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Philippe Choné, Laurent Linnemer

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The quasilinear quadratic utility model: an overview

Abstract

The quasi-linear quadratic utility model is widely used in economics. The knowledge of its exact origin is less widespread. A first contribution of the paper is to explain the genesis of this model. Next, we review the main properties of the general model, mainly following the previous literature. Finally, it is shown that all the tractable versions of the model used in practice are (almost) identical and have a mean variance structure. We provide ready-to-use formulae for this symmetric model.

JEL-Codes: L130.

Philippe Choné
CREST, ENSAE
Institut Polytechnique de Paris
France
Philippe.Chone@ensae.fr

Laurent Linnemer
CREST, ENSAE
Institut Polytechnique de Paris
France
laurent.linnemer@ensae.fr

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1 Introduction

The usage of a Quasilinear Quadratic Utility Model (hereafter QQUM) is widespread in oligopoly theory. A QQUM helps modelling price or quantity competition in a differentiated-product environment while keeping the (relative) simplicity of a linear demand system. This approach is particularly fruitful when a closed-form solution is needed.

As Amir, Erickson, and Jin (2017) point out “this framework has become so widely invoked that virtually no author nowadays cites any of the(se) early works when adopting this convenient setting.” This lack of reference is not new, however. Some economists find it so natural to use linear (direct or indirect) demands (and to derive them from QQUM) that they do not try to find a source. By analogy, no sane economist would look for a reference when using a linear demand like $D(p) = a - bp$.[1] This lack of reference, however, can confuse other economists who try to cite a source but have difficulties coördinating on the correct one.

The goal of this paper is threefold. First, it recounts how the QQUM was introduced and extensively used by Richard E. Levitan and Martin Shubik[2] in the 1960s and shows they deserve credit for having paved the way. In fact, by analogy with the Cobb-Douglas utility function, it would not be farfetched to name QQUM after Levitan and Shubik. Second, a brief survey of the QQUM literature is provided. This literature can be divided into two type of papers: Those which make few restrictive assumptions besides strict concavity and those which focus on a symmetric utility function. We show that models presented as different are in fact isomorphic. Finally, we provide ready-to-use formulae of the symmetric QQUMs used in practice, emphasizing their mean-variance structure and that many results follow from this property.

The structure of the paper is as follows. First, in section 2, we trace back the origin of the model to the early 1960s. Next, in section 3, we survey the main results obtained for the general form. Finally, comparisons between the various variants found in the literature are made in section 4 and useful formulae from the two main symmetric models used in practice are gathered in section 5.

2 Genesis of QQUM

In the second edition of his book Martin (2002) states that part of the QQUM originates in Bowley (1924) (see section 3.6 of Martin's book).[3] However, a close look at page 56 of Bowley's book shows that it is farfetched, especially in the perspective of an oligopoly model with differentiated goods. Bowley, indeed, considers (for two commodities q_1 and q_2) the following quadratic utility (changing his notations in order to be in harmony with the rest of this paper):

$$U = a_1 q_1 + a_2 q_2 - \sigma q_1 q_2 - \frac{b_1}{2} q_1^2 - \frac{b_2}{2} q_2^2 \tag{1}$$

which is the most general way to write it. The same expression (also for two goods) can be found in Dixit (1979) or in Singh and Vives (1984). Yet, Bowley simply uses (1) to derive demand functions. Most importantly, he does not make the assumption of quasi-linearity of the utility function. Bowley solves the standard microceconomics consumer problem: $\max U$ w.r.t. q_1 and q_2 subject to $p_1 q_1 + p_2 q_2 = m$ where m is the consumer's income. Instead, in the QQUM literature the consumer problem is $\max U + q_0$ w.r.t. q_0, q_1, and q_2 subject to $q_0 + p_1 q_1 + p_2 q_2 = m$. Needless to say that the two problems take different turns and do not have the same solutions. In Bowley's case, demands are not even linear in prices.

[1] Although, Cournot (1838) might be a decent try.

[2] March 24, 1926 – August 22, 2018.

[3] In the 1993 first edition of Martin's book there is no such reference to Bowley and only a slight reference to Levitan and Shubik.

Fact 1 (Dawn of QQUM)**.** *Martin Shubik introduced a linear demand system to model a differentiated good oligopoly (business) game in the early 1960s. In 1961, he joined forces with Richard E. Levitan at I.B.M. to develop a program to solve the (business) game.*

Indeed, after an extensive search, we could only traced back the usage of QQUM in oligopoly theory to the 60s and to the long collaboration between Richard E. Levitan and Martin Shubik. We found the following quote which is of particular interest:

> "I went to IBM in October 1961 and started to work with Dick Levitan to move my game from an IBM 650 to a bigger, better machine-also to add any new features to it. (Our first game used a template as we had no way to print out the format. We could only print numbers.) Levitan did his thesis with Dorfmann and me on a quadratic programming method for allocating demand among oligopolistic firms with product differentiation."
>
> Martin Shubik (see Smith (1992), page 252)

From Vernon Smith's perspective the work of Levitan and Shubik is one of the first (if not the first) laboratory experiment in oligopoly theory. The quote seems to imply that before coming to IBM Shubik had already started to use QQUM. But there is no doubt that the model is central to Levitan and Shubik collaboration between 1961 and 1980.[4]

So the first available reference is Shubik (1961),[5] where in the description of his game he first introduces a more general demand function which is then simplified[6] into a linear one *à la* QQUM. That is (again changing the notations):

$$q_i = \frac{1}{n}\left[a - b\left(1 + \sigma - \frac{\sigma}{n}\right)p_i + b\frac{\sigma}{n}\left(\sum_{j\neq i} p_j\right)\right] = \frac{1}{n}\left[a - bp_i + b\sigma\left(\overline{\mathbf{p}} - p_i\right)\right] \tag{2}$$

where $\overline{\mathbf{p}} = \frac{1}{n}\sum p_i$ is the average price. Note that throughout our paper, whenever there is no ambiguity the subscript is skipped in the sum terms. That is the lengthy $\sum_{i=1}^{n} x_i$ is simply written $\sum x_i$. The expression (2) has been (with or without the $\frac{1}{n}$) a workhorse of product differentiation oligopoly models. In Shubik (1961), the main elements of the linear demand functions for differentiated goods are there, however, there is no reference to a representative consumer and therefore no hint of a quadratic utility function. Notice, already, Shubik's idea of putting a coefficient $\frac{1}{n}$ in front of the demand expression. His point is that if all prices are equal, i.e. for all i, $p_i = \overline{p}$, then total demand is independent of n, i.e. $\sum q_i = a - b\overline{\mathbf{p}}$. A point well explained in Martin's book. This is potentially relevant when doing comparative statics on n, but as shown in section 5 even with that assumption consumers' surplus increases with n (i.e. even for a fixed p) because consumers value diversity. So it is not obvious which normalization is the most convenient: keeping total demand constant or consumers' surplus constant.[7]

Levitan's name appears, however, immediately after. First, in two IBM research reports also describing the same business/experimental game: Levitan and Shubik (1962a) and Levitan and Shubik (1962b). Unfortunately, these two reports cannot be found online today. Next, in Shubik (1964) the collaboration with Levitan is also made clear: "This paper is part of a continuing study done by the author in coöperation with Richard Levitan of the IBM corporation." Demands are again given by (2) (multiplied by terms depending on advertising and conjuncture). The content of the

[4] Maybe Levitan's role at the beginning was more of the nature of a research assistant than a research collaboration which might explain that at first Shubik does not refer to Levitan in his 1961 working paper.

