustment is negative, reflecting concave marginal damages. Fourth, the adjustments for the other two uncertainties in (A5) are now multiplied by correction factors that are greater than unity, reflecting rising marginal damages, as in the deterministic case. The same applies to the terms adjusting for correlations in (A7)-(A8), with new correlation terms with the carbon stock arising there. Finally, Result 3 allows for a higher-order term (A6), which is normally small but may be non-negligibly small if $\theta_{\chi T}$ is large enough.

The effect of deterministic temperature lags ($\chi_0 \neq \bar{\chi}$) is captured as follows. The flow damage coefficient Θ is evaluated at the initial (low) temperature. The third term in the round brackets in (A2) is positive and captures this delayed deterministic temperature rise. Similarly, all the adjustments are corrected by their respective correction factors to take this delayed temperature increase into account.

Appendix B: Transformation to Non-Dimensional Form (For Online Publication)

We define the non-dimensional variables

$$\text{(B1)}\qquad \hat{K}\equiv\frac{K}{K_0},\hat{E}\equiv\frac{E}{E_0},\hat{\chi}\equiv\frac{\chi}{\overline{\chi}},\hat{\lambda}\equiv\frac{\lambda}{\overline{\lambda}},\hat{F}\equiv\frac{F}{F_0},\hat{C}\equiv\frac{C}{C_0},\hat{I}\equiv\frac{I}{C_0},\hat{\Phi}\equiv\frac{\Phi}{C_0},\hat{t}\equiv g_0 t \text{ and}$$

$$\hat{J}\equiv g_0 J/(C_0)^{1-\eta},$$

where zero subscripts refer to initial values ($t=0$), except for $F_0\equiv A(E_0)^{\frac{1}{\alpha}}\left((1-\alpha)/b\right)^{\frac{1}{\alpha}}K_0$ and $C_0\equiv g_0K_0$, so that all hatted quantities are $O(1)$ initially, assuming $\chi_0/\overline{\chi}=O(1)$ and $\lambda_0/\overline{\lambda}=O(1)$. We define $g_0\equiv g\left(E=E_0\right)$ to be the growth rate of the economy without additional climate change, $\phi\equiv\phi/g_0$ and $\hat{i}\equiv i/g_0$, where $i\equiv I/K$. The HJB equation (16) becomes in non-dimensional terms

$$\begin{aligned}
0 \quad = \quad & \max_{\hat{C},\hat{F}}\left[\frac{1}{1-\gamma}\frac{\hat{C}^{1-\gamma}-\hat{\rho}\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}}+\hat{J}_{\hat{t}}+\hat{J}_{\hat{K}}\phi(\hat{i})\hat{K}+\hat{J}_{\hat{E}}(\hat{\mu}\hat{F}e^{-\hat{g}\hat{t}}-\hat{\varphi}\hat{E})\right.\\
\text{(B2)}\qquad & \left.+\hat{J}_{\hat{\chi}}\hat{v}_{\chi}(1-\hat{\chi})+\hat{J}_{\hat{\lambda}}\hat{v}_{\lambda}(1-\lambda)+\frac{1}{2}\hat{J}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2+\frac{1}{2}\hat{J}_{\hat{E}\hat{E}}\hat{\sigma}_E^2+\frac{1}{2}\hat{J}_{\hat{\chi}\hat{\chi}}\hat{\sigma}_{\chi}^2+\frac{1}{2}\hat{J}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2\right]\\
& +\hat{J}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E+\hat{J}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi}+\hat{J}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}+\hat{J}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}\\
& +\hat{J}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}+\hat{J}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda},
\end{aligned}$$

where $\hat{I}=\hat{Y}-\hat{b}\hat{F}-\hat{C}=\hat{A}(\hat{E},\hat{\chi})\hat{K}^{\alpha}\hat{F}^{1-\alpha}-\hat{b}\hat{F}-\hat{C}$, $\hat{Y}\equiv Y/C_0$ and $\phi=\hat{i}-(1/2)\hat{\omega}\hat{i}^2-\delta$. The resulting non-dimensional expressions are

$$\text{(B3)}\qquad \begin{aligned}
&\hat{\rho}\equiv\frac{\rho}{g_0},\ \hat{b}\equiv\frac{bF_0}{g_0K_0},\hat{\omega}\equiv g_0\omega,\ \hat{\delta}\equiv\frac{\delta}{g_0},\hat{g}\equiv\frac{g}{g_0},\ \hat{\mu}\equiv\frac{\mu F_0}{g_0E_0},\ \hat{\varphi}\equiv\frac{\varphi}{g_0},\ \hat{v}_{\chi}\equiv\frac{v_{\chi}}{g_0},\\
&\hat{v}_{\lambda}\equiv\frac{v_{\lambda}}{g_0},\ \hat{\sigma}_K\equiv\frac{\sigma_K}{\sqrt{g_0}},\ \hat{\sigma}_E\equiv\frac{\sigma_E}{\sqrt{g_0}E_0},\ \hat{\sigma}_{\chi}\equiv\frac{\sigma_{\chi}}{\sqrt{g_0}\,\overline{\chi}}\text{ and }\hat{\sigma}_{\lambda}\equiv\frac{\sigma_{\lambda}}{\sqrt{g_0}\,\overline{\lambda}},
\end{aligned}$$

with $\hat{A}\left(\hat{E},\hat{\chi}\right) \equiv A(E,\chi)F_0^{1-\alpha} / g_0 K_0^{1-\alpha}$. Damages and total factor productivity become

(B4) $\hat{D}(\hat{E},\hat{\chi},\hat{\lambda}) \equiv \epsilon \hat{\lambda}^{1+\theta_\lambda} \hat{\chi}^{1+\theta_{\chi T}} \hat{E}^{1+\theta_{ET}}$ and $\hat{A} \equiv \hat{A}^*(1-\hat{D}) = \hat{A}^*(1-\epsilon \hat{\lambda}^{1+\theta_\lambda} \hat{\chi}^{1+\theta_{\chi T}} \hat{E}^{1+\theta_{ET}})$,

where the damage fraction $\hat{D} \equiv D$ is already non-dimensional, $\hat{A}^* \equiv A^* F_0^{1-\alpha} / g_0 K_0^{1-\alpha}$ and the final non-dimensional parameter is

$$\text{(B5)} \qquad \epsilon \equiv \bar{\lambda}^{1+\theta_\lambda} \bar{\chi}^{1+\theta_{\chi T}} \left(\frac{E_0}{S_{PI}}\right)^{1+\theta_{ET}}.$$

The first-order conditions of (B2) with respect to $\hat{C}$ and $\hat{F}$ are, respectively,

$$\text{(B6)} \quad \frac{\hat{C}^{-\gamma}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}} - \phi'\left(\hat{i}\right)\hat{J}_{\hat{K}} = 0 \;\;\Rightarrow\;\; \hat{C} = \left(\phi'\left(\hat{i}\right)\hat{J}_{\hat{K}}\right)^{-\frac{1}{\gamma}} \left((1-\eta)\hat{J}\right)^{-\frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}} \quad \text{and}$$

$$\text{(B7)} \quad \hat{J}_{\hat{K}}\left((1-\alpha)\hat{A}\hat{K}^{\alpha}\hat{F}^{-\alpha} - \hat{b}\right)\phi'\left(\hat{i}\right) + \hat{J}_{\hat{E}} e^{-\hat{g}\hat{t}} \hat{\mu} = 0 \;\;\Rightarrow\;\; \hat{F} = \left(\frac{1-\alpha}{\hat{b} + \hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K},$$

where we have defined the optimal SCC in non-dimensional terms as

$$\text{(B8)} \qquad \hat{P} \equiv \left(\frac{F_0}{g_0 K_0}\right) P = -\hat{\mu} \frac{\hat{J}_{\hat{E}}}{\phi'(\hat{i})\hat{J}_{\hat{K}}},$$

and use (B7) to write the production function as

$$\text{(B9)} \qquad \hat{Y} = \hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{K}^{\alpha}\hat{F}^{1-\alpha} = \hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)^{\frac{1}{\alpha}} \left(\frac{1-\alpha}{\hat{b} + \hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1-\alpha}{\alpha}} \hat{K}.$$

Appendix C: Derivation of Zeroth-Order Solution (For Online Publication)

In non-dimensional terms, the truncated series solutions for the value function and the forward-looking control variables (18) is

$$\text{(C1)}\qquad \begin{aligned} \hat{J}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) &= \hat{J}^{(0)}\left(\hat{K},\epsilon\hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\right)+\epsilon\hat{J}^{(1)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})+O(\epsilon^2),\\ \hat{F}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) &= \hat{F}^{(0)}\left(\hat{K},\epsilon\hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\right)+\epsilon\hat{F}^{(1)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})+O(\epsilon^2),\\ \hat{C}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) &= \hat{C}^{(0)}\left(\hat{K},\epsilon\hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\right)+\epsilon\hat{C}^{(1)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})+O(\epsilon^2). \end{aligned}$$

At $O(1)$ the Hamilton-Jacobi-Bellman equation (B2) can be written as

$$\text{(C2)}\qquad \begin{aligned} &\frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}^{(0)}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}\frac{\eta-\gamma}{1-\eta}}-\hat{\rho}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}}}{(1-\gamma)\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}-1}}+\hat{J}^{(0)}_{\hat{t}}+\hat{J}^{(0)}_{\hat{K}}\hat{\phi}(\hat{i}^{(0)})\hat{K}\\ &\qquad +\frac{1}{2}\hat{J}^{(0)}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2=0, \end{aligned}$$

where we have substituted for the forward-looking variables $\hat{C}$ and $\hat{F}$ at $O(1)$ from (B6) and (B7) and we have used

$$\text{(C3)}\qquad \underbrace{\frac{1}{d\hat{t}}E_t\left[d\hat{K}\right]}_{O(1)}=\hat{\phi}(\hat{i}^{(0)})\hat{K}\,.$$

In (C2)-(C3), $\hat{i}^{(0)}$ is the (constant) optimally chosen investment rate. Equation (C2) has a power-law solution of the form $J^{(0)}=\psi_0\hat{K}^{1-\eta}$, and following some manipulation we obtain

$$\text{(C4)}\quad \hat{J}^{(0)}=\psi_0\hat{K}^{1-\eta}\ \text{ with }\ \psi_0=\frac{1}{1-\eta}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-(1-\eta)}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_K^2\right)\right)^{-\gamma\frac{1-\eta}{1-\gamma}}.$$

From the first-order condition (B6), we obtain

$$\text{(C5)} \qquad \hat{C}^{(0)} = \hat{c}^{(0)}\hat{K} \ \text{ with } \ \hat{c}^{(0)} = \frac{1}{\hat{\phi}'(\hat{i}^{(0)})}\left(\hat{\rho} - (1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)}) - \frac{1}{2}\eta\hat{\sigma}_K^2\right)\right),$$

where $q\left(\hat{i}\right) = 1\big/\phi'\left(\hat{i}\right)$ denotes Tobin's $q,$ the price of capital in consumption terms.[1] We can thus write the value function (C4) as

$$\text{(C6)} \qquad J^{(0)} = \frac{1}{1-\eta}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}}\hat{K}^{1-\eta}.$$

Substituting in for $\hat{F}$ from (B7) and for $\hat{Y}$ from (B9), we obtain from $\hat{I} = \hat{Y} - \hat{C} - \hat{b}\hat{F}$:

$$\text{(C7)} \qquad \hat{i}^{(0)} = \hat{r}^{(0)}_{\text{mpk}} + \hat{\delta} - \hat{c}^{(0)} = \hat{r}^{(0)}_{\text{mpk}} + \hat{\delta} - \hat{q}^{(0)}\left(\hat{\rho} - (1-\gamma)\left(\hat{\phi}(\hat{i}^{(0)}) - \frac{1}{2}\eta\hat{\sigma}_K^2\right)\right),$$

where $\hat{r}^{(0)}_{\text{mpk}} \equiv \hat{Y}_{\hat{K}}\left(\hat{P} = 0\right) - \hat{\delta} = \alpha\hat{A}\left(\hat{E}, \hat{\chi}, \hat{\lambda}\right)^{\frac{1}{\alpha}}\left((1-\alpha)\big/\hat{b}\right)^{\frac{1-\alpha}{\alpha}} - \hat{\delta}$ denotes the marginal productivity of capital net of depreciation.[2] Equation (C7) implicitly defines the optimally chosen investment rate $\hat{i}^{(0)}.$ From (C3), the leading-order endogenous growth rate of capital and hence of consumption is

$$\text{(C8)} \qquad \hat{g}^{(0)} = \underbrace{\frac{1}{\hat{K}}\frac{1}{d\hat{t}}E_t\left[d\hat{K}\right]}_{O(1)} = \hat{\phi}(\hat{i}^{(0)}) \quad \text{and hence} \quad \hat{g}^{(0)} = \hat{\phi}\left(\hat{i}^{(0)}\right) = 1.$$

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[1] The value of the capital stock is $\hat{q}\hat{K},$ or dimensionally $qK,$ where $\hat{q} = 1/\hat{\phi}'(\hat{i}) = 1/\phi'(i)$ is already a fraction and is left unchanged by the scaling (cf. $\hat{q} = q$ or $\omega i = \hat{\omega}\hat{i}$).