[5] Notice that in the book Shubik (1959) the linear demands for differentiated goods are not mentioned.

[6] The whole game is fairly complicated as it aims to incorporate not only the effect of advertising on demand, but also inventory constraints, and financial variables like loans, and dividends.

[7] Of course with more firms the equilibrium prices would change. The idea of the normalization is to disentangle this competitive effect from the pure diversity effect.

missing IBM reports is probably used in the Cowles Foundation research papers published later, in particular: Levitan and Shubik (1967a) (part of which is published as Levitan and Shubik (1971b)) and Levitan and Shubik (1967b).

Fact 2 (Quasilinear Quadratic Utility)**.** *At least from the mid 1960s, Levitan and Shubik founded their linear demand system on a quasilinear quadratic utility. Moreover, the model also started to be used for both price and quantity competition.*

Indeed, the understanding of QQUM by Levitan and Shubik had evolved from the astute but *ad hoc* linear demands given by (2) to a more structural model with a representative consumer. "We assume that consumer preferences can be represented by a general quadratic utility function. Our somewhat strong special assumption is that to a first approximation there is no income effect between this class of goods and the remainder of the consumer's purchases." (Levitan and Shubik (1967b), page 2). They also refer to the PhD thesis Levitan (1966) for a detailed analysis.[8] So they must have written QQUM between 1964 and 1965. They write:[9]

$$U = a\sum q_i - \frac{1}{2\beta}\left[2\sigma \sum_i \sum_{j>i} q_i q_j + \sum \left(\sigma + \frac{1-\sigma}{w_i}\right) q_i^2\right] - \sum p_i q_i \tag{3}$$

where they call w_i the weight reflecting the size of the ith firm. That is, for all i, $0 < w_i < 1$ and $\sum w_i = 1$. It is also interesting to notice that in both Levitan and Shubik (1967a) and Levitan and Shubik (1967b) matrix notations are used to solve for the Nash equilibrium (both for price and quantity competition).

After mastering this version of the QQUM, Levitan and Shubik seem to have been less involved with it. They have another working paper together (duopoly model) Levitan and Shubik (1969) (later published in the *Journal of Economic Theory*: Levitan and Shubik (1971a)). A duopoly variant of (3) is used in Shapley and Shubik (1969) where they (strangely) do not cite Levitan and Shubik (1967a) nor Levitan and Shubik (1967b). Levitan and Shubik published several articles on related topics (always in a duopoly setting): Levitan and Shubik (1971b), Levitan and Shubik (1972), and Levitan and Shubik (1978).

To summarize this examination of Levitan and Shubik's working papers, articles, and book: as early as 1961 they were using linear demand functions. At first, they did not offer a microeconomic foundation for them. But by the year 1967, they derived systematically these linear demands from a quasilinear quadratic utility function. All along they did not pay too much attention to second order conditions. They not only proposed foundations for QQUM, they used the model to solve oligopoly games with either Bertrand or Cournot competition, and finally they compared equilibrium prices under both form of competition. They do not compute consumers' surplus, nor aggregate profits, nor welfare.

Fact 3 (Book and slow diffusion)**.** *In 1980, Levitan and Shubik gathered their previous work on oligopoly, and QQUM in particular, in their book Shubik and Levitan (1980). The reception of the book in the academic arena was cold.*

The first five chapters of the book Shubik and Levitan (1980) can be seen as an update of the book Shubik (1959). Chapter 6 introduces QQUM for a symmetric duopoly (admittedly they do not

[8] "Demand in an oligopolistic market and the theory of rationing" Harvard U. Officially, the two advisors were Robert Dorfman and Hendrik Houthakker.

[9] Again, adjusting the notations to keep formulae homogeneous throughout this paper. From their formula page 2, the following notational changes have been made: $V \to a$ and $\gamma \to \frac{\sigma}{1-\sigma}$.

check second order conditions), chapter 7 extends to a n firm symmetric oligopoly, and chapter 9 to an asymmetric oligopoly[10] where their chosen quadratic utility (9.5) page 132 is written in the compact form:

$$U = a\sum q_i - \frac{1}{2\beta}\left[(1-\sigma)\sum\frac{q_i^2}{w_i} + \sigma\left(\sum q_i\right)^2\right] - \sum p_i q_i$$

which can be rearranged as (3).

Despite their thorough work on QQUM, it would not be an understatement to say that Levitan and Shubik's approach was not immediately popular. In particular, the 1980 book *Market structure and behavior* by Levitan and Shubik which from today's perspective has certainly been a success (most academic libraries hold the book and it is still in print) received mixed reviews to say the least. In fact, it is almost painful to read some reviews written at the time. Rothschild (1982) in the *Journal of Economic Literature* is unmerciful: "Much of the book is devoted to computing the solutions of different variants of a single model. Although it is interesting and important to know that this can be done, it is difficult to stay awake while watching the process. I found it hard to make anything of the many numerical results that are presented and so, I suspect, did the authors. The results of an attempt to apply the oligopoly model to a real problem can charitably be described as eccentric. The U.S. automobile industry (in 1965) is modeled as a three-firm industry in which each firm sells a single product. Apparently implausible conclusions –in particular that small price reductions would have greatly increased sales and that GM cars were more expensive than those of Ford or Chrysler– are laid to inadequacies of data rather than to deficiencies of the model." In the *Journal of Political Economy*, Telser (1982) is not enthusiastic either: "A mere catalog of some models of oligopoly does not constitute a useful contribution to economics. Readers deserve a coherent set of principles that can relate the theories, a demonstration of their explanatory power, if any, and a statement of which survives these tests." Telser is also quite harsh on the empirical part: "The book also contains a section purporting to apply the theories to the study of the automobile industry, but it does not pass even the loosest standards of econometric rigor." In the *Economic Journal*, Reid (1982) is more positive and spends more space than the previous two reviews on praising the book, concluding "On balance, however, the reading of the book is a tonic. It stimulates, fascinates and informs, and will repay frequent re-reading."

Chapter 9 is praised (both Reid and Telser find chapter 9 the most ambitious) but also criticized: "…but the great weight of attention is still given to pure theory and occasionally, as in chapter 9, to one of its least attractive varieties, namely the intricate manipulation of specialized functional forms." (Reid (1982)). As well as by Pagoulatos (1983) in *Southern Economic Journal* "Finally, the mathematical manipulations of different linear functions presented in Chapter 9 leave the reader in strong doubt about the usefulness of following every step of the various exercises. Relegating the nonessential manipulations to an appendix would have added considerably to the enjoyment of the book."