[2] Dimensionally, we have $r^{(0)}_{\text{mpk}} = \hat{r}^{(0)}_{\text{mpk}} g_0.$

In equilibrium, the marginal propensity to consume $\hat{c}^{(0)}/\hat{q}^{(0)}$ equals the expected return on investment $\hat{r}^{(0)}$ minus the growth rate of the economy $\hat{g}^{(0)}$. In turn, the expected return on investment equals the sum of the risk-free rate $\hat{r}_{\mathrm{rf}}^{(0)}$ and the risk premium $\Delta\hat{r}^{(0)}$. Hence, $\hat{c}^{(0)}/\hat{q}^{(0)} = \hat{r}^{(0)} - \hat{g}^{(0)} = \hat{r}_{\mathrm{rf}}^{(0)} + \Delta\hat{r}^{(0)} - \hat{g}^{(0)}$ and with a risk premium of $\Delta\hat{r}^{(0)} = \eta\hat{\sigma}_K^2$ in the absence of any climate risk at zeroth-order, the risk-free rate is:

(C9) $$\hat{r}_{\mathrm{rf}}^{(0)} = \hat{\rho} + \gamma\hat{g}^{(0)} - (1+\gamma)\eta\hat{\sigma}_K^2/2 .$$

Although $\hat{J}_{\hat{E}}^{(0)}$ can be computed from (C6), a consistent leading-order estimate of the optimal SCC also requires $\hat{J}_{\hat{E}}^{(1)}$ and thus the next order in the perturbation expansion, i.e., $\hat{P} = -\hat{\mu}\left(\hat{J}_{\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{E}}^{(1)}\right)\Big/\phi'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}$.

Appendix D: Derivation of First-Order Solution (For Online Publication)

D.1. Solution to multi-variate Ornstein-Uhlenbeck process

We define $\hat{k} \equiv k \equiv \log\left(K / K_0\right)$, so the vector of states $d\mathbf{x} = \left\{d\hat{k}, d\hat{E}, d\hat{\chi}, d\hat{\lambda}\right\}^T$ is described by a multi-variate Ornstein-Uhlenbeck process (11), which in non-dimensional terms is

(D1.1) $$d\mathbf{x} = \boldsymbol{\alpha} - \mathbf{v} \circ \left(\mathbf{x} - \boldsymbol{\mu}\right) d\hat{t} + \mathbf{S}d\mathbf{W}_{\hat{t}} .$$

The growth rate vector (12), relevant to the capital and atmospheric carbon stock processes only, is given in non-dimensional terms by

(D1.2) $$\boldsymbol{\alpha} = \left(\frac{1}{d\hat{t}}\frac{E_t\left[d\hat{K}\right]}{\hat{K}} - \frac{1}{2}\hat{\sigma}_K^2, \frac{1}{d\hat{t}}E_t\left[d\hat{E}\right], 0, 0\right)^T = \left(\hat{\phi}(\hat{\imath}) - \frac{1}{2}\hat{\sigma}_K^2, \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}\hat{t}}, 0, 0\right)^T ,$$

the mean reversion rate vector by $\mathbf{v} = \left(0, \hat{\varphi}, \nu_\chi, \nu_\lambda\right)^T$, the vector of means by $\boldsymbol{\mu}^T = (0,0,1,1)^T$, and the covariance matrix $\mathbf{SS}^T$ has the form

$$\text{(D1.3)} \qquad \frac{1}{d\hat{t}} E_t\left[d\mathbf{x} d\mathbf{x}^T\right] = \mathbf{SS}^T = \begin{pmatrix} \hat{\sigma}_K^2 & \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda \\ \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \hat{\sigma}_E^2 & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda \\ \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \hat{\sigma}_\chi^2 & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda \\ \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda & \hat{\sigma}_\lambda^2 \end{pmatrix}.$$

We begin by integrating the multi-variate Ornstein-Uhlenbeck process (D1.1), including only terms at zeroth order, so that the coefficients are constant, and a closed-form solution is available. Specifically, $\boldsymbol{\alpha}^{(0)} = \left(\hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2, \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{1/\alpha} \hat{A}^{1/\alpha}\hat{K}_0, 0, 0\right)^T$, where we have relied on the solution for $\hat{K}$ from the zeroth-order problem (cf. (C8)). The slow dependence of productivity $\hat{A}$ on the states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ can be neglected when integrating with respect to time, consistent with the multiple-scales nature of our perturbation expansion. For constant coefficients, (D1.1) can be integrated to give:

$$\text{(D1.4)} \qquad \mathbf{x}(t) = \boldsymbol{\mu} + \boldsymbol{\alpha} t + e^{\mathbf{v}\hat{t}} \circ \left(\mathbf{x}_0 - \boldsymbol{\mu}\right) + \int_0^{\hat{t}} e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S} d\mathbf{W}_{\hat{u}}.$$

The quantity $\mathbf{x}(t)$ is normally distributed with covariance matrix $\boldsymbol{\Sigma}(t)$:

$$\text{(D1.5)} \qquad \boldsymbol{\Sigma}(t) = \int_0^{\hat{t}} \left(e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\right)\left(e^{\mathbf{v}(\hat{u}-\hat{t})} \circ \mathbf{S}\right)^T d\hat{u} =$$

$$\begin{pmatrix} \hat{\sigma}_K^2 \hat{t} & \frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\nu_\chi}\left(1-e^{-\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\nu}_\lambda}\left(1-e^{-\hat{\nu}_\lambda\hat{t}}\right) \\ \frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\hat{\sigma}_E^2}{2\hat{\varphi}}\left(1-e^{-2\hat{\varphi}\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\hat{\nu}_\chi}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\hat{t}}\right) \\ \frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\nu_\chi}\left(1-e^{-\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\nu_\chi}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\hat{t}}\right) & \frac{\hat{\sigma}_\chi^2}{2\nu_\chi}\left(1-e^{-2\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\hat{t}}\right) \\ \frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\nu}_\lambda}\left(1-e^{-\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\hat{t}}\right) & \frac{\hat{\sigma}_\lambda^2}{2\hat{\nu}_\lambda}\left(1-e^{-2\hat{\nu}_\lambda\hat{t}}\right) \end{pmatrix}.$$

D.2. Evolution equations for $\hat{K}$ and $\hat{E}$

We consider the expected evolution equations of the states $\hat{K}$ and $\hat{E}$ at $O(\epsilon)$ and $O(1)$, respectively. At this order, we have for the expected evolution of $\hat{K}$:

(D2.1) $$\underbrace{\frac{1}{d\hat{t}}E_t\left[d\hat{K}\right]}_{O(\epsilon)} = \hat{\phi}'(\hat{i}^{(0)})\epsilon\hat{I}^{(1)} = -\hat{\phi}'(\hat{i}^{(0)})\epsilon\hat{C}^{(1)} = \frac{\hat{\phi}'(\hat{i}^{(0)})\hat{c}^{(0)}}{\gamma - \frac{\hat{c}^{(0)}\phi''}{\hat{\phi}'(\hat{i}^{(0)})}}\hat{K}\left(\frac{\epsilon\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} + \frac{\eta-\gamma}{1-\eta}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right),$$

where the first identity makes use of $\hat{\Phi} = \epsilon\hat{I}^{(1)} - \hat{\omega}\epsilon\hat{I}^{(1)}\hat{I}^{(0)}/\hat{K} = \epsilon\hat{I}^{(1)}\hat{\phi}'(\hat{i}^{(0)})$ at $O(\epsilon)$. We further note from $\hat{I} = \hat{Y} - \hat{b}\hat{F} - \hat{C}$ that $\hat{I}^{(1)} = -\hat{C}^{(1)}$, since production net of fossil fuel costs is unaffected by the SCC in our formulation:

(D2.2) $$\frac{\partial}{\partial\hat{P}}\left[\hat{Y} - \hat{b}\hat{F}\right]\Big|_{\hat{P}=0} = \frac{\partial}{\partial\hat{P}}\left[\hat{A}^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1-\alpha}{\alpha}}\hat{K} - \hat{b}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}\right]\Bigg|_{\hat{P}=0} = 0.$$

The identity in (D2.2) relies on the Cobb-Douglas nature of the production function. The third identity in (D2.1) follows from a Taylor-series expansion of $\hat{C}$, given by (B6), with respect to the small parameter ϵ (about $\epsilon = 0$):

(D2.3) $$\hat{c}^{(1)} = \hat{c}^{(0)} \left(-\frac{1}{\gamma} \frac{\phi''}{\hat{\phi}'\left(\hat{i}^{(0)}\right)} \epsilon \hat{i}^{(1)} - \frac{1}{\gamma} \frac{\epsilon \hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} - \frac{1}{\gamma} \frac{\eta - \gamma}{1-\eta} \frac{\epsilon \hat{J}^{(1)}}{\hat{J}^{(0)}} \right).$$

Noting that $\hat{i}^{(1)} = -\hat{c}^{(1)}$, we can rearrange this linear equation to give

(D2.4) $$\hat{c}^{(1)} = \frac{\hat{c}^{(0)}}{1 - \frac{1}{\gamma} \frac{\hat{c}^{(0)} \phi''}{\hat{\phi}'(\hat{i}^{(0)})}} \left(-\frac{1}{\gamma} \frac{\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} - \frac{1}{\gamma} \frac{\eta - \gamma}{1-\eta} \frac{\hat{J}^{(1)}}{\hat{J}^{(0)}} \right),$$

which is used in the third identity in (D2.1). For $\hat{E}$, we have at $O(1)$:

(D2.5) $$\underbrace{\frac{1}{d\hat{t}} E_t\left[d\hat{E} \right]}_{O(1)} = \hat{\mu} \left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K} e^{-\hat{g}^{(0)}\hat{t}} - \hat{\varphi}\hat{E}.$$

D.3. The Hamilton-Jacobi-Bellman equation

Substituting for the forward-looking variables $\hat{C}$ from (B6) and $\hat{F}$ from (B7), the Hamilton-Jacobi-Bellman equation (B2) becomes at $O(\epsilon)$:

(D3.1) $$\underbrace{\hat{f}^*(\hat{J})}_{O(\epsilon)} + \epsilon \hat{J}_{\hat{t}}^{(1)} + \epsilon \hat{J}_{\hat{K}}^{(1)} \hat{K} \hat{\phi}(\hat{i}^{(0)}) + \frac{\hat{\phi}'\left(\hat{i}^{(0)}\right) \hat{c}^{(0)}}{\gamma - \hat{c}^{(0)} \phi'' / \hat{\phi}'(\hat{i}^{(0)})} \left(\hat{K} \hat{J}_{\hat{K}}^{(1)} + (\eta - \gamma) \hat{J}^{(1)} \right)$$
$$+ \left(\hat{J}_{\hat{E}}^{(0)} + \epsilon \hat{J}_{\hat{E}}^{(1)} \right) \left(\hat{\mu} \left(\frac{1-\alpha}{\hat{b}} \right)^{1/\alpha} \hat{A}^{1/\alpha} \hat{K} e^{-\hat{g}^{(0)}\hat{t}} - \hat{\varphi}\hat{E} \right) + \left(\hat{J}_{\hat{\chi}}^{(0)} + \epsilon \hat{J}_{\hat{\chi}}^{(1)} \right) \hat{v}_{\chi} (1 - \hat{\chi})$$

$$+\left(\hat J^{(0)}_{\hat\lambda}+\epsilon\hat J^{(1)}_{\hat\lambda}\right)\hat\nu_\lambda(1-\hat\lambda)+\frac{1}{2}\epsilon\hat J^{(1)}_{\hat K\hat K}\hat K^2\hat\sigma_K^2+\frac{1}{2}\left(\hat J^{(0)}_{\hat E\hat E}+\epsilon\hat J^{(1)}_{\hat E\hat E}\right)\hat E^2\hat\sigma_E^2$$

$$+\frac{1}{2}\left(\hat J^{(0)}_{\hat\chi\hat\chi}+\epsilon\hat J^{(1)}_{\hat\chi\hat\chi}\right)\hat\sigma_\chi^2+\frac{1}{2}\left(\hat J^{(0)}_{\hat\lambda\hat\lambda}+\epsilon\hat J^{(1)}_{\hat\lambda\hat\lambda}\right)\hat\sigma_\lambda^2+\left(\hat J^{(0)}_{\hat K\hat E}+\epsilon\hat J^{(1)}_{\hat K\hat E}\right)\hat K\rho_{KE}\hat\sigma_K\hat\sigma_E$$

$$+\left(\hat J^{(0)}_{\hat K\hat\chi}+\epsilon\hat J^{(1)}_{\hat K\hat\chi}\right)\hat K\rho_{K\chi}\hat\sigma_K\hat\sigma_\chi+\left(\hat J^{(0)}_{\hat K\hat\lambda}+\epsilon\hat J^{(1)}_{\hat K\hat\lambda}\right)\hat K\rho_{K\lambda}\hat\sigma_K\hat\sigma_\lambda$$

$$+\left(\hat J^{(0)}_{\hat E\hat\chi}+\epsilon\hat J^{(1)}_{\hat E\hat\chi}\right)\rho_{E\chi}\hat\sigma_E\hat\sigma_\chi+\left(\hat J^{(0)}_{\hat E\hat\lambda}+\epsilon\hat J^{(1)}_{\hat E\hat\lambda}\right)\rho_{E\lambda}\hat\sigma_E\hat\sigma_\lambda+\left(\hat J^{(0)}_{\hat\chi\hat\lambda}+\epsilon\hat J^{(1)}_{\hat\chi\hat\lambda}\right)\rho_{\chi\lambda}\hat\sigma_\chi\hat\sigma_\lambda=0,$$

where we have used the identity $\partial/\partial\hat k=\hat K\,\partial/\partial\hat K$ (chain rule), substituted the evolution equations for $\hat K$ at subsequent orders ((C3) and (D2.1)) and $\hat E$ at zeroth-order (D2.5), and defined $\hat f^*(J)\equiv\hat f\left(\hat C^*,\hat J\right)$ with $\hat C$ optimally chosen. From (3) and (B6), $\hat f^*(J)$ is

(D3.2) $$\hat f^*=\frac{1}{1-\gamma}\left(\hat\phi'(\hat\imath)\hat J_{\hat K}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat J\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}-\frac{1-\eta}{1-\gamma}\hat\rho\hat J.$$

A Taylor-series expansion for $\hat f^*(J)$ in ϵ (about $\epsilon=0$) gives

(D3.3)

$$\underset{O(\epsilon)}{\hat f^*}=\frac{\left(\hat\phi'(\hat\imath^{(0)})\hat J^{(0)}_{\hat K}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat J^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\gamma}\left(-\frac{\hat\phi''}{\hat\phi'(\hat\imath^{(0)})}\epsilon\hat\imath^{(1)}-\frac{\epsilon\hat J^{(1)}_{\hat K}}{\hat J^{(0)}_{\hat K}}+\frac{\gamma-\eta}{(1-\gamma)(1-\eta)}\frac{\epsilon\hat J^{(1)}}{\hat J^{(0)}}\right)-\frac{1-\eta}{1-\gamma}\hat\rho\epsilon\hat J^{(1)}$$

$$\frac{\hat\phi'(\hat\imath^{(0)})\hat c^{(0)}}{\gamma}\left(-\frac{\hat\phi''}{\hat\phi'\left(\hat\imath^{(0)}\right)}\frac{\hat c^{(0)}}{\gamma-\frac{\hat c^{(0)}\phi''}{\hat\phi'(\hat\imath^{(0)})}}\left(\epsilon\hat K\hat J^{(1)}_{\hat K}+(\eta-\gamma)\epsilon\hat J^{(1)}\right)-\epsilon\hat K\hat J^{(1)}_{\hat K}+\frac{\gamma-\eta}{(1-\gamma)}\epsilon\hat J^{(1)}\right)-\frac{1-\eta}{1-\gamma}\hat\rho\epsilon\hat J^{(1)},$$

where we have substituted for $\hat\imath^{(1)}=-\hat c^{(1)}$ from (D2.4) and used the identity:

(D3.4) $$\frac{\left(\hat\phi'(\hat\imath^{(0)})\hat J^{(0)}_{\hat K}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat J^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\hat K\hat J^{(0)}_{\hat K}}=\hat\phi'(\hat\imath^{(0)})\hat c^{(0)}.$$

Substituting from (D3.2), two of the terms in (D3.1) simplify to

$$\text{(D3.5)}\qquad \underset{O(\epsilon)}{\hat{f}^{*}} + \hat{J}_{\hat{K}}^{(0)} \underbrace{\frac{1}{d\hat{t}} E_t\left[d\hat{K} \right]}_{O(\epsilon)} = -\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}(\eta-\gamma) + (1-\eta)\hat{\rho} \right]\epsilon\hat{J}^{(1)}.$$

Using (D3.5), (D3.1) can be rewritten as a forced equation:

$$\text{(D3.6)}\qquad \begin{aligned} &-\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}\left(\eta-\gamma\right) + \left(1-\eta\right)\hat{\rho} \right]\epsilon\hat{J}^{(1)} + \epsilon\hat{J}_{\hat{t}}^{(1)} + \epsilon\hat{J}_{\hat{K}}^{(1)}\hat{K}\hat{\phi}(\hat{\imath}^{(0)}) \\ &\epsilon\hat{J}_{\hat{E}}^{(1)}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}} - \varphi\hat{E} \right) + \epsilon\hat{J}_{\hat{\chi}}^{(1)}\hat{v}_{\chi}\left(1-\chi\right) + \epsilon\hat{J}_{\hat{\lambda}}^{(1)}\hat{v}_{\lambda}\left(1-\hat{\lambda}\right) \\ &+\frac{1}{2}\epsilon\hat{J}_{\hat{K}\hat{K}}^{(1)}\hat{K}^2\hat{\sigma}_K^2 + \frac{1}{2}\epsilon\hat{J}_{\hat{E}\hat{E}}^{(1)}\hat{E}^2\hat{\sigma}_E^2 + \frac{1}{2}\epsilon\hat{J}_{\hat{\chi}\hat{\chi}}^{(1)}\hat{\sigma}_{\chi}^2 + \frac{1}{2}\epsilon\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(1)}\hat{\sigma}_{\lambda}^2 \\ &\epsilon\hat{J}_{\hat{K}\hat{E}}^{(1)}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \epsilon\hat{J}_{\hat{K}\hat{\chi}}^{(1)}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + \epsilon\hat{J}_{\hat{K}\hat{\lambda}}^{(1)}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} \\ &+\epsilon\hat{J}_{\hat{E}\hat{\chi}}^{(1)}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \epsilon\hat{J}_{\hat{E}\hat{\lambda}}^{(1)}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} + \epsilon\hat{J}_{\hat{\chi}\hat{\lambda}}^{(1)}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda} = -\hat{G}\left(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}\right), \end{aligned}$$

where the forcing is defined as

$$\text{(D3.7)}\qquad \begin{aligned} \hat{G}(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}) \equiv{}& \hat{J}_{\hat{E}}^{(0)}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} Ke^{-\hat{g}^{(0)}\hat{t}} - \varphi\hat{E} \right) + \hat{J}_{\hat{\chi}}^{(0)}\hat{v}_{\chi}(1-\hat{\chi}) + \\ & \hat{J}_{\hat{\lambda}}^{(0)}\hat{v}_{\lambda}(1-\hat{\lambda}) + \frac{1}{2}\hat{J}_{\hat{E}\hat{E}}^{(0)}\hat{\sigma}_E^2 + \frac{1}{2}\hat{J}_{\hat{\chi}\hat{\chi}}^{(0)}\hat{\sigma}_{\chi}^2 + \frac{1}{2}\hat{J}_{\hat{\lambda}\hat{\lambda}}^{(0)}\hat{\sigma}_{\lambda}^2 + \hat{J}_{\hat{K}\hat{E}}^{(0)}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \hat{J}_{\hat{K}\hat{\chi}}^{(0)}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} \\ &+\hat{J}_{\hat{K}\hat{\lambda}}^{(0)}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} + \hat{J}_{\hat{E}\hat{\chi}}^{(0)}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \hat{J}_{\hat{E}\hat{\lambda}}^{(0)}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} + \hat{J}_{\hat{\chi}\hat{\lambda}}^{(0)}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}. \end{aligned}$$

To obtain derivatives of the zeroth-order value function with respect to $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$, we first differentiate with respect to the marginal productivity of capital $\hat{r}_{\text{mpk}}^{(0)}$, which depends on these three variables (via the chain rule of differentiation). From (C6), we obtain:

$$\text{(D3.8)} \qquad \frac{\partial \hat{J}^{(0)}}{\partial \hat{r}^{(0)}_{\text{mpk}}} = \hat{J}^{(0)}\left(-(1-\eta)\frac{\hat{\phi}''(\hat{i}^{(0)})}{\hat{\phi}'(\hat{i}^{(0)})} + \gamma\frac{1-\eta}{\hat{c}^{(0)}}\right)\frac{\partial \hat{i}^{(0)}}{\partial \hat{r}^{(0)}_{\text{mpk}}}.$$

Since the investment rate is implicitly defined, we get from (C7) by implicit differentiation:

$$\text{(D3.9)} \qquad \frac{\partial \hat{i}^{(0)}}{\partial \hat{r}^{(0)}_{\text{mpk}}} = \frac{1}{\gamma - \hat{c}^{(0)}\hat{\phi}''(\hat{i}^{(0)}) / \hat{\phi}'(\hat{i}^{(0)})}.$$

Combining (D3.8) and (D3.9), we obtain

$$\text{(D3.10)} \qquad \frac{\partial \hat{J}^{(0)}}{\partial \hat{r}^{(0)}_{\text{mpk}}} = \hat{J}^{(0)}\frac{1-\eta}{\hat{c}^{(0)}} = \left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}.$$

Using the chain rule of differentiation, we find the individual terms that contribute to the forcing (D3.7):

$$\text{(D3.11)} \qquad \begin{aligned} \underset{O(\epsilon)}{\hat{J}^{(0)}_{\hat{E}}} &= \left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}\,\frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}} \text{ and} \\ \underset{O(\epsilon)}{\hat{J}^{(0)}_{\hat{E}\hat{E}}} &= \left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}\,\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}^2}, \end{aligned}$$

and similarly for derivatives with respect to $\hat{\chi}$ and $\hat{\lambda}$, as well as cross-derivatives. From the zeroth-order solution $\hat{r}^{(0)}_{\text{mpk}} = \alpha\hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)^{1/\alpha}\left((1-\alpha)/\hat{b}\right)^{(1-\alpha)/\alpha} - \hat{\delta}$ and the non-dimensional total factor productivity (B4), we obtain

$$\text{(D3.12a)} \qquad \begin{aligned} \frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}} &= -\epsilon\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\hat{A}^{*}\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}), \\ \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}^2} &= -\epsilon\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\hat{A}^{*}\theta_{ET}\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}-1}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}), \end{aligned}$$

$$\text{(D3.12b)}\quad \begin{aligned}\frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\chi}} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),\\ \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\chi}^2} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),\end{aligned}$$

$$\text{(D3.12c)}\quad \begin{aligned}\frac{\partial \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\lambda}} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda}),\\ \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\lambda}^2} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda}),\end{aligned}$$

$$\text{(D3.12d)}\quad \begin{aligned}\frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}\partial \hat{\chi}} &= -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda}),\\ \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{E}\partial \hat{\lambda}} &= -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),\\ \frac{\partial^2 \hat{r}^{(0)}_{\text{mpk}}}{\partial \hat{\chi}\partial \hat{\lambda}} &= -\epsilon \hat{A}^* \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\Lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),\end{aligned}$$

where have used the following short-hands $\hat{\mathrm{X}}(\hat{\chi}) \equiv (\hat{\chi})^{1+\theta_{\chi T}}$ and $\hat{\Lambda}(\hat{\lambda}) \equiv \left(\hat{\lambda}\right)^{1+\theta_{\lambda}}$, so $\hat{D} = \epsilon \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda})$. Equations (D3.11) and (D3.12) can be substituted into (D3.7):

$$\text{(D3.13)}\quad \begin{aligned}\hat{G}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) = -\epsilon \hat{A}(\hat{E},\hat{\chi})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^*\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\\ \Big[\Big(-(1+\theta_{ET})\hat{\varphi}\hat{\mathrm{X}}\hat{\Lambda} + \hat{v}_{\chi}(1-\hat{\chi})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda} + \hat{v}_{\lambda}(1-\hat{\lambda})\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}} + \frac{1}{2}\hat{\sigma}^2_{\chi}\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}\hat{\Lambda} + \frac{1}{2}\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}^2_{\lambda}\\ +(1-\eta)\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + (1-\eta)\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} + \hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\Big)\hat{K}^{1-\eta}\hat{E}^{1+\theta_{ET}}\\ +(1+\theta_{ET})\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}}e^{-\hat{g}^{(0)}\hat{t}} + \frac{1}{2}\theta_{ET}\left(1+\theta_{ET}\right)\hat{\sigma}^2_E\hat{\mathrm{X}}\hat{\Lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1}\\ +\Big((1-\eta)(1+\theta_{ET})\hat{\mathrm{X}}\hat{\Lambda}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + (1+\theta_{ET})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\\ +(1+\theta_{ET})\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\Big].\end{aligned}$$