Fact 4 (QQUM ignored in the late 1980s IO surveys)**.** *Levitan and Shubik's work on oligopoly, and QQUM in particular, was not mentioned in the main IO surveys which flourished at the end of the 1980s.*

Indeed, in the IO Bible, Tirole (1988), it is the address models, Hotelling and Salop, (see chap. 7) which are put forward to deal with product differentiation. Similarly, in the "Product Differentiation" chapter of the Handbook of Industrial Organization (vol. 1, chap. 12) Eaton and Lipsey (1989) focus on address models and when briefly discussing the representative consumer approach they do not

[10] They have two sources of asymmetry: i) firms have heterogenous marginal costs c_i, and ii) the "weights" w_i. There is no heterogeneity in terms of a. This is not really a drawback as what mostly matters is the difference $a_i - c_i$ but they do not indicate that they knew this property.

mention Levitan and Shubik (see also their Figure 12.4. "Historical perspective", page 762). They only refer to the seminal papers of monopolistic competition: Spence (1976a), Spence (1976b) and Dixit and Stiglitz (1977). When they use (1) they refer to Spence (1976a).[11] Finally, in Anderson, De Palma, and Thisse (1992) there is no reference to QQUM and (therefore) no reference to Levitan and Shubik.[12]

How to explain the relative lack of success of Levitan and Shubik's work on QQUM? First, as shown by the surveys by Tirole and Eaton and Lipsey, by the end of the 1980s the representative consumer approach was not seen as having appropriate microeconomic foundations. Consumers have different tastes and each individual buys only a tiny subset of all available varieties. The aggregate demand could possibly be linear in prices but QQUM does not provide a micro-foundation capturing this idea that consumers are different in these dimensions. Second, the representative consumer approach became popular in the monopolistic competition literature but there, following Dixit and Stiglitz (1977), it is not the quadratic utility function which is used but the Constant Elasticity of Substitution (CES) utility function.

A minor confusion comes from the fact that QQUM was independently introduced by Spence (1976a) (n firms, completely symmetric model) and Dixit (1979) (a general duopoly setting). In fact, in Spence (1976a) there is no representative consumer. Linear demands are assumed and Spence derives (in a footnote) the consumers' surplus (which takes the QQUM form). In Dixit (1979), a general (two-good) quasi-linear utility function is introduced (see (1) page 21) and used to derive inverse demands (see (2) page 22). In order to derive comparative statics results, Dixit assumes a quadratic form (see (4) page 26). He gives the precise conditions under which the utility is concave. It is Spence (1976a) which is referred to in the seminal work of Singh and Vives (1984) (they also cite Shubik and Levitan (1980), but they do not present it as a predecessor of Dixit). Both Dixit and Spence were very active researchers in the late 1970s and 1980s and their articles, published in top journals, were probably more visible than a book and a series of old working papers.

Fact 5 (QQUM usage in IO)**.** *From the 1980s up until today, QQUM has been used in IO, typically when closed-form formulae are needed.*

However, despite the dominant view that address models were sounder, and despite the competing references of Dixit and Spence, the spirit of QQUM endured and proved itself useful in IO and some authors started to cite the book Shubik and Levitan (1980) and also, but to a lesser extend, the chapter Levitan and Shubik (1971b). Prominent examples are Deneckere and Davidson (1985),[13] Vives (1985),[14] Shaked and Sutton (1990), Bagwell and Ramey (1991), Shaffer (1991), and Sutton (1997). In, Motta (2004), influential book "Competition Policy: Theory and Practice", the Levitan and Shubik's model is used (in particular in chapter 5 on horizontal mergers) to illustrate some properties with a closed-form model.

Among the articles relying on QQUM, there is a literature on comparing prices, quantities, profits, welfare, between Bertrand and Cournot competition. Levitan and Shubik themselves have compared prices when all goods are substitutes, see Levitan and Shubik (1967b) page 7, but this strand of the

[11]In this article, linear demands for differentiated goods are introduced (see (2) page 411) and a hint toward a quadratic utility function is given in footnote 6, page 412.

[12]Finally, almost 30 years later, in the Special Issue in Honor of Martin Shubik published by *Games and Economic Behavior* (Volume 65, Issue 1, Pages 1-288, January 2009) the QQUM is not cited once (and Levitan and Shubik 1980 book only twice) in 16 articles.

[13]They give Shubik as the sole author of the book because on the book cover, of the first editions, the author is "Martin Shubik with Richard Levitan". Many authors give credit to both authors and we follow this tradition here.

[14]There Levitan and Shubik's book is cited although the publication year is wrong: 1971 instead of 1980. The same mistake is made in Vives (2001). Maybe a confusion between the 1980 book and the 1971 chapter. In Vives (2008) the year is correct but the QQUM origin is attributed to Shapley and Shubik (1969).

literature really started with Singh and Vives (1984) and Vives (1985), the main reference remains Amir and Jin (2001).[15]

Table 1 lists a few articles on various topics where (by and large) the same type of modelization strategy is followed. A general product differentiation oligopoly framework is used at the start of the paper, some results are derived, and at the end the QQUM is introduced in order to derive more specific results which are unclear in the general framework. In all these examples only a symmetric QQUM is used (the few exceptions are in a duopoly or triopoly setting). That is, a utility given by (2) or the symmetric version of Spence but not the asymmetric (3). Also often the terms a_i or c_i are assumed to be homogenous. As shown in section 5, however, there is no difficulty in including heterogenous a_i and c_i.

The QQUM is not particularly popular among econometricians probably because in its general form it involves too many coefficients to estimate. Pinkse, Slade, and Brett (2002) is an exception. There QQUM is presented as a second order approximation of a general demand model. This is a clever remark which could very well explain the success of QQUM in practice when a result cannot be shown with a general (nonlinear) demand function.

[15] Häckner (2000) presents new results, in particular for $\sigma < 0$. See Chang and Peng (2012) for a survey.

Table 1: Sample of articles using QQUM

Article	Journal	year	Nb firms	Type of **B**	Hetero. **a** or **c**	L&S
Spence	AER	1976	n	Symmetric	No	No
Dixit	BJE	1979	2	General	Yes	No
Friedman	BJE	1983	n	Symmetric	Yes	Yes
Singh and Vives	RJE	1984	2	General	Yes	Yes
Deneckere and Davidson	RJE	1985	n	Symmetric	No	Yes
Vives	JET	1985	n	Symmetric	No	Yes
Shaked and Sutton	RJE	1990	n	Symmetric	No	Yes
Bagwell and Ramey	RJE	1991	n	Symmetric	No	Yes
Shaffer	RJE	1991	2	Symmetric	No	Yes
Besanko and Perry	RJE	1993	3	Symmetric	No	Yes
Röller and Tombak	MS	1993	n	Symmetric	No	Yes
Raju, Sethuraman, and Dhar	MS	1995	n	Symmetric	No	Yes
Sutton	RJE	1997	n	Symmetric	No	Yes
Sayman, Hoch, and Raju	MkS	2002	3	Asymmetric	Yes	Yes
Pinkse and Slade	EER	2004	n	Symmetric	Yes	No
Motta	Book	2004	n	Symmetric	No	Yes
Daughety and Reinganum	RJE	2008	n	Symmetric	Yes	No
Fumagalli and Motta	EJ	2008	n	Symmetric	No	Yes
Vives	JINDEC	2008	n	Symmetric	Yes	Yes
Foros, Hagen, and Kind	MS	2009	n	Symmetric	No	Yes
Kind, Nilssen, and Sørgard	MkS	2009	n	Symmetric	No	Yes
Subramanian, Raju, Dhar, and Wang	MS	2010	2	Symmetric	Yes	Yes
Inderst and Valletti	EER	2011	n	Symmetric	No	Yes
Calzolari and Denicolo	AER	2015	2	Symmetric	No	Yes
Edelman and Wright	QJE	2015	n	Symmetric	No	Yes
Abhishek, Jerath, and Zhang	MS	2016	2	Symmetric	No	Yes
Allain, Henry, and Kyle	MS	2016	n	Symmetric	No	Yes
Cho and Wang	MS	2016	n	Symmetric	No	Yes

3 QQUM itself

At the core of QQUM there is a paradox of sort. On the one hand, the equilibrium expressions (price, quantity, profit, surplus, welfare) are not simple and are often hard to read and write (see the criticisms of chapter 9 of Levitan and Shubik's book). On the other hand, the model is used in practice because the linear demand functions allow to solve the equilibrium (price or quantity competition) in closed-form. That is, variants of the model are simple enough to be a subgame of a larger game. For example, in a first stage firms invest in R&D and in a second stage they compete and competition is modelled with a QQUM.