Because we are ultimately interested in $\hat{J}_{\hat{E}}^{(1)}$ for the computation of the social cost of carbon, we first differentiate (D3.6) with respect to $\hat{E}$ and seek a solution for $\hat{J}_{\hat{E}}^{(1)}$ of the form $\hat{J}_{\hat{E}}^{(1)} = \psi_1 (1+\theta_{ET}) \hat{\Omega}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t})$, which gives (from (D3.6)):[3]

$$\text{(D3.14)} \qquad \hat{J}_{\hat{E}}^{(1)} = \psi_1 \left(1+\theta_{ET}\right) \hat{\Omega}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) \Rightarrow -\hat{r}_{\Omega}\hat{\Omega} + \frac{1}{d\hat{t}} E_t \left[d\hat{\Omega} \right] = -\hat{\Gamma}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}),$$

where we have introduced the effective discount rate

$$\text{(D3.15)} \qquad \hat{r}_{\Omega} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} + (1-\eta)\left(\hat{\phi}(\hat{i}^{(0)}) - \frac{1}{2}\eta \hat{\sigma}_K^2 \right) + \hat{\varphi},$$

and the coefficient

$$\text{(D3.16)} \qquad \psi_1 \equiv \hat{A}^* \hat{A}(\hat{E}, \hat{\chi}, \lambda)^{\frac{1}{\alpha}-1} \left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1-\alpha}{\alpha}} \left(\hat{c}^{(0)} \right)^{-\gamma \frac{1-\eta}{1-\gamma} - 1} \left(\hat{\phi}'(\hat{i}^{(0)}) \right)^{-\frac{1-\eta}{1-\gamma}}.$$

The scaled forcing is defined by[4]

$$\begin{aligned}
\text{(D3.17)} \quad \Gamma(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t}) \equiv \Big(& (1+\theta_{ET})\hat{\varphi}\hat{X}\hat{\Lambda} - \hat{\nu}_{\chi}(1-\hat{\chi})\hat{X}_{\hat{\chi}}\hat{\Lambda} - \hat{\nu}_{\lambda}(1-\hat{\lambda})\hat{X}\hat{\Lambda}_{\hat{\lambda}} - \frac{1}{2}\hat{\sigma}_{\chi}^2 \hat{X}_{\hat{\chi}\hat{\chi}}\hat{\Lambda} \\
& - \frac{1}{2}\hat{X}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 - (1-\eta)\hat{X}_{\hat{\chi}}\hat{\Lambda}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} - (1-\eta)\hat{X}\hat{\Lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} - \hat{X}_{\hat{\chi}}\hat{\Lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda} \Big) \hat{K}^{1-\eta}\hat{E}^{\theta_{ET}} \\
& - \theta_{ET}\hat{\mu}\left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{X}\hat{\Lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}-1} e^{-\hat{g}^{(0)}\hat{t}} - \frac{1}{2}(\theta_{ET}-1)\theta_{ET}\hat{\sigma}_E^2 \hat{X}\hat{\Lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-2} \\
& - \theta_{ET}\Big((1-\eta)\hat{X}\hat{\Lambda}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \hat{X}_{\hat{\chi}}\hat{\Lambda}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \hat{X}\hat{\Lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} \Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1}.
\end{aligned}$$

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[3] Dimensionally, we have $\Omega = E_0^{\theta_{ET}} \bar{\chi}^{1+\theta_{\chi T}} \bar{\lambda}^{1+\theta_{\lambda}} K_0^{1-\eta} \hat{\Omega}$.

[4] Dimensionally, we have $\Gamma = E_0^{1+\theta_{ET}} \bar{\chi}^{1+\theta_{\chi T}} \bar{\lambda}^{1+\theta_{\lambda}} K_0^{1-\eta} g_0 \hat{\Gamma}$.

Equation (D3.14) has the closed-form solution:

$$\text{(D3.18)} \qquad \hat{\Omega} = E_t\left[\int_{\hat{t}}^{\infty} \hat{\Gamma}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{s}) e^{-\hat{r}_{\Omega}(\hat{s}-\hat{t})} d\hat{s}\right].$$

We can now compute the SCC according to $\hat{P} = -\hat{\mu}\left(\hat{J}_{\hat{E}}^{(0)} + \epsilon \hat{J}_{\hat{E}}^{(1)}\right) \Big/ \phi'(\hat{i}^{(0)}) \hat{J}_{\hat{K}}^{(0)}$:

$$\text{(D3.19)} \qquad \hat{P} = \frac{\hat{\mu}\ \hat{\Theta}\left(\hat{E}, \hat{\chi}, \hat{\lambda}\right)\hat{Y}\Big|_{\hat{P}=0}}{\hat{r}^*}\left(1 - \frac{\hat{\Omega}(\hat{K}, \hat{E}, \hat{\chi}, \hat{\lambda}, \hat{t})}{\hat{E}^{\theta_{ET}}\, \hat{\mathrm{X}}(\hat{\chi})\hat{\Lambda}(\hat{\lambda})\hat{K}^{1-\eta}}\right) \text{ with } \hat{\Theta} \equiv \frac{\hat{D}_{\hat{E}}\left(\hat{E}, \hat{\chi}, \hat{\lambda}\right)}{1 - \hat{D}\left(\hat{E}, \hat{\chi}, \hat{\lambda}\right)},$$

where we introduced $\hat{r}^* \equiv \hat{r}^{(0)} - \hat{g}^{(0)}$. Dimensionally, (D3.19) corresponds to Result 1.

Appendix E: Leading-Order Effects of Uncertainty (For Online Publication)

Assuming that the future atmospheric carbon stock does not inherit any of the uncertainty from new emissions through their dependence on the stochastic capital stock (assumption I), examining only the leading-order effects of uncertainty (assumption II), setting the initial value of the damage ratio but not of the climate sensitivity parameter at its steady-state ($\hat{\mu}_{\lambda}(\hat{t}) = 1$, $\hat{\mu}_{\chi}(\hat{t}) = \hat{\chi}_0$) (assumption III), this appendix derives closed-form solutions for the optimal risk-adjusted SCC based on Result 1. In doing so, we derive Result 3 (and its special case Result 2 for $\theta_{ET} = 0$).

E.1. Carbon stock dynamics

The expected value of the carbon stock is governed by the differential equation (D2.5) with solution

$$\text{(E1.1)} \quad E_t\left[\hat{E}(\hat{s})\right] = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s}) + \hat{\mu}^*\hat{K}(\hat{t})\left[1 - \exp(-\hat{\varphi}\Delta\hat{s})\right] / \hat{\varphi} = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s})\hat{e}(\Delta\hat{s}),$$

with $\hat{\mu}^* \equiv \hat{\mu}\left((1-\alpha)/\hat{b}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}}$, $\Delta\hat{s} \equiv \hat{s} - \hat{t}$ and $\hat{e}(\Delta\hat{s}) = 1 + (\hat{\mu}^* \hat{K}(\hat{t})/\hat{E}(\hat{t}))(\exp(\hat{\varphi}\Delta\hat{s}) - 1)/\hat{\varphi}$. Dimensionally, we define μ^* so that $\mu F^{(0)} = \mu^* K$, where μ does not have units and μ^* has units TtC^{-1}year^{-1}. We can then obtain $\mu^* = \mu\left(A(1-\alpha)/b\right)^{1/\alpha}$ or $\hat{\mu}^* = \left(K_0/g_0 E_0\right)\mu^*$.

E.2. Leading-order forcing

To identify leading-order terms only, we expand in $\Delta\hat{\chi} \equiv \hat{\chi} - \hat{\mu}_\chi$, $\Delta\hat{\lambda} \equiv \hat{\lambda} - \hat{\mu}_\lambda$ and $\Delta E \equiv E - E_t\left[E\right]$ with the corresponding covariance matrix given by (D1.5) (assumption II). We begin by considering terms that only involve capital stock uncertainty, which can be evaluated without further approximation. The probability density function for time $\hat{s}$, but with the expectation operator evaluated at time $\hat{t}$, is

$$f_k = \frac{1}{\sqrt{2\pi\hat{\sigma}_K^2(\hat{s}-\hat{t})}} \exp\left(-\frac{1}{2}\left(\frac{(\hat{k}-\hat{\alpha}_k\hat{s})^2}{\hat{\sigma}_K^2(\hat{s}-\hat{t})}\right)\right), \tag{E2.1}$$

where $\hat{\alpha}_k = \hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2$. Combining with the discount factor in (D3.18) and an additional factor accounting for the decay of the atmospheric carbon stock, we have without further approximation

$$\begin{aligned} &E_t\left[\left(\hat{K}(\hat{s})\right)^{1-\eta}\right]\exp\left(-\left(\hat{r}_\Omega + \theta_{ET}\hat{\varphi}\right)(\hat{s}-\hat{t})\right) = \left(\hat{K}(\hat{t})\right)^{1-\eta}\exp(-\hat{r}^\star\Delta\hat{s}) \quad \text{and} \\ &E_t\left[\left(\hat{K}(\hat{s})\right)^{2-\eta}\right]\exp\left(-(\hat{r}_\Omega + \hat{g}^{(0)} + (\theta_{ET}-1)\hat{\varphi})\Delta\hat{s}\right) = \left(\hat{K}(\hat{t})\right)^{2-\eta}\exp(-\hat{r}^{\star\star}\Delta\hat{s}), \end{aligned} \tag{E2.2}$$

where $\hat{r}^\star \equiv \hat{r}^* + (1+\theta_{ET})\hat{\varphi} = \hat{r}^{(0)} - \hat{g}^{(0)} + (1+\theta_{ET})\hat{\varphi}$ and $\hat{r}^{\star\star} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} - (1-\eta)\hat{\sigma}_K^2 + \theta_{ET}\hat{\varphi} \;\; + \hat{r}^\star - (1-\eta)\hat{\sigma}_K^2 - \hat{\varphi}$. We use alternative star symbols to denote rates corrected for atmospheric carbon stock decay. To leading order, we have for the terms involving the carbon stock:

(E2.3)
$$E_t\left[\hat{E}^{\theta_{ET}}\right]=\left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}}\left[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right]+O(\hat{\Sigma}_E^4),$$
$$E_t\left[\hat{E}^{\theta_{ET}-1}\right]=\left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}-1}\left[1+\frac{1}{2}\left(\theta_{ET}-1\right)\left(\theta_{ET}-2\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right]+O(\hat{\Sigma}_E^4),$$

$$E_t\left[\hat{E}^{\theta_{ET}-2}\right]=\left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}-2}\left[1+\frac{1}{2}\left(\theta_{ET}-2\right)\left(\theta_{ET}-3\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right]+O(\hat{\Sigma}_E^4),$$

where we let the subscript on Σ^2 denote the relevant elements of the covariance matrix $\mathbf{\Sigma}$ (D1.5) and we have ignored any contributions to uncertainty from new emissions through their dependence on uncertain future GDP (assumption I). The following terms also make a contribution to the forcing (D3.17): $\hat{\mathrm{X}}\hat{\Lambda}$, $\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}$, $(\hat{\chi}-\hat{\mu}_\chi)\hat{\mathrm{X}}_{\hat{\chi}}\hat{\Lambda}$, $\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}$, $(\hat{\lambda}-\hat{\mu}_\lambda)\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}}$, $\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}\hat{\Lambda}$ and $\hat{\mathrm{X}}\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}$. Keeping only those terms contributing at leading order, we have

(E2.4a)
$$E_t\left[\hat{\mathrm{X}}(\hat{\chi})\right]=\hat{\mu}_\chi^{1+\theta_{\chi T}}\left[1+\frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_\chi}{\hat{\mu}_\chi}\right)^2\right]+O(\hat{\Sigma}_\chi^4),$$
$$E_t\left[\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\right]=\hat{\mu}_\chi^{\theta_{\chi T}}\left[(\theta_{\chi T}+1)+\frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}(\theta_{\chi T}-1)\left(\frac{\hat{\Sigma}_\chi}{\hat{\mu}_\chi}\right)^2\right]+O(\hat{\Sigma}_\chi^4),$$