Since Levitan and Shubik's specific QQUM model, academics have generalized it. In this section, we use this general framework to introduce formally QQUM and derive some properties. This section builds on previous articles which can be divided into two groups. First, Economics oriented articles: Jin (1997), Amir and Jin (2001), Bernstein and Federgruen (2004), Choné and Linnemer (2008), Chang and Peng (2012), and Amir, Erickson, and Jin (2017). Second, Operation Research oriented ones: Farahat and Perakis (2009), Farahat and Perakis (2011a), Farahat and Perakis (2011b), Kluberg and Perakis (2012). Cross-citations between the two groups tend to be rare.

Notations Mostly for compactness in presentation, it is convenient to use the following notations. Let $\mathbf{x}$ (bold font) denote a vector of size n: $\mathbf{x} = (x_1, \cdots, x_n)'$ where the $'$ stands for transposition. A capital bold letter, as $\mathbf{X}$, denotes a $n \times n$ matrix which elements are x_{ij}.

Levitan and Shubik already resorted to matrix notations but mostly for their proofs. As economists are not always at ease with matrix notations, standard expressions are (most of the time) also given throughout this paper.

Quasi-linear Quadratic utility Let q_i denote the quantity of good i, $i = 1$ to n, consumed and let $\mathbf{q} = (q_1, \cdots, q_n)'$ denote the column vector of such quantities. The quasi-linear quadratic utility model (QQUM) first assumes quasi-linearity. That is, there is a numéraire good q_0 which price is normalized to 1 and the utility function (of the representative consumer) writes $U(\mathbf{q}) + q_0$. Let $\mathbf{p} = (p_1, \cdots, p_n)'$ denote the column vector of prices.[16] The maximization problem of the consumer writes: $\max_{q_0,\mathbf{q}} U(\mathbf{q}) + q_0$ s.t. $q_0 + \mathbf{p}'\mathbf{q} = m$ where m denotes the wealth of the consumer. Eliminating[17] q_0 and dropping the constant term m, leads to $\max_{\mathbf{q}} U(\mathbf{q}) - \mathbf{p}'\mathbf{q}$. Assuming a quadratic form for $U(.)$ allows to write the maximization problem as

$$\max_{\mathbf{q}} U(\mathbf{q}) - \mathbf{p}'\mathbf{q} = \max_{\mathbf{q}} (\mathbf{a} - \mathbf{p})' \mathbf{q} - \frac{1}{2}\mathbf{q}'\mathbf{B}\mathbf{q} \tag{4}$$

where $\mathbf{a}$ is the column vector of the (marginal) quality (or utility) indexes, a_i, one for each variety i, and $\mathbf{B}$ is a $n \times n$ positive definite matrix[18] a necessary condition for U to be strictly concave. The $\mathbf{B}$ matrix captures the complementarity/substitution patterns that exist between the different varieties.

Without matrix notations, the objective function of (4) writes (noting $b_{ii} = b_i$):

$$U(\mathbf{q}) - \mathbf{p}'\mathbf{q} = \sum (a_i - p_i) q_i - \sum_i \sum_{j>i} b_{ij} q_i q_j - \frac{1}{2} \sum b_i q_i^2 \tag{4 bis}$$

Some authors normalize the b_i to one. This is possible (by changing the units with which each quantity is measured, i.e. using $x_i = q_i/\sqrt{b_i}$) but it is not always innocuous. Indeed, down the road,

[16] It goes without saying that throughout prices and quantities are non negative.

[17] This cannot be done for all values of m. If m is too small the constraint $q_0 \geq 0$ could be binding and the optimal quantities would depend on m. See Varian (1992) (chapter 10 section 3) and Amir, Erickson, and Jin (2017).

[18] That is, $\mathbf{B}$ is symmetric and for any $\mathbf{q} \neq 0$, the scalar $\mathbf{q}'\mathbf{B}\mathbf{q}$ is positive.

additional normalizations are (most of the time) introduced and it could be confusing to not make all normalizations at the same time.

Fact 6 (Limiting cases)**.** *The utility function $U(\mathbf{q})$ can be rewritten in order to emphasize two limiting cases: perfect substitutes and perfect complements. In the neighborhood of perfect complements, the no-income-effect assumption (i.e. $q_0 > 0$) cannot hold.*

Indeed,

$$\begin{aligned} U(\mathbf{q}) &= \sum a_i q_i - \sum_i \sum_{j>i} b_{ij} q_i q_j - \frac{1}{2} \sum b_i q_i^2 \\ &= \sum a_i q_i - \sum_i \sum_{j>i} \left(b_{ij} - \sqrt{b_i b_j} \right) q_i q_j - \frac{1}{2} \left(\sum \sqrt{b_i}\, q_i \right)^2 \end{aligned}$$

As the second line makes it clear, if for all i, j, $b_i = b_{ij} = b > 0$, then $U - \sum p_i q_i = \sum (a_i - p_i) q_i - \frac{b}{2} \left(\sum q_i \right)^2$. This utility is maximized by buying all units from the seller offering the largest surplus $a_i - p_i$ (Bertrand competition for vertically differentiated goods). The case $a_i = a > 0$ being Bertrand competition for an homogeneous good where only the $\sum q_i$ is relevant and it corresponds to the perfect substitutes case. In these cases, however, the matrix $\mathbf{B}$ is not definite positive. But one can be arbitrarily close to $b_i = b_{ij} = b > 0$ and still have a definite positive $\mathbf{B}$.

The polar case is when all goods are perfect complements. Formally, this case corresponds to a Leontief utility function (which is concave but not strictly concave), for example, $q_0 + a \min \{q_1, \cdots, q_n\}$ with $a > 1$ then maximizing under a budget constraint $q_0 + \sum p_i q_i \leq m$ leads to $q_0 = 0$ and, for all i, $q_i = m / \sum p_i$. This shows that for perfect complements (or in the neighborhood) the quantities cannot be independent of the available income. But as it cannot be realistic for a consumer to spend all his income on a particular set of goods. It means that this model does not capture fully the economic environment of the consumer. Indeed, the choice of m should be modelled. QQUM can "in spirit" replicate this case (with the same drawback). Indeed,

$$\begin{aligned} U(\mathbf{q}) &= \sum a_i q_i - \sum_i \sum_{j>i} b_{ij} q_i q_j - \frac{1}{2} \sum b_i q_i^2 \\ &= \sum a_i q_i - \sum_i \sum_{j>i} \left(b_{ij} + \frac{\sqrt{b_i b_j}}{n-1} \right) q_i q_j - \frac{1}{2(n-1)} \sum_i \sum_{j>i} \left(\sqrt{b_i}\, q_i - \sqrt{b_j}\, q_j \right)^2 \end{aligned}$$

now if for all i, $a_i = a > 1$, $b_i = b > 0$ and for $j \neq i$, $b_{ij} = \frac{-b}{n-1}$, then $U = a \sum q_i - \frac{b}{2(n-1)} \sum_i \sum_{j>i} (q_i - q_j)^2$. Maximizing $U(\mathbf{q}) + q_0$ under a budget constraint $q_0 + \sum p_i q_i \leq m$ also leads to $q_0 = 0$ and, for all i, $q_i = m / \sum p_i$.