(E2.4b)
$$E_t\left[(\hat{\chi}-\hat{\mu}_\chi)\hat{\mathrm{X}}_{\hat{\chi}}\left(\hat{\chi}\right)\right]=\hat{\mu}_\chi^{1+\theta_{\chi T}}\left[(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_\chi}{\hat{\mu}_\chi}\right)^2\right]+O(\hat{\Sigma}_\chi^4),$$
$$E_t\left[\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}(\hat{\chi})\right]=\hat{\mu}_\chi^{\theta_{\chi T}-1}\left[(\theta_{\chi T}+1)\theta_{\chi T}\right]+O(\hat{\Sigma}_\chi^2),$$

(E2.5a)

$$E_t\left[\hat{\Lambda}(\hat{\lambda})\right] = \hat{\mu}_\lambda^{1+\theta_\lambda}\left[1+\frac{1}{2}(\theta_\lambda+1)\theta_\lambda\left(\frac{\hat{\Sigma}_\lambda}{\hat{\mu}_\lambda}\right)^2\right]+O(\hat{\Sigma}_\lambda^4),$$

$$E_t\left[\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda})\right] = \hat{\mu}_\lambda^{\theta_\lambda}\left[(\theta_\lambda+1)+\frac{1}{2}(\theta_\lambda+1)\theta_\lambda(\theta_\lambda-1)\left(\frac{\hat{\Sigma}_\lambda}{\hat{\mu}_\lambda}\right)^2\right]+O(\hat{\Sigma}_\lambda^4),$$

(E2.5b)

$$E_t\left[(\hat{\lambda}-\hat{\mu}_\lambda)\hat{\Lambda}_{\hat{\lambda}}(\hat{\lambda})\right] = \hat{\mu}_\lambda^{1+\theta_\lambda}\left[(\theta_\lambda+1)\theta_\lambda\left(\frac{\hat{\Sigma}_\lambda}{\hat{\mu}_\lambda}\right)^2\right]+O(\hat{\Sigma}_\lambda^4),$$

$$E_t\left[\hat{\Lambda}_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda})\right] = \hat{\mu}_\lambda^{\theta_\lambda-1}\left[(\theta_\lambda+1)\theta_\lambda\right]+O(\hat{\Sigma}_\lambda^2).$$

Using (E2.2)-(E2.5), we now consider the terms in the forcing (D3.17) consecutively and let the subscript indices correspond to the sequence of terms in (D3.17) (left to right). To consider the covariance terms in the forcing (D3.17), we also expand in $\Delta\hat{k} \equiv \hat{k} - \left(\hat{\phi}(\hat{i}^{(0)}) - \hat{\sigma}_K^2/2\right)\hat{t}$ and only consider deviations from the zeroth-order mean consistent with our search for leading-order terms only. The following terms arise:

(E2.6)

$$E_t\left[\Gamma_1\right] = (1+\theta_{ET})\hat{\varphi}\left[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(E_t\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}+\frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\lambda^2\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+\left(1-\eta\right)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}}+\left(1-\eta\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi}$$

$$+\left(1-\eta\right)\left(1+\theta_{\lambda}\right)\frac{\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\hat{\nu}_{\lambda}\Delta\hat{s}\right)}{\hat{\nu}_{\lambda}}+\theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_{E}\hat{\sigma}_{\chi}}{E_{t}\left[\hat{E}\right]\hat{\mu}_{\chi}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_{\chi}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_{\chi}}$$

$$+\theta_{ET}\left(1+\theta_{\lambda}\right)\frac{\rho_{E\lambda}\hat{\sigma}_{E}\hat{\sigma}_{\lambda}}{E_{t}\left[\hat{E}\right]\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_{\lambda}}$$

$$\left.+\left(1+\theta_{\chi T}\right)\left(1+\theta_{\lambda}\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\chi}\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}}\right]E_{t}\left[\hat{K}^{1-\eta}\right]\left(E_{t}\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(E2.7)

$$E_{t}\left[\Gamma_{2}\right]=\hat{\nu}_{\chi}(1+\theta_{\chi T})\left(\theta_{\chi T}\frac{\hat{\sigma}_{\chi}^{2}}{\hat{\mu}_{\chi}^{2}}\frac{1-\exp\left(-2\hat{\nu}_{\chi}\Delta\hat{s}\right)}{2\hat{\nu}_{\chi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\theta_{\lambda}(1+\theta_{\lambda})\frac{\hat{\sigma}_{\chi}^{2}\hat{\sigma}_{\lambda}^{2}}{\hat{\mu}_{\chi}^{2}\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{\nu}_{\chi}\Delta\hat{s}\right)}{2\hat{\nu}_{\chi}}\frac{1-\exp\left(-2\hat{\nu}_{\lambda}\Delta\hat{s}\right)}{2\hat{\nu}_{\lambda}}$$

$$\left(1-\eta\right)\frac{\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}}{\hat{\mu}_{\chi}}\frac{1-\exp\left(-\hat{\nu}_{\chi}\Delta\hat{s}\right)}{\hat{\nu}_{\chi}}+\theta_{ET}\frac{\rho_{E\chi}\hat{\sigma}_{E}\hat{\sigma}_{\chi}}{E_{t}\left[\hat{E}\right]\hat{\mu}_{\chi}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_{\chi}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_{\chi}}$$

$$\left.+\left(1+\theta_{\lambda}\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\chi}\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}}\right)\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}}E_{t}\left[\hat{K}^{1-\eta}\right]\left(E_{t}\left[\hat{E}\right]\right)^{\theta_{ET}}$$

$$+\hat{\nu}_{\chi}(1+\theta_{\chi T})\left(\hat{\chi}(\hat{t})-1\right)e^{-\hat{\nu}_{\chi}\Delta\hat{s}}\left[1+\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_{E}^{2}}{\left(E_{t}\left[\hat{E}\right]\right)^{2}}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}-1\right)\frac{\hat{\sigma}_{\chi}^{2}}{\hat{\mu}_{\chi}^{2}}\frac{1-\exp\left(-2\hat{\nu}_{\chi}\Delta\hat{s}\right)}{2\hat{\nu}_{\chi}}+\frac{1}{2}\theta_{\lambda}\left(1+\theta_{\lambda}\right)\frac{\hat{\sigma}_{\lambda}^{2}}{\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{\nu}_{\lambda}\Delta\hat{s}\right)}{2\hat{\nu}_{\lambda}}$$

$$+\frac{1}{4}\theta_{\chi T}\left(\theta_{\chi T}-1\right)\theta_{\lambda}\left(1+\theta_{\lambda}\right)\frac{\hat{\sigma}_{\lambda}^{2}\hat{\sigma}_{\chi}^{2}}{\hat{\mu}_{\chi}^{2}\hat{\mu}_{\lambda}^{2}}\frac{1-\exp\left(-2\hat{\nu}_{\chi}\Delta\hat{s}\right)}{2\hat{\nu}_{\chi}}\frac{1-\exp\left(-2\hat{\nu}_{\lambda}\Delta\hat{s}\right)}{2\hat{\nu}_{\lambda}}$$

$$+\left(1-\eta\right)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_{K}\hat{\sigma}_{E}}{E_{t}\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}}+\left(1-\eta\right)\theta_{\chi T}\frac{\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}}{\hat{\mu}_{\chi}}\frac{1-\exp\left(-\hat{\nu}_{\chi}\Delta\hat{s}\right)}{\hat{\nu}_{\chi}}$$

$$+(1-\eta)(1+\theta_{\lambda})\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\frac{1-\exp(-\hat{\nu}_{\lambda}\Delta\hat{s})}{\hat{\nu}_{\lambda}}+\theta_{ET}\theta_{\chi T}\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}}{E_t\left[\hat{E}\right]\hat{\mu}_{\chi}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_{\chi}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_{\chi}}$$

$$+\theta_{ET}(1+\theta_{\lambda})\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}}{E_t\left[\hat{E}\right]\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_{\lambda}}$$

$$\left.+\theta_{\chi T}(1+\theta_{\lambda})\frac{\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\chi}\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}}\right]E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}},$$

(E2.8)

$$E_t\left[\Gamma_3\right]=\hat{\nu}_{\lambda}(1+\theta_{\lambda})\left(\theta_{\lambda}\frac{\hat{\sigma}_{\lambda}^2}{\hat{\mu}_{\lambda}^2}\frac{1-\exp(-2\hat{\nu}_{\lambda}\Delta\hat{s})}{2\hat{\nu}_{\lambda}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\theta_{\lambda}\frac{\hat{\sigma}_{\chi}^2\hat{\sigma}_{\lambda}^2}{\hat{\mu}_{\chi}^2\hat{\mu}_{\lambda}^2}\frac{1-\exp(-2\hat{\nu}_{\chi}\Delta\hat{s})}{2\hat{\nu}_{\chi}}\frac{1-\exp(-2\hat{\nu}_{\lambda}\Delta\hat{s})}{2\hat{\nu}_{\lambda}}$$

$$(1-\eta)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}}{\hat{\mu}_{\lambda}}\frac{1-\exp(-\hat{\nu}_{\lambda}\Delta\hat{s})}{\hat{\nu}_{\lambda}}+\theta_{ET}\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}}{E_t\left[\hat{E}\right]\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_{\lambda}}$$

$$\left.(1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}}{\hat{\mu}_{\chi}\hat{\mu}_{\lambda}}\frac{1-\exp\left(-\left(\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}\right)\Delta\hat{s}\right)}{\hat{\nu}_{\chi}+\hat{\nu}_{\lambda}}\right)E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_{\chi}^{1+\theta_{\chi T}}\hat{\mu}_{\lambda}^{1+\theta_{\lambda}}$$

$$+\hat{\nu}_{\lambda}(1+\theta_{\lambda})\left(\hat{\lambda}(\hat{t})-1\right)e^{-\hat{\nu}_{\lambda}\Delta\hat{s}}\left[1+\frac{1}{2}\theta_{ET}(\theta_{ET}-1)\frac{\hat{\sigma}_E^2}{\left(E_t\left[\hat{E}\right]\right)^2}\frac{1-\exp(-2\hat{\varphi}\Delta\hat{s})}{2\hat{\varphi}}\right.$$

$$+\frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\frac{\hat{\sigma}_{\chi}^2}{\hat{\mu}_{\chi}^2}\frac{1-\exp(-2\hat{\nu}_{\chi}\Delta\hat{s})}{2\hat{\nu}_{\chi}}+\frac{1}{2}\theta_{\lambda}(\theta_{\lambda}-1)\frac{\hat{\sigma}_{\lambda}^2}{\hat{\mu}_{\lambda}^2}\frac{1-\exp(-2\hat{\nu}_{\lambda}\Delta\hat{s})}{2\hat{\nu}_{\lambda}}$$

$$+\frac{1}{4}\theta_{\chi T}(1+\theta_{\chi T})\theta_{\lambda}(\theta_{\lambda}-1)\frac{\hat{\sigma}_{\lambda}^2\hat{\sigma}_{\chi}^2}{\hat{\mu}_{\chi}^2\hat{\mu}_{\lambda}^2}\frac{1-\exp(-2\hat{\nu}_{\chi}\Delta\hat{s})}{2\hat{\nu}_{\chi}}\frac{1-\exp(-2\hat{\nu}_{\lambda}\Delta\hat{s})}{2\hat{\nu}_{\lambda}}$$

$$
\begin{aligned}
&+(1-\eta)\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}}+(1-\eta)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{v}_\chi\Delta\hat{s}\right)}{\hat{v}_\chi}\\
&+(1-\eta)\theta_\lambda\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{v}_\lambda\Delta\hat{s}\right)}{\hat{v}_\lambda}+\theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{E_t\left[\hat{E}\right]\hat{\mu}_\chi}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_\chi}\\
&+\theta_{ET}\theta_\lambda\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{E_t\left[\hat{E}\right]\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_\lambda}\\
&\left.+\left(1+\theta_{\chi T}\right)\theta_\lambda\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{v}_\chi+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{v}_\chi+\hat{v}_\lambda}\right]E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},
\end{aligned}
$$

(E2.9)
$$
\begin{aligned}
E_t\left[\Gamma_4\right]=&\left[-\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}-\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda}\right]\\
&E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},
\end{aligned}
$$

(E2.10)
$$
\begin{aligned}
E_t\left[\Gamma_5\right]=&\left[-\frac{1}{2}\left(1+\theta_\lambda\right)\theta_\lambda\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}-\frac{1}{4}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_\lambda\left(1+\theta_\lambda\right)\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{v}_\chi\Delta\hat{s}\right)}{2\hat{v}_\chi}\right]\\
&E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},
\end{aligned}
$$

(E2.11) $$E_t\left[\Gamma_6\right]=-(1-\eta)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(E2.12) $$E_t\left[\Gamma_7\right]=-(1-\eta)\left(1+\theta_\lambda\right)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{\theta_\lambda},$$

(E2.13) $$E_t\left[\Gamma_8\right]=-\left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}\hat{\mu}_\chi^{\theta_{\chi T}}\hat{\mu}_\lambda^{\theta_\lambda},$$

(E2.14) $$E_t\left[\Gamma_9\right]=-\theta_{ET}\hat{\mu}^*\Big[1+\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\frac{\hat{\sigma}_\chi^2}{\hat{\mu}_\chi^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}$$

$$+\frac{1}{2}\theta_\lambda\left(\theta_\lambda+1\right)\frac{\hat{\sigma}_\lambda^2}{\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}+\frac{1}{4}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\theta_\lambda\left(\theta_\lambda+1\right)\frac{\hat{\sigma}_\chi^2\hat{\sigma}_\lambda^2}{\hat{\mu}_\chi^2\hat{\mu}_\lambda^2}\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda}$$

$$+\left(2-\eta\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\mu}_\chi}\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi}+\left(2-\eta\right)\left(1+\theta_\lambda\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\mu}_\lambda}\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda}$$

$$+\left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\mu}_\chi\hat{\mu}_\lambda}\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Bigg]E_t\left[\hat{K}^{2-\eta}\right]e^{-\hat{g}^{(0)}\Delta\hat{s}}\left(E_t\left[\hat{E}\left(\hat{s}\right)\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(E2.15) $$E_t\left[\Gamma_{10}\right]=-\frac{1}{2}\theta_{ET}(\theta_{ET}-1)\hat{\sigma}_E^2E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-2}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(E2.16) $$E_t\left[\Gamma_{11}\right]=-(1-\eta)\theta_{ET}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_EE_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(E2.17) $$E_t\left[\Gamma_{12}\right]=-\theta_{ET}(1+\theta_{\chi T})\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{\theta_{\chi T}}\hat{\mu}_\lambda^{1+\theta_\lambda},$$

(E2.18) $$E_t\left[\Gamma_{13}\right]=-\theta_{ET}(1+\theta_\lambda)\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1}\hat{\mu}_\chi^{1+\theta_{\chi T}}\hat{\mu}_\lambda^{\theta_\lambda},$$

where elements of the covariance matrix have been substituted from (D1.5).

E.3. Leading-order solution

Combining all the leading-order terms in the forcing equation (E2.6)-(E2.18) and substituting into (D3.19), further assuming that $\hat{\mu}_\lambda(\hat{t})=1$ but not $\hat{\mu}_\chi(\hat{t})=1$ (assumption III), we obtain after considerable manipulation including integrating by parts:

(E3.1)

$$\hat{P} = \frac{\hat{\mu}\,\hat{\Theta}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{Y}\Big|_{\hat{P}=0}}{\hat{r}^{\star}}\left(1+\theta_{ET}\hat{\mu}^{*}\frac{\hat{K}}{\hat{E}}\frac{1}{\hat{r}^{\star\star}}\Upsilon_{\theta_{ET}\neq 0}+(1+\theta_{\chi T})\frac{\hat{\nu}_{\chi}}{\hat{r}^{\star}}\frac{1-\hat{\chi}}{\hat{\chi}}\Upsilon_{\chi_0\neq\bar{\chi}}+\Delta_{EE}+\Delta_{\chi\chi}+\Delta_{\lambda\lambda}+\Delta_{CK}+\Delta_{CC}+\Delta_{\chi\times\lambda}\right),$$

where the adjustments for uncertainty are

(E3.2)

$$\Delta_{EE}\equiv\frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\hat{E}^2}\frac{1}{\hat{r}^{\star}-2\hat{\varphi}}\Upsilon_{EE},\quad \Delta_{\chi\chi}\equiv\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\frac{\hat{\sigma}_{\chi}^2}{\hat{\mu}_{\chi}^2}\frac{1}{\hat{r}^{\star}+2\hat{\nu}_{\chi}}\Upsilon_{\chi\chi},$$

$$\Delta_{\lambda\lambda}\equiv\frac{1}{2}\theta_{\lambda}\left(1+\theta_{\lambda}\right)\frac{\hat{\sigma}_{\lambda}^2}{\hat{r}^{\star}+2\hat{\nu}_{\lambda}}\Upsilon_{\lambda\lambda},\ \Delta_{\chi\times\lambda}\equiv\frac{1}{16}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\theta_{\lambda}\left(1+\theta_{\lambda}\right)\frac{\hat{\sigma}_{\lambda}^2\hat{\sigma}_{\chi}^2}{\hat{\nu}_{\chi}\hat{\nu}_{\lambda}\hat{\mu}_{\chi}^2}\frac{\hat{r}^{\star}}{\hat{r}^{\star}-(1+\theta_{ET})\hat{\varphi}}$$
$$\times\left(\frac{(1+\theta_{ET})\hat{\varphi}+2\hat{\nu}_{\chi}}{\hat{r}^{\star}+2\hat{\nu}_{\chi}}+\frac{(1+\theta_{ET})\hat{\varphi}+2\hat{\nu}_{\lambda}}{\hat{r}^{\star}+2\hat{\nu}_{\lambda}}-\frac{(1+\theta_{ET})\hat{\varphi}+2(\hat{\nu}_{\chi}+\hat{\nu}_{\lambda})}{\hat{r}^{\star}+2\hat{\nu}_{\lambda}+2\hat{\nu}_{\chi}}-\frac{(1+\theta_{ET})\hat{\varphi}}{\hat{r}^{\star}}\right)\Upsilon_{\chi\times\lambda},$$

$$\Delta_{CK}\equiv-(\eta-1)\hat{\sigma}_K\left(\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_E}{\hat{E}\left(\hat{r}^{\star}-\hat{\varphi}\right)}\Upsilon_{KE}+\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}}{\hat{r}^{\star}+\hat{\nu}_{\chi}}\frac{\hat{\sigma}_{\chi}}{\hat{\mu}_{\chi}}\Upsilon_{K\chi}+\left(1+\theta_{\lambda}\right)\frac{\rho_{K\lambda}\hat{\sigma}_{\lambda}}{\hat{r}^{\star}+\hat{\nu}_{\lambda}}\Upsilon_{K\lambda}\right),$$

$$\Delta_{CC}\equiv\theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}}{\hat{E}\hat{\mu}_{\chi}}\frac{\hat{r}^{\star}}{\hat{r}^{\star}-\hat{\varphi}}\frac{1}{\hat{r}^{\star}+\hat{\nu}_{\chi}}\Upsilon_{E\chi}$$
$$+\left(1+\theta_{\lambda}\right)\left(\left(1+\theta_{\chi T}\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}}{\hat{r}^{\star}+\hat{\nu}_{\lambda}+\hat{\nu}_{\lambda}}\frac{1}{\hat{\mu}_{\chi}}\Upsilon_{\chi\lambda}+\theta_{ET}\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}}{\hat{E}}\frac{\hat{r}^{\star}}{\hat{r}^{\star}-\hat{\varphi}}\frac{1}{\hat{r}^{\star}+\hat{\nu}_{\lambda}}\Upsilon_{E\lambda}\right).$$

We distinguish two types of correction factors, for $\theta_{ET}\neq 0$ and for $\chi_0\neq\bar{\chi}$, which can be linearly combined, for example: $\Upsilon_{\chi\chi}\equiv 1+\Upsilon_{\chi\chi,\theta_{ET}\neq 0}+\Upsilon_{\chi\chi,\chi_0\neq\bar{\chi}}$. The combined correction factors are equal to unity when $\theta_{ET}\neq 0$ and $\chi_0\neq\bar{\chi}$ (e.g. $\Upsilon_{\chi\chi}\equiv 1$). We give the correction factors in terms of dimensional quantities below (using the definitions summarized in (B1) and (B3) and $\hat{\mu}^{*}\hat{K}=\mu F^{(0)}/(g_0E_0)$), so that they can be used directly in Result 3 given dimensionally in Appendix A. The correction factors for $\theta_{ET}\neq 0$ are given by

(E3.4)

$$\Upsilon_{\theta_{ET}\neq 0} = \frac{r^{\star\star}}{1-(1+\theta_{ET})\varphi/r^{\star}} \int_0^{\infty} \left(e^{-r^{\star\star}\Delta s} - \frac{(1+\theta_{ET})\varphi}{r^{\star}} e^{-(r^{\star}-\varphi)\Delta s} \right) \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}} d\Delta s,$$

$$\Upsilon_{Ki,\theta_{ET}\neq 0} = \theta_{ET}\mu^{*} \frac{K(t)}{E(t)} \frac{1}{1-(1+\theta_{ET})\varphi/r^{\star}} \left[\left(1+\frac{(1+\theta_{ET})\varphi}{\nu_i} \right) \int_0^{\infty} e^{-(\nu_i + r^{\star}-\varphi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-\mathrm{I}_{\chi}(i)} d\Delta s \right.$$

$$-(1+\theta_{ET}) \frac{\varphi}{r^{\star}} \frac{r^{\star}+\nu_i}{\nu_i} \int_0^{\infty} e^{-(r^{\star}-\varphi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-\mathrm{I}_{\chi}(i)} d\Delta s +$$

$$\left. \frac{2-\eta}{1-\eta} \frac{\nu_i + r^{\star}}{\nu_i} \int_0^{\infty} \left(e^{-r^{\star\star}\Delta \hat{s}} - e^{-(r^{\star\star}+\nu_i)\Delta s} \right) \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-\mathrm{I}_{\chi}(i)} d\Delta s \right] \text{ for } i=\chi,\lambda,$$

$$\Upsilon_{ij,\theta_{ET}\neq 0} = 1+\theta_{ET}\mu^{*} \frac{K}{E} \frac{1}{1-(1+\theta_{ET})\varphi/r^{\star}} \left[\left(1+\frac{(1+\theta_{ET})\varphi}{\nu_i+\nu_j} \right) \int_0^{\infty} e^{-(r^{\star}+\nu_i+\nu_j-\varphi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}-1} \right.$$

$$\times \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-\mathrm{I}_{\chi}(i)-\mathrm{I}_{\chi}(j)} d\Delta s - \frac{(1+\theta_{ET})\varphi}{\nu_i+\nu_j} \frac{r^{\star}+\nu_i+\nu_j}{r^{\star}} \int_0^{\infty} e^{-(r^{\star}-\varphi)\Delta s} \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-\mathrm{I}_{\chi}(i)-\mathrm{I}_{\chi}(j)} d\Delta s$$

$$\left. + \frac{r^{\star}+\nu_i+\nu_j}{\nu_i+\nu_j} \int_0^{\infty} \left(e^{-r^{\star\star}\Delta s} - e^{-(r^{\star\star}+\nu_i+\nu_j)\Delta \hat{s}} \right) \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}-\mathrm{I}_{\chi}(i)-\mathrm{I}_{\chi}(j)} d\Delta s \right] \text{ for } i,j=\chi,\lambda,$$

$$\Upsilon_{EE,\theta_{ET}\neq 0} = -\frac{\left(r^{\star}-2\varphi\right)}{2\varphi} \frac{\mu^{*}\frac{K}{E}\left(\theta_{ET}-2\right)}{1-(1+\theta_{ET})\varphi/r^{\star}} \left[\int_0^{\infty} \left(\frac{(\theta_{ET}-1)\varphi}{r^{\star}-2\varphi} e^{-(r^{\star}-3\varphi)\Delta s} - \frac{(1+\theta_{ET})\varphi}{r^{\star}} e^{-(r^{\star}-\varphi)\Delta s} \right. \right.$$

$$\left. -e^{-(r^{\star\star}-2\varphi)\Delta s} + e^{-r^{\star\star}\Delta s} \right) \left(e(\Delta s)\right)^{\theta_{ET}-3} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{1+\theta_{\chi T}} ds,$$

$$\Upsilon_{\chi\times\lambda,\theta_{ET}\neq 0} = 1+ \frac{\theta_{ET}\mu^{*} K(t)/E(t)}{\frac{(1+\theta_{ET})\varphi}{r^{\star}} - \frac{(1+\theta_{ET})\varphi+2\nu_{\chi}}{r^{\star}+2\nu_{\chi}} - \frac{(1+\theta_{ET})\varphi+2\nu_{\lambda}}{r^{\star}+2\nu_{\lambda}} + \frac{(1+\theta_{ET})\varphi+2(\nu_{\chi}+\nu_{\lambda})}{r^{\star}+2(\nu_{\lambda}+\nu_{\chi})}}$$