More generally, as discussed by Varian (1992) (chapter 10 section 3), for a quasi-linear utility function, the available income should be large enough in order to have demand functions which are independent of income. This assumption can be mild for substitutes but, as pointed out by Amir, Erickson, and Jin (2017) (see their Proposition 14), it is, indeed, incompatible with perfect complements. For these goods, all the available income is spent. Or, in the absence of constraint on income, consumption would go to infinity. That is, to analyze a market where perfect complements are sold, one should first model how consumers allocate their revenue between these particular goods and the other goods they consume.

In between perfect substitutes and perfect complements, one can put independent goods. That is, assuming for all i, j, $j \neq i$, $b_{ij} = 0$. If each variant i is sold by a monopoly (it would not really change the problem emphasized here if each good is sold at marginal cost) and denoting c_i the constant marginal cost of production, demand is $q_i = (a_i - p_i)/b_i$, the price $p_i = (a_i + c_i)/2$, and the expenditure of the consumer would be $e = \sum p_i q_i = \sum \left(a_i^2 - c_i^2\right)/(4b_i)$. Clearly, without restricting further the values of a_i, c_i, and b_i this sum would diverge when the number of good, n, tends to infinity, and the constraint $e \leq m$ cannot be verified. For example, if $b_i = b$ for all i, and assuming the existence of $\varepsilon > 0$ such that $a_i^2 - c_i^2 > \varepsilon$, then $e > n\frac{\varepsilon}{4b}$ which tends to infinity with n. This problem would occur by continuity if $\mathbf{B}$ is in a neighbourhood of such a diagonal matrix. Assuming $b \sim n$ would eliminate the problem but rather artificially. Therefore one should keep in mind that QQUM is not fit to model an infinite number of (almost) independent goods.

Demand and inverse demand functions The first-order condition of the maximization of U with respect to $\mathbf{q}$ provides immediately the expression of inverse demand:

$$\mathbf{Bq} = \mathbf{a} - \mathbf{p} \text{ i.e. } \mathbf{p}(\mathbf{q}) = \mathbf{a} - \mathbf{Bq} \tag{5}$$

this linear relationship between prices and demands is the main purpose of choosing a quadratic utility function. Without matrix notations, it writes:

$$\text{for all } i, \; p_i = a_i - b_i q_i - \sum_{j \neq i} b_{ij} q_j \tag{5 bis}$$

which shows that, up to this point, the matrix notations do not simplify particularly the writing of the model. However, to characterize the direct demand functions, one needs $\mathbf{B}^{-1}$ the inverse of $\mathbf{B}$, and here the matrix notations are useful.[19] Inverting (5) gives the direct demand functions:

$$\mathbf{q}(\mathbf{p}) = \mathbf{B}^{-1}(\mathbf{a} - \mathbf{p}) \tag{6}$$

denoting β_{ij} the elements[20] of $\mathbf{B}^{-1}$, and using β_i for β_{ii}, the direct demands can be given without matrix notations

$$\text{for all } i, \; q_i = \sum_j \beta_{ij}(a_j - p_j) = \beta_i(a_i - p_i) + \sum_{j \neq i} \beta_{ij}(a_j - p_j) \tag{6 bis}$$

Fact 7 (Heterogeneity of the consumers)**.** *The aggregation of heterogenous individual QQUM into a representative consumer's QQUM is not always possible.*

As we have seen one criticism of QQUM is that it assumes a representative consumer (or identical consumers) and not heterogenous consumers whose demands have to be summed. However, it can be presented as the aggregation of individual demands in special cases. Indeed, let parameterize a population of consumers by θ, with a distribution function $F(.)$, and assume each consumer has a quasilinear quadratic utility function. The individual demands are (as above)

$$q_i(\theta) = \sum_j \beta_{ij}(\theta)(a_j(\theta) - p_j) \text{ and total demand is } q_i = \int q_i(\theta) dF(\theta)$$

Can the aggregate demand be obtained from a QQUM? One should be careful to use the above formula only for prices where quantities are positive for all consumers. To illustrate this possibility, assume $a_j(\theta) = a_j$ for all θ, assume that the prices belong to a range such that for all i, $q_i(\theta) > 0$. Then on aggregate:

$$q_i = \sum_j \beta_{ij}(a_j - p_j) \text{ where } \beta_{ij} = \int \beta_{ij}(\theta) dF(\theta)$$

[19]Recall here that as $\mathbf{B}$ is positive definite it has an inverse and $\mathbf{B}^{-1}$ is also positive definite.

[20]Formally, let $\mathbf{B}^{ij}$ be the matrix where lign j and column i are deleted from $\mathbf{B}$ then $\beta_{ij} = (-1)^{i+j}\det\left(\mathbf{B}^{ij}\right)/\det(\mathbf{B})$.

if the matrix $(\beta_{ij})_{i,j}$ is positive definite, then a representative consumer exists. This would be the case if all $\mathbf{B}^{-1}(\theta)$ are diagonal dominant.

Another issue, is that all consumers do not value all goods (e.g. for some i, $a_i(\theta) < c_i$). This is not always inconsistent with a representative consumer. Each individual consumer could have a quasi-linear quadratic utility over a subset of varieties. For example, a fraction of consumers could only value good 1 (with a demand $a_1 - p_1$), another group only value good 2 (with a demand $a_2 - p_2$), and the remaining consumers value both goods (with $\mathbf{B_3} = \begin{pmatrix} 1 & \sigma \\ \sigma & 1 \end{pmatrix}$ and $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$). Then it is easy to show that as long as all three groups consume the aggregate demand can be derive from a QQUM with $\mathbf{a}$ and $\mathbf{B} = \begin{pmatrix} 2-\sigma & \sigma \\ \sigma & 2-\sigma \end{pmatrix}$. To conclude this point, although it requires some care, it is possible to have both heterogenous consumers and aggregate demand given by QQUM.

Firms To complete the description of the oligopoly two elements are needed. First, production costs have to be introduced. Most of the time, they are assumed to be linear (i.e. constant marinal cost): let c_i denote the marginal cost of production of firm i and let $\mathbf{c} = (c_1, \ldots, c_n)'$ denote the column vector of these marginal costs. Assuming convex costs is mainly problematic when one is looking for closed-form expressions. See, however, Bernstein and Federgruen (2004) for existence and uniqueness of equilibrium with convex costs, and various comparative statics results.Second, an ownership structure has to be specified.

Fact 8 (Ownership structure)**.** *While most existing studies consider oligopolies with single-product firms, the model naturally extends to multi-product competition, delivering simple expressions for first-order conditions at Cournot-Nash and Bertrand Nash equilibria.*

This question appears naturally in the context of mergers, see Choné and Linnemer (2008) for example. It is also considered in Farahat and Perakis (2009), Farahat and Perakis (2010), and Farahat and Perakis (2011b). Let $N = \{1, \ldots, n\}$ denote the set of all brands. The structure of the industry is described by a partition of N into r subsets: $\{I_1, \cdots, I_r\}$ where I_k denotes the set of brands owned by firm k.

More generally, one could assume cross-ownership. In that case, let $\alpha_{kj} \in [0,1]$ denote the share of profit generated by the sales of good j owned by firm k, with for all j, $\sum_k \alpha_{kj} = 1$. One would also need to specify how the price (or quantity) of a multi-owned good is chosen. For example, it could be chosen by the firm with the largest share but when this share is less than 0.5 there might be no obvious choice.