$$\times \int_0^{\infty} \left(\frac{(1+\theta_{ET})\varphi}{r^{\star}} e^{-(r^{\star}-\varphi)\Delta s} - e^{-r^{\star\star}\Delta s} - \frac{(1+\theta_{ET})\varphi+2\nu_{\chi}}{r^{\star}+2\nu_{\chi}} e^{-(r^{\star}+2\nu_{\chi}-\varphi)\Delta s} \right.$$

$$+e^{-(r^{\star\star}+2\nu_{\chi})\Delta s} - \frac{(1+\theta_{ET})\varphi+2\nu_{\lambda}}{r^{\star}+2\nu_{\lambda}} e^{-(r^{\star}+2\nu_{\lambda}-\varphi)\Delta s} + e^{-(r^{\star\star}+2\nu_{\lambda})\Delta s}$$

$$\left. \frac{(1+\theta_{ET})\varphi+2(\nu_{\chi}+\nu_{\lambda})}{r^{\star}+2\nu_{\lambda}+2\nu_{\chi}} e^{-(r^{\star}+2\nu_{\lambda}+2\nu_{\chi}-\varphi)\Delta s} - e^{-(r^{\star\star}+2\nu_{\chi}+2\nu_{\lambda})\Delta s} \right) \times \left(e(\Delta s)\right)^{\theta_{ET}-1} \left(\frac{\mu_{\chi}(\Delta s)}{\mu_{\chi}(t)} \right)^{\theta_{\chi T}-1} d\Delta s,$$

where $\mathrm{I}_\chi(i)=1$ for $i=\chi$ and $\mathrm{I}_\chi(i)=0$ for $i\neq\chi$ (cf. indicator function), the function that takes into account future changes to the mean carbon stock $e(\Delta s)=1+(\mu^* K(t)/E(t))(\exp(\varphi\Delta s)-1)/\varphi$, and the time-varying mean climate sensitivity $\mu_\chi(\Delta s)=\mu_\chi(t)\exp(-\nu_\chi\Delta s)+\bar{\mu}_\chi(1-\exp(-\nu_\chi\Delta s))$. The correction factors for $\chi_0\neq\bar{\chi}$ are:

(E3.5)

$$\Upsilon_{\chi_0\neq\bar{\chi}}=r^\star\int_0^\infty\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}}e^{-(r^\star+\nu_\chi)\Delta s}d\Delta s,$$

$$\Upsilon_{ij,\chi_0\neq\bar{\chi}}=\left(1+\theta_{\chi T}-\mathrm{I}_\chi(i)-\mathrm{I}_\chi(j)\right)\nu_\chi\frac{\bar{\chi}-\mu_\chi(t)}{\mu_\chi(t)}\frac{r^\star+\nu_i+\nu_j}{\nu_i+\nu_j}\int_0^\infty\left(e^{-(r^\star+\nu_\chi)\Delta s}-\frac{r^\star}{r^\star+\nu_i+\nu_j}e^{-(r^\star+\nu_\chi+\nu_i+\nu_j)\Delta s}\right)$$
$$\times\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-\mathrm{I}_\chi(i)-\mathrm{I}_\chi(j)}d\Delta\hat{s},$$

$$\Upsilon_{Ki,\chi_0\neq\bar{\chi}}=\left(1+\theta_{\chi T}-\mathrm{I}_\chi(i)\right)\nu_\chi\frac{\bar{\chi}-\mu_\chi(t)}{\mu_\chi(t)}\frac{r^\star+\nu_i}{\nu_i}\int_0^\infty\left(e^{-(r^\star+\nu_\chi)\Delta s}-\frac{r^\star}{r^\star+\nu_\chi}e^{-(\nu_\chi+\nu_i)\Delta s}\right)\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-\mathrm{I}_\chi(i)}d\Delta s,$$

$$\Upsilon_{EE,\chi_0\neq\bar{\chi}}=\left(1+\theta_{\chi T}\right)\nu_\chi\frac{\left(\bar{\chi}-\mu_\chi(t)\right)}{\mu_\chi(t)}\int_0^\infty\left(\frac{r^\star}{2\varphi}e^{-(r^\star+\nu_\chi-2\varphi)\Delta s}-\frac{r^\star-2\varphi}{2\varphi}e^{-(r^\star+\nu_\chi)\Delta s}\right)\left(e(\Delta s)\right)^{\theta_{ET}-2}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}}d\Delta s,$$

$$\Upsilon_{\chi\times\lambda,\chi_0\neq\bar{\chi}}=\frac{\left(1-\frac{(1+\theta_{ET})\varphi}{r^\star}\right)\left(\theta_{\chi T}-1\right)\nu_\chi\frac{\bar{\chi}-\mu_\chi(t)}{\mu_\chi(t)}}{-\frac{(1+\theta_{ET})\varphi}{r^\star}+\frac{(1+\theta_{ET})\varphi+2\nu_\chi}{r^\star+2\nu_\chi}+\frac{(1+\theta_{ET})\varphi+2\nu_\lambda}{r^\star+2\nu_\lambda}-\frac{(1+\theta_{ET})\varphi+2(\nu_\chi+\nu_\lambda)}{r^\star+2(\nu_\lambda+\nu_\chi)}}$$
$$\int_0^\infty\left(1-\frac{r^\star}{r^\star+2\nu_\chi}e^{-2\nu_\chi\Delta s}-\frac{r^\star}{r^\star+2\nu_\lambda}e^{-2\nu_\lambda\Delta s}+\frac{r^\star}{r^\star+2\nu_\lambda+2\nu_\chi}e^{-2(\nu_\lambda+\nu_\chi)\Delta s}\right)$$
$$\times e^{-(r^\star+\nu_\chi)\Delta s}\left(e(\Delta s)\right)^{\theta_{ET}}\left(\frac{\mu_\chi(\Delta s)}{\mu_\chi(t)}\right)^{\theta_{\chi T}-2}d\Delta s.$$

Equation (E3.1) together with (E3.2)-(E3.5) gives the optimal SCC. We do not explicitly give the correction factors for the correlation terms involving carbon stock uncertainty.

Appendix F: Calibration (For Online Publication)

F.1. Asset returns, risk aversion and intertemporal substitution

We follow the calibration of Pindyck and Wang (2013), but ignore the effect of catastrophic shocks.[5,6] Using monthly asset data from the S&P 500 for the period 1947-2008, we obtain an annual return on assets (capital gains plus dividends) of $r^{(0)} =$ 7.2%/year with annual volatility of $\sigma_K = 12\%$. For a return on safe assets of 0.80%/year based on the annualized monthly return on 3-months T-bills, we obtain a risk premium of $\Delta r^{(0)} \equiv r^{(0)} - r_{\mathrm{rf}}^{(0)} = 6.4\%$/year and calibrate the coefficient of relative risk aversion as $\eta =$ 4.3 (cf. $\Delta r^{(0)} = \eta \sigma_K^2$). Taking the growth rate to be equal to the historical growth rate of $g^{(0)} =$ 2.0%/year, the equation $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - (1+\gamma)\eta \sigma_K^2 / 2$ (cf. (B9)) defines the combinations of ρ and γ that are consistent with historical asset returns. Setting the coefficient of elasticity of intertemporal substitution $\mathrm{EIS} = 2/3$, we obtain $\gamma = \mathrm{EIS}^{-1} = 1.5$ and thus a rate of time preference is $\rho = 5.8\%$/year.

F.2. Productivity, fossil fuel, adjustment costs and the depreciation rate

To calibrate total factor productivity, we consider the production function in the absence of climate damage that can be obtained by setting $P = 0$ (i.e. at zeroth order), namely $Y^{(0)} = A^* K$ with $A^* = A^{1/\alpha} \left((1-\alpha)/b\right)^{(1-\alpha)/\alpha}$ (cf. (B9)). Pindyck and Wang (2013) use empirical estimates of the physical, human and intangible capital stocks and find

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[5] Pindyck and Wang (2013) use Poisson shocks to capture small risks of large disasters (cf. Barro, 2016) and thus match skewness and kurtosis of asset returns. These shocks are responsible for approximately 1%-point of the risk premium.

[6] The alternative is to calibrate our AK model to the observed volatility of consumption or output (cf. Gollier, 2012), which are generally much less volatile than capital (asset returns). Because the volatilities of capital, consumption and output are equal to the volatility of capital in an AK model, this alternative calibration gives a much lower volatility and, consequently, a higher coefficient of relative risk aversion to match the equity premium (see also the discussion in Pindyck and Wang, 2013). Historical data for the growth rate of world GDP for 1961-2015 imply a volatility of σ_K = 1.5%/year$^{1/2}$ and thus a much higher value of risk aversion of $\eta = 2.8 \times 10^2$ for an equity premium of 6.4%/year. Kocherlota (1996) obtains σ_K = 3.6%/year$^{1/2}$ from US annual consumption growth during 1889-1978, which gives $\eta = 49$. We use σ_K = 1.5%/year$^{1/2}$, but not the corresponding high values of risk aversion.

$A^* = 0.113$ /year, which we adopt. Based on emissions of $F_0^{(0)} = 9.1$ GtC/year in 2015, energy costs making up a share $1-\alpha = 4.3\%$ of world GDP at PPP in 2015 of \$116 trillion/year, we estimate the fossil fuel cost to be $b = Y_0^{(0)}(1-\alpha)/F_0^{(0)} = \$5.4 \times 10^2$ /tC.[7] The gross marginal productivity of capital is thus $Y_K^{(0)}\big|_{t=0} = \alpha A^* = 0.11$/year.[8] Using Pindyck and Wang's (2013) consumption-investment ratio $c^{(0)}/i^{(0)} = 2.84$ and the identity $\alpha A^* = c^{(0)} + i^{(0)}$, we obtain initial values of $c^{(0)} =$ 8.0%/year and $i^{(0)} = 2.8\%$/year. Using $q^{(0)} = c^{(0)}/(r^{(0)} - g^{(0)}) =$ 1.5 and $q^{(0)} = (1-\omega i^{(0)})^{-1}$, we get the adjustment-cost parameter $\omega = 12.5$ year. Finally, we find the depreciation rate that is consistent with the assumed rate of economic growth: $\delta = i^{(0)} - \omega(i^{(0)})^2/2 - g^{(0)} = 0.33\%$/year.

F.3. Atmospheric carbon stock and uncertainty

Here we calibrate our carbon stock model (6) to the Law Dome Ice Core 2000-year data set and historical emissions. The first column of Figure F1 shows maximum-likelihood estimates, from which it is evident that estimates displaying a certain linear relationship between φ and μ are of comparable likelihood.[9]

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[7] We estimate the share of energy costs from data for energy use and energy costs from BP Statistical Review of World Energy 2017. Data for emissions are obtained from the same source available online at https://www.bp.com/en/global/corporate/energy-economics/statistical-review-of-world-energy.html. Our estimate of energy costs as a percentage of GDP is in good agreement with data from the U.S. Energy Information Administration available online at https://www.eia.gov/totalenergy/data/annual/showtext.php?t=ptb0105.

[8] This is in line with Caselli and Feyrer (2007), who estimate annual marginal products of capital of 8.5% for rich countries and 6.9% for poor countries, and an observed annual risk premium of 5-7%. They use a depreciation rate of 6.0% to calculate the capital stock from investment, include the share of reproducible capital rather than the share of total capital, account for differences in prices between capital and consumption goods and correct for inflation.

[9] Annual data from the Law Dome firn and ice core records and the Cape Grim record are available online at ftp://ftp.ncdc.noaa.gov/pub/data/paleo/icecore/antarctica/law/law2006.txt.This data is based on spline fits to different dataset with different spline windows across time reflecting changes in the temporal resolution of the data. The discrete nature of the fitted data is evident for the early years. Annual carbon emissions from fossil fuel consumption and cement production are available online at http://cdiac.ornl.gov/trends/emis/tre_glob_2013.html.