Parameter space For the model to have economic sense, its parameters have to be constrained. The model allows for four types of heterogeneities: the qualities $\mathbf{a}$, the marginal costs $\mathbf{c}$. Most importantly, the elements of matrix $\mathbf{B}$ have to be such that $\mathbf{B}$ is positive definite. The diagonal terms b_i of matrix $\mathbf{B}$ correspond to $-\partial^2 U/\partial^2 q_i$ and capture the concavity of U with respect to q_i (or how quickly is the marginal utility of good i decreasing). The b_{ij}, $i \neq j$ correspond to $-\partial^2 U/\partial q_i \partial q_j$ and capture the (possibly rich) pattern of complementarity and substitutability among the goods.

Some authors, while working within the general framework, assume that all elements of $\mathbf{B}$ are positive and they also add the assumption that $\mathbf{B}$ is diagonal dominant.[21] These assumptions imply that $\mathbf{B}$ is positive definite.

[21] Bernstein and Federgruen (2004), Farahat and Perakis (2009), Farahat and Perakis (2010), Farahat and Perakis (2011b), and Kluberg and Perakis (2012).

Notice that some authors start with $U = (\mathbf{a}-\mathbf{p})'\,\mathbf{q} - \frac{1}{2}\mathbf{q}'\mathbf{B}^{-1}\mathbf{q}$ inverting (in the notation only) the role played by $\mathbf{B}$ and $\mathbf{B}^{-1}$. This is a question of taste but this can be slightly confusing as the off diagonal elements of $\mathbf{B}^{-1}$ are negative when those of $\mathbf{B}$ are positive.

In addition, the following two assumptions are often made: $\mathbf{a}-\mathbf{c} > \mathbf{0}$ (Positive primary markups) and $\mathbf{q}(\mathbf{c}) \geq \mathbf{0}$ (Positive primary outputs, i.e. all varieties are produced when all prices are set at marginal costs). Both names are introduced by Amir and Jin (2001) see Chang and Peng (2012) for a discussion. Both assumptions are quite natural. However, as in equilibrium prices are larger than marginal costs, a variety can be profitable even if it would not be sold at the first best. See Zanchettin (2006) for a detailed comparison of Betrand and Cournot competition when this assumption is not made in a duopoly setting.

Equilibrium The purpose of establishing direct and indirect demand functions is to use them to find the Nash equilibrium of an oligopoly game where each product is produced by one firm. As firms either compete in prices or quantities there are two games: a Bertrand-like one (i.e. competition in prices) and a Cournot-like one (i.e. competition in quantities).

In both type of competition, price margins $\mathbf{p}-\mathbf{c}$ and quantities $\mathbf{q}$ depend only on the marginal surpluses $\mathbf{a}-\mathbf{c}$. That is, they do not depend separately on $\mathbf{a}$ and $\mathbf{c}$. Therefore it is possible to normalize one of these two vectors. For example, choosing $a_i = a$ for all i or $c_i = c$ for all i. Doing so, unfortunately, only marginally simplifies the expressions at the expense of intuition. It is, however, useful to introduce a new notation for these marginal surpluses:

$$\text{let } v_i = a_i - c_i\,, \quad \text{or in matrix form } \mathbf{v} = \mathbf{a} - \mathbf{c}$$

Before characterizing the Nash equilibrium of the oligopoly games. We present briefly two benchmarks: the first-best and the monopoly.

First-best Obviously, to maximize welfare, each good should be priced at marginal cost and the first-best quantities, using (5), are given by

$$\mathbf{B}\mathbf{q}^* = \mathbf{a} - \mathbf{p} = \mathbf{a} - \mathbf{c} = \mathbf{v} \tag{7}$$

using (A.3), we have $W^* = \frac{1}{2}\mathbf{v}'\mathbf{B}^{-1}\mathbf{v}$.

Monopoly It can be useful (e.g. to study collusion,[22] or merger to monopoly) to characterize the quantities (or equivalently in this case the prices) that a monopoly controlling all goods would choose. Its profit is

$$\Pi = \sum \pi_i = (\mathbf{p}-\mathbf{c})'\,\mathbf{q} = (\mathbf{a}-\mathbf{c})'\,\mathbf{q} - \mathbf{q}'\mathbf{B}\,\mathbf{q}$$

which is maximized here with respect to $\mathbf{q}$ and the f.o.c. writes in matrix notations

$$2\mathbf{B}\mathbf{q}^{\mathbf{m}} = \mathbf{a} - \mathbf{c} = \mathbf{v}$$

Fact 9 (Monopoly)**.** *Monopoly quantities are half the first-best quantities. Monopoly prices are invariant with the elements of* $\mathbf{B}$*, they are simply given by* $\mathbf{p}^{\mathbf{m}} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$*. Or equivalently,* $\mathbf{p}^{\mathbf{m}} - \mathbf{c} = \frac{1}{2}\mathbf{v}$*.*

Moreover (almost like in the standard one good linear demand),

$$\Pi^m = \frac{1}{4}\mathbf{v}'\mathbf{B}^{-1}\mathbf{v}\ , \text{ and } \ W^m = \frac{3}{8}\mathbf{v}'\mathbf{B}^{-1}\mathbf{v} = \frac{3}{4}W^* \tag{8}$$

[22] See Deneckere (1983) for $n = 2$, Deneckere (1984) for a key correction, and Majerus (1988) for n firms.

Cournot competition: f.o.c. Following Jin (1997) and Amir and Jin (2001),[23] one can elegantly write the f.o.c. of the maximization of the profit of firm i quite generally (i.e. for an arbitary positive definite $\mathbf{B}$ matrix). Indeed, the profit of firm i being

$$\pi_i = (p_i - c_i)q_i = \left(a_i - c_i - \sum_j b_{ij}q_j\right) q_i$$

which is maximized here with respect to q_i. The f.o.c. (the s.o.c. is satisfied as b_{ii} which is noted b_i is positive) writes

$$p_i - c_i + \frac{\partial(p_i - c_i)}{\partial q_i} q_i = p_i - c_i - b_i q_i = 0$$

or in matrix form

$$\mathbf{p} - \mathbf{c} = \mathbf{diag}(\mathbf{b})\mathbf{q}$$

where $\mathbf{diag}(\mathbf{b})$ is the diagonal matrix which elements are the diagonal elements of $\mathbf{B}$ (i.e. the b_i). Now, using $\mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} - \mathbf{Bq}$ the f.o.c. collected in matrix form are

$$(\mathbf{B} + \mathbf{diag}(\mathbf{b}))\, \mathbf{q}^C = \mathbf{a} - \mathbf{c} = \mathbf{v} \tag{9}$$

Equation (9) helps addressing the questions or the existence of an equilibrium and of its uniqueness in the Cournot competition game. The answer is simple. As the $\mathbf{B}$ matrix and the $\mathbf{diag}(\mathbf{b})$ matrix are both positive definite their sum $\mathbf{B} + \mathbf{diag}(\mathbf{b})$ is positive definite and then invertible. Therefore an equilibrium exists and it is unique. In Appendix B, f.o.c. are similarly computed for the case where firms are multi-products.

Bertrand competition: f.o.c. Again following Jin (1997) and Amir and Jin (2001), one can also write the f.o.c. of the maximization of the profit of firm i with respect to price quite generally (i.e. for an arbitary positive definite $\mathbf{B}$ matrix). Indeed, the profit of firm i being

$$\pi_i = (p_i - c_i)q_i = (p_i - c_i)\left(\beta_i(a_i - p_i) + \sum_{j \neq i} \beta_{ij}(a_j - p_j)\right)$$

which is maximized here with respect to p_i. The f.o.c. (the s.o.c. is satisfied as β_i is positive) writes

$$q_i + \frac{\partial q_i}{\partial p_i}(p_i - c_i) = q_i - \beta_i(p_i - c_i) = 0$$

or in matrix form

$$\mathbf{q} = \mathbf{diag}(\boldsymbol{\beta})(\mathbf{p} - \mathbf{c})$$

where $\mathbf{diag}(\boldsymbol{\beta})$ is the diagonal matrix which elements are the diagonal elements of $\mathbf{B^{-1}}$ (i.e. the β_i). Now, using $\mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} - \mathbf{Bq}$ the f.o.conditions collected in matrix form are

$$\left(\mathbf{B} + \mathbf{diag}(\boldsymbol{\beta})^{-1}\right) \mathbf{q}^B = \mathbf{a} - \mathbf{c} = \mathbf{v} \tag{10}$$

the prices have been eliminated in order to show the similarity of the Bertrand and Cournot characterization. Here also as $\mathbf{B}$ and $\mathbf{diag}(\boldsymbol{\beta})^{-1}$ are both positive definite (as the β_i are positive), also is their sum $\mathbf{B} + \mathbf{diag}(\boldsymbol{\beta})^{-1}$ and thus inversible. Hence the existence of a unique equilibrium. In Appendix B, f.o.c. are similarly computed for the case where firms are multi-products.

[23] In Levitan and Shubik (1967b), the same derivation is done, for a particular, $\mathbf{B}$ matrix. The writing is slightly messy but the idea is sound.

To move from the Cournot characterization to the Bertrand one, the $\mathbf{diag}(\mathbf{b})$ matrix has to be replaced by $\mathbf{diag}(\boldsymbol{\beta})^{-1}$. In Jin (1997) (see Appendix A), it is shown that $b_i > 1/\beta_i$, that is the element of the $\mathbf{diag}(\mathbf{b})$ matrix are larger than the ones of $\mathbf{diag}(\boldsymbol{\beta})^{-1}$. In that sense, the intuition is that the Bertrand quantities should be larger than the Cournot ones. However, this is not that mechanical and counterexamples exist.

In Appendix A, we show how consumers' surplus, total profit, and welfare can be written in this general framework. Another approach is to introduce the following function which allows to describe all equilibrium quantities.

Fact 10 (Equilibrium quantities)**.** *Let*

$$\mathbf{q}^*(\mathbf{X}) \equiv \mathbf{q}^*(\mathbf{X};\mathbf{B},\mathbf{v}) = (\mathbf{B}+\mathbf{X})^{-1}\mathbf{v} \tag{11}$$

where $\mathbf{X}$ *is a positive definite matrix. Then the First-best, Monopoly, Cournot, and Bertrand equilibrium quantities are respectively given by*

$$\mathbf{q}^* = \mathbf{q}^*(\mathbf{0}), \ \mathbf{q^m} = \mathbf{q}^*(\mathbf{B}), \ \mathbf{q}^C = \mathbf{q}^*(\mathbf{diag}(\mathbf{b})), \quad \textit{and} \ \ \mathbf{q}^B = \mathbf{q}^*\left(\mathbf{diag}(\boldsymbol{\beta}^{-1})\right).$$

The case of multi-product firms can also be described with this function, see (B.1) and (B.2).

For example, in the case of Cournot (resp. Bertrand), as the matrix $\mathbf{X}$ is diagonal, it is particularly easy to move from the expression of the first-best quantity q_i^* to the expression of q_i^C. On has to change all the b_j in $2b_j$, $j=1$ to n.[24] This change applies *only* to the quantity expressions, the prices are still given by (5), and the expressions of consumers' surplus, aggregate profits, and welfare by (A.1), (A.2), and (A.3), respectively. For example, the welfare at the Cournot equilibrium is

$$2W^C = \mathbf{v}'\mathbf{q^C} - \mathbf{q^C}'\mathbf{B}\,\mathbf{q^C}$$

4 Various forms encountered in the literature

The general case is, however, seldom encountered (except for $n=2$ where it remains manageable) and at least three main specifications of the general case have been used by economists. The differences come mostly from the $\mathbf{B}$ matrix. Authors also differ in terms of their heterogeneity choices of the parameters $\mathbf{a}$ and $\mathbf{c}$ but there are two broad possibilities: either no heterogeneity, i.e. $\mathbf{a}=(a,\cdots,a)'$ and $\mathbf{c}=(c,\cdots,c)'$ or heterogeneity for $\mathbf{a}$, $\mathbf{c}$, or both. Indeed, as shown by (9) and (10) the equilibrium quantities depend only on the difference $\mathbf{a}-\mathbf{c}$.

First, there are the Levitan and Shubik's forms which they call asymmetric and nonsymmetric in their book and which dates back to Levitan and Shubik (1967a) (part of which is published as Levitan and Shubik (1971b)) and Levitan and Shubik (1967b) and the chapter 9 of their 1980 book. Their nonsymmetric case, already given by (3), is:

$$U^{\mathrm{LS}} - \sum p_i q_i = \sum (a-p_i)\, q_i - \frac{1}{2\beta}\left[2\sigma \sum_i \sum_{j>i} q_i q_j + \sum \left(\sigma + \frac{1-\sigma}{w_i}\right) q_i^2\right]$$

with, for all i, $0 < w_i < 1$ and $\sum w_i = 1$, in addition marginal costs are heterogeneous (chapter 9 of their book). The symmetric case being $w_i = 1/n$ for all i and marginal costs are homogeneous (chapter 7 of their book). The symmetric Levitan and Shubik's formulation is used, for example, by Motta (2004) and Wang and Zhao (2007). Levitan and Shubik always

[24] For Bertrand $b_j \to b_j + 1/\beta_j$.

consider $\sigma > 0$ but with their symmetric specification, the matrix $\mathbf{B}$ is positive definite for any $\sigma \in \left]-\infty, 1\right[$.[25]

Second, there is the version of Spence (1976a):

$$U^{\text{Sp}} - \sum p_i q_i = \sum (a - p_i)\, q_i - \sigma \sum_i \sum_{j>i} q_i q_j - \frac{b}{2} \sum q_i^2 \tag{12}$$

Spence assumes homogenous marginal costs. But this formulation has also been used in the literature with heterogenous $\mathbf{a}$ and/or $\mathbf{c}$. This model is the most popular. For example, Majerus (1988) and later Häckner (2000) (see also Hsu and Wang (2005), and many others) who all "simplify" further by assuming $b = 1$. One as to assume $\sigma \in \left]\frac{-b}{n-1}, b\right[$ for the matrix $\mathbf{B}$ to be definite positive.

Third, there is the formulation of Sutton (1997) (see also Sutton (1996) but the working paper of the 1997 RAND article pre-dates the 1996 EER article):

$$U^{\text{Su}} - \sum \tilde{p}_i q_i = \sum (1 - \tilde{p}_i)\, x_i - \sigma \sum_i \sum_{j>i} \frac{x_i}{u_i} \frac{x_j}{u_j} - \frac{1}{2} \sum_i \frac{x_i^2}{u_i^2}$$

with $u_i > 0$ (and where the usual $1/2$ that Sutton does not use is introduced). See also Symeonidis (1999), Symeonidis (2003b) (duopoly), and Symeonidis (2003a).