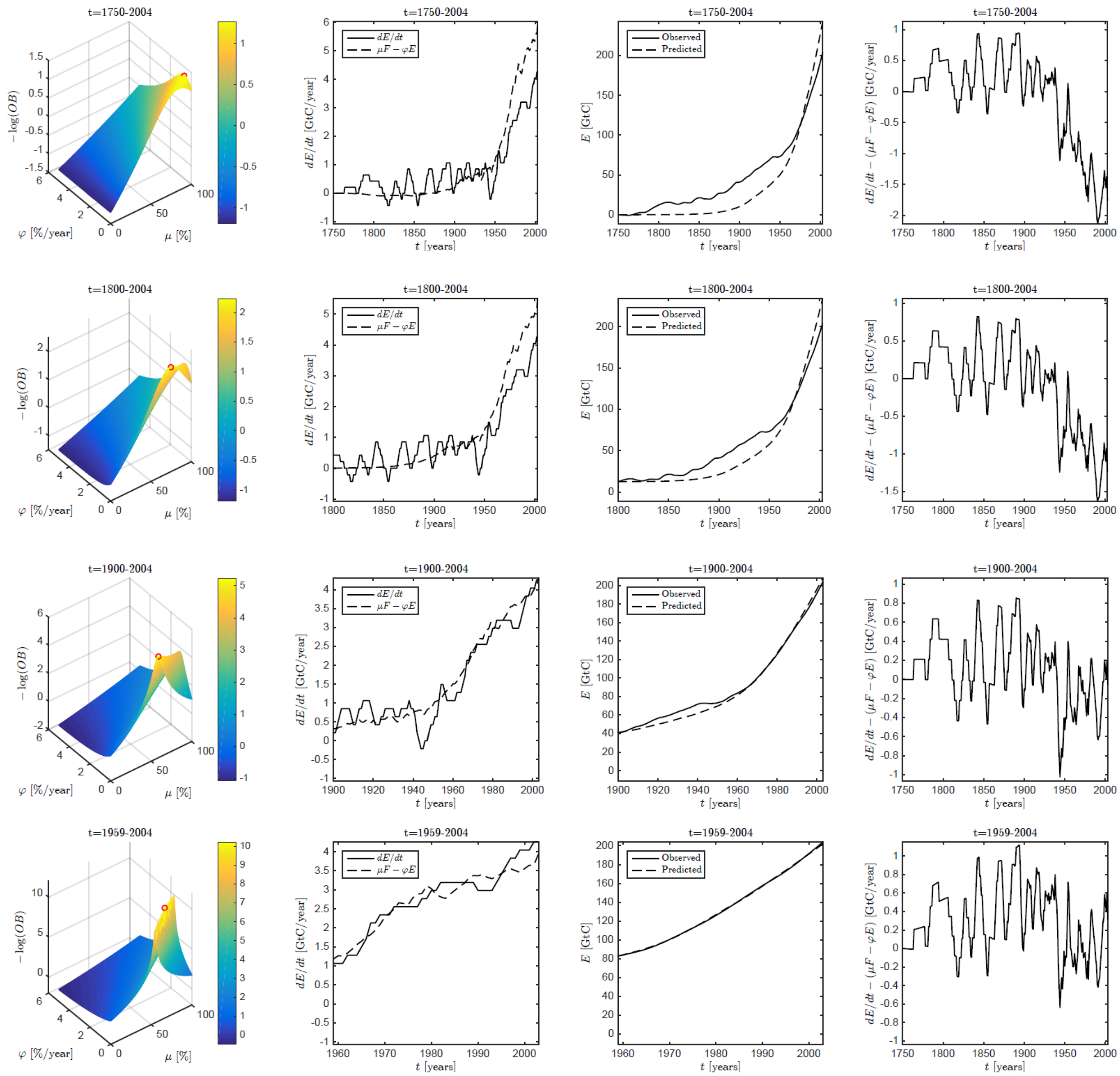

FIGURE F1. HISTORICAL ATMOSPHERIC CARBON STOCK CALIBRATION

These loci of maximum likelihood are shown separately in Figure F2, with the overall maximum denoted by a red circle and corresponding values given in Table F1. The remaining columns in Figure F1 show the predicted and observed rate of change of the atmospheric carbon stock (second column), the predicted and observed atmospheric carbon stock (third column) and the remaining variability (fourth column).[10]

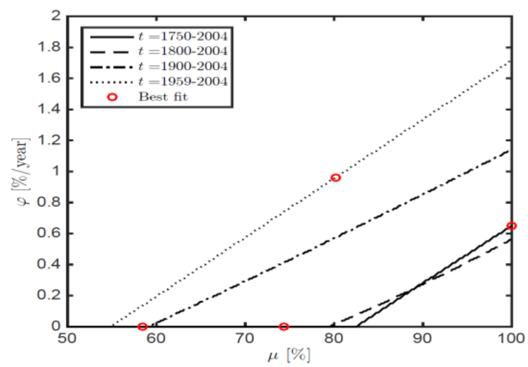

FIGURE F.2. LOCI OF BEST FIT OF TMOSPHERIC STOCK CALIBRATION

Figure F1 indicates that our model (6) captures the observed historical variations in the atmospheric carbon stock reasonably well, including for very long time periods. The final column in Table F1 shows volatility as percentage of the initial carbon stock, from which we note that the stochastic carbon stock correction to the optimal SCC will be tiny if estimated from historical emissions.

TABLE F.1 – ATMOSPHERIC CARBON STOCK CALIBRATION

Time	μ	φ [%/year]	σ_E [GtC/year$^{1/2}$]	σ_E/S_0 [%/year$^{1/2}$]	σ_E/E_0 [%/year$^{1/2}$]
1750-2004	1.0	0.66	0.31	0.036	0.12
1800-2004	0.75	0.00	0.26	0.029	0.10
1900-2004	0.59	0.00	0.21	0.025	0.081
1959-2004	0.79	0.91	0.23	0.027	0.089

[10] By setting $\varphi = 0$, we can estimate the fraction μ of emissions that stays in the atmosphere forever, whilst the remainder is instantaneously absorbed by the oceans and other carbon sinks. Calibrating to this data, we find μ = 0.68, 0.64, 0.56 and 0.43 for the periods 1750-2004, 1800-2004, 1900-2004 and 1959-2004, respectively. Performing a similar analysis, Le Quéré et al. (2009) find that, between 1959 and 2008, 43% of each year's CO_2 emissions remained in the atmosphere on average.

F.4. Calibration of the curvature of the temperature-carbon stock relationship

The curvature of our temperature relationship (7), $T(E,\chi)=\chi^{1+\theta_\chi}(E/S_{\mathrm{PI}})^{1+\theta_E}$, is constant: $\theta_E \equiv ET_{EE}(E,\chi)/T_E(E,\chi)$. The radiative law for global mean temperature, $T \propto \ln(S/S_{\mathrm{PI}})/\ln(2) \propto \ln\left((E+S_{\mathrm{PI}})/S_{\mathrm{PI}}\right)/\ln(2)$ (Arrhenius, 1854)[11] gives $\theta_E = -E/(E+S_{PI})$. If we evaluate the temperature relationship at double (quadruple) the pre-industrial stock $E=S_{\mathrm{PI}}$ ($E=3S_{\mathrm{PI}}$), we obtain $\theta_E = -0.50$ (or $\theta_E = -0.75$).[12] For $S_0 =$ 0.854 TtC or $E_0 = 0.258$ TtC (given $S_{\mathrm{PI}} = 0.596$ TtC), we obtain $\theta_E = -0.30$. We set $\theta_E = -0.36$ for our base case calibration.

F.5. Climate sensitivity and uncertainty

If climate sensitivity parameter χ is normally distributed with mean μ_χ and standard deviation Σ_χ, the climate sensitivity $T_2 = \chi^{1+\theta_\chi}$ is described by the probability density function

$$\text{(F1)}\quad f_{T_2}\left(T_2;\mu_\chi,\Sigma_\chi,\theta_\chi\right)=\frac{1}{\sqrt{2\pi}\Sigma_\chi\left(1+\theta_\chi\right)}T_2^{-\frac{\theta_\chi}{1+\theta_\chi}}\exp\left(-\left(T_2^{\frac{1}{1+\theta_\chi}}-\mu_\chi\right)^2/2\Sigma_\chi^2\right).$$

Unlike for fat-tailed distributions, which typically have algebraically-decaying tails, all moments of (F1) are defined due to its exponential tail (for $\theta_\chi \geq -1$), so that Weitzman's (2009) 'dismal theorem' does not apply. Positive values of θ_χ result in a positively-skewed

[11] In their table 6.2, IPCC (2001) propose a logarithmic relationship for radiative forcing as a function CO_2, also given in IPCC (1990, chapter 2, where original sources are cited), among two other non-logarithmic, but generally concave parametrizations. IPCC (1990, chapter 2, page 51) note that for "low/moderate/high concentrations, the form is well approximated by a linear/square-root/logarithmic dependence", where the limit of validity of the logarithmic calibration is said to be 1000 ppm. For other greenhouse gases alternative parametrizations are proposed: a square-root dependence for methane and a linear dependence for halocarbons.

[12] Whereas the normalized curvature of Arrhenius's (1854) logarithmic radiative law with respect to the atmospheric carbon stock *S*, namely $ST_{SS}(S)/T_S(S)$ is constant and equal to -1, this limit is only reached for large carbon stock in our case, in which $\theta_E \equiv ET_{EE}(E,\chi)/T_E(E,\chi)$.

(non-Gaussian) distribution with more probability mass at high temperatures. Leading-order central moments of climate sensitivity can be obtained from performing Taylor-series expansions of $T_2 = \chi^{1+\theta_\chi}$ about its mean μ_χ:

(F2a) $$E\left[T_2\right] = \mu_\chi^{1+\theta_\chi}\left(1+\frac{1}{2}\theta_\chi(1+\theta_\chi)(\Sigma_\chi / \mu_\chi)^2\right) + O(\Sigma_\chi^4),$$

(F2b) $$\mathrm{var}\left[T_2\right] \equiv E\left[\left(T_2 - E\left[T_2\right]\right)^2\right] = (1+\theta_\chi)^2 \mu_\chi^{2(1+\theta_\chi)}(\Sigma_\chi / \mu_\chi)^2 + O(\Sigma_\chi^4),$$

(F2c) $$\mathrm{skew}\left[T_2\right] \equiv E\left[\left(T_2 - E\left[T_2\right]\right)^3\right] = 3\theta_\chi(1+\theta_\chi)^3 \mu_\chi^{3(1+\theta_\chi)}(\Sigma_\chi / \mu_\chi)^4 + O(\Sigma_\chi^6),$$

(F2d) $$\mathrm{skew}^*\left[T_2\right] \equiv \mathrm{skew}\left[T_2\right] / \left(\mathrm{var}\left[T_2\right]\right)^{3/2} = 3\theta_\chi(\Sigma_\chi / \mu_\chi) + O(\Sigma_\chi^3).$$

Our calibration of the distribution of the climate sensitivity are based on a wide range of distributions reported and used by the IPCC (2014, AR5) (see Figure 2 in section IV.B of the paper). We would like to add that the 5-95% confidence ranges of 0.7-2.0ºC/TtC reported by Gillett et al. (2013), 1.0-2.1ºC/TtC reported by Matthews et al. (2009) and 1.4-2.5ºC/TtC reported by Allen et al. (2009) are in line with these distributions of the TCR and our calibration.

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Appendix G: Accuracy of Results 3 (For Online Publication)

Result 1 is evaluated numerically by discretization in time before evaluating the expectation operator numerically exactly and summing up the discounted contributions of every time step. Whereas the stochastic processes for χ and λ are autonomous, the stochastic process for K remains autonomous in Result 1, and all three have (independent) probability distributions available in closed form, the probability distribution of E at any time period in the future must combine all uncertain emissions (proportional to K) before that time. As the time integral of a Geometric Brownian motion does not have a closed-form solution, we update the probability distribution function of E every time step with the stochastic emissions and the decay in that period according to the differential equation for E and project on a fixed grid for E to enable transfer of the probability density function between time periods. Of course, the validity of Result 1 itself still relies on the parameter ϵ being small. Consistent with our perturbation scheme, all our optimal risk-adjusted carbon prices in Results 1 and 3 are evaluated along the business-as-usual path for which $P = 0$. We assess the accuracy of Result 3 for a number of the calibrations examined in section V. By choosing the grid size to be sufficiently small and the grid to be sufficiently large in each case, we ensure that discretization errors associated with Result 1 are negligible. Two factors determine the accuracy of using Result 3 instead of Result 1.

TABLE G1 – ACCURACY OF RESULT 3 COMPARED TO RESULT 1

Impatience ρ [/year]	5.8%	1.5%	0.1%	0.1%	0.1%
Economic volatility σ_K [/year$^{1/2}$]	12%	12%	1.5%	1.5%	1.5%
Damages	Proportional	Proportional	Proportional	Convex	Highly convex (AS12)
Total error in risk-adjusted SCC	-0.02%	-2.0%	0.73%	1.9%	-1.3%

First, in Result 3 we ignore any uncertainty in the atmospheric carbon stock that arises because of the uncertain nature of future economic growth and thus of future emissions. For our base case calibration with *proportional* damages ($\theta_{ET} = 0$), the stochastic nature of E does not lead to a change in the SCC. Second, in Result 3 we only consider leading-order terms in the climate sensitivity uncertainty.

We can confirm from Table G1 that the combined effect of these two errors is sufficiently small to be ignored for all practical purposes. As expected, it is larger for low discount rates, higher economic volatility and *convex* damages.