Fact 11 (Similarity of the common models)**.** *A natural simplification of the general model* (4 bis) *encompasses these three models. It takes the following form:*

$$U^{\mathbf{b}} = \sum (a_i - p_i)\, q_i - \sigma \sum_i \sum_{j>i} q_i q_j - \frac{1}{2} \sum b_i q_i^2 \tag{13}$$

Proof. To see that $U^{\mathbf{q}}$ encompasses U^{Su}, just write $q_i = \frac{x_i}{u_i}$ and $p_i = u_i \tilde{p}_i$ (the marginal cost should also be normalized)[26] it comes that Sutton's expression writes:

$$U^{\text{Su}} = \sum (u_i - p_i)\, q_i - \sigma \sum_i \sum_{j>i} q_i q_j - \frac{1}{2} \sum_i q_i^2$$

which is exactly $U^{\mathbf{q}}$ with $a_i = u_i$ and $b_i = 1$. □

In (13), the substitution effects are assumed to be the same for any two goods, i.e. for all i, j, $i \neq j$, $\frac{\partial^2 U}{\partial q_i \partial q_j} = -\sigma$ as in all the other simplified forms, but the concavity with respect to each q_i are allowed to vary, as $\frac{\partial^2 U}{\partial q_i^2} = -b_i$. The attractiveness of a variety in this model is therefore bi-dimensional. The higher $v_i = a_i - c_i$, the more (marginal) surplus is offered by good i and the more should be consumed in equilibrium. The second aspect is b_i. A larger b_i implies that additional units are less and less valued which limits consumption. The best products are those with large $a_i - c_i$ and low b_i.[27]

In comparison, in Levitan and Shubik formulation, it is assumed that $b_i \equiv \frac{\sigma}{\beta} + \frac{1-\sigma}{\beta w_i}$, a rather awkward assumption. The heterogeneity comes from the w_i but the fact that b_i vary with σ is not

[25] In fact, they use a parameter γ which correspond to $\sigma/(1-\sigma)$ and they (implicitly) assume γ between 0 and $+\infty$. The model allows γ between -1 and and $+\infty$.

[26] The normalization is innocuous. For example, if x_i is measured in grams, and $u_i = 1000$, then q_i is measured in kilograms. For the price, if $\tilde{p}_i$ is the price for one gram, then $1000\tilde{p}_i$ is indeed the price for one kilogram.

[27] Notice that for $n = 2$, this formulation is the most general one and this $n = 2$-version is used by Dixit (1979) and Singh and Vives (1984).

intuitive as σ measures already the differentiation between two goods. Therefore one would prefer not to mix the two. Furthermore as they impose $\sum w_i = 1$ it also means that the b_i vary with n.

Figure 1 summarizes the links between the various QQUM encountered in practice. At first, it seems that Levitan-Shubik nonsymmetric model encompasses Spence's one. Indeed, choosing $\beta = 1$, keeping σ and making $w_i = w = \frac{b-\sigma}{1-\sigma}$ allows to move from Levitan-Shubik to Spence. The only caveat is that in Levitan-Shubik to have $b_i = b$ for all i you need $\sum w_i = 1$ and this fixes a particular value of b, i.e. $b = \frac{1+(n-1)\sigma}{n}$. So in fact, it is Spence's formulation which encompasses the symmetric formulation of Levitan-Shubik. It is also straightforward to see that Spence's formulation encompasses Sutton's one.

A fourth model is presented in Amir, Erickson, and Jin (2017) (see their section 7) where it is attributed to Bresnahan (1987). They assume that the matrix $\mathbf{B}$ is such that $b_{ij} = \sigma^{|i-j|}$. They call it "a linear demand with local interaction" or the KMS model after the name of the matrix which is a Kac–Murdock–Szegö matrix or an asymmetric n-Toeplitz matrix. They show that, in addition to its own price, the demand of firm i, $2 \le i \le n-1$ only depends on the prices of $i-1$ and $i+1$, whereas the demand for firm 1 (resp. n) depends only on its price and the price of firm 2 (resp. $n-1$). This is reminiscent of an Hotelling model. The asymmetry between the extreme varieties $i = 1$ and $i = n$ and the varieties in the middle $2 \le i \le n-1$ is not necessarily intuitive outside particular cases. Formally,

$$U^{\text{KMS}} = \sum (a_i - p_i)\, q_i - \sum_i \sum_{j>i} \sigma^{|i-j|} q_i q_j - \frac{1}{2} \sum_i q_i^2 \ , \text{ and}$$

$$\begin{aligned}
(1-\sigma^2)q_1 &= a_1 - p_1 - \sigma(a_2 - p_2) \\
(1-\sigma^2)q_i &= -\sigma(a_{i-1} - p_{i-1}) + (1+\sigma^2)(a_i - p_i) - \sigma(a_{i+1} - p_{i+1}) \ \text{ for } 2 \le i \le n-1 \\
(1-\sigma^2)q_n &= -\sigma(a_{n-1} - p_{n-1}) + a_n - p_n
\end{aligned}$$

The local interaction property would disappear if $b_{ii} = b \neq 1$, however.

5 Symmetric formulations: ready-to-use formulae

As in practice the most used models are the symmetric ones (either Spence's/Sutton's utility, or Levitan and Shubik's symmetric utility), in this section we solve completely for the Nash equilliria of the Bertrand and Cournot oligopoly games with heterogenous $\mathbf{a}$ and $\mathbf{c}$.[28] The main results are presented in a compact way in Table 2. Proofs are in the Appendix.

By symmetric formulations, we mean for all i, $b_i = b$ and for all i, j, $j \neq i$, $b_{ij} = \sigma$ (these two constants b and σ are not the same in the two formulations which make them different). Heterogeneity is allowed for $\mathbf{a}$ and $\mathbf{c}$. That is, there is nothing to gain, in term of simplification, by assuming both that $a_i = a_j = a$ and $c_i = c_j = c$, on the contrary.

Fact 12 (Mean-Variance structure of equilibrium values)**.** *Equilibrium surplus, profit, and welfare are all usefully written in terms of mean, $\overline{\mathbf{v}}$, and variance, $Var(\mathbf{v})$, of the marginal surpluses $\mathbf{v}$.*[29] *Table 2 summarizes the symmetric Levitan-Shubik's and Spence's formulations.*[30]

[28] In another working paper, we solve for closed-form equilibrium expression for the utility function given by (13) (the flexible nonsymmetric utility of Figure 1), also with heterogenous $\mathbf{a}$ and $\mathbf{c}$.

[29] Using, for any vectors $\mathbf{x}$, $\mathbf{y}$ the notation $\overline{\mathbf{x}} = \frac{1}{n}\sum_i x_i$ for the mean and $\text{Var}(\mathbf{x}) = \frac{1}{n}\sum_i (x_i - \overline{\mathbf{x}})^2$ for the variance and $\text{Cov}(\mathbf{x}, \mathbf{y}) = \frac{1}{n}\sum_i (x_i - \overline{\mathbf{x}})(y_i - \overline{\mathbf{y}})$ for the covariance.

[30] Individual equilibrium profits are not given in the Table but they are easily obtained by multiplying the price margin with the quantity which are proportional with one another.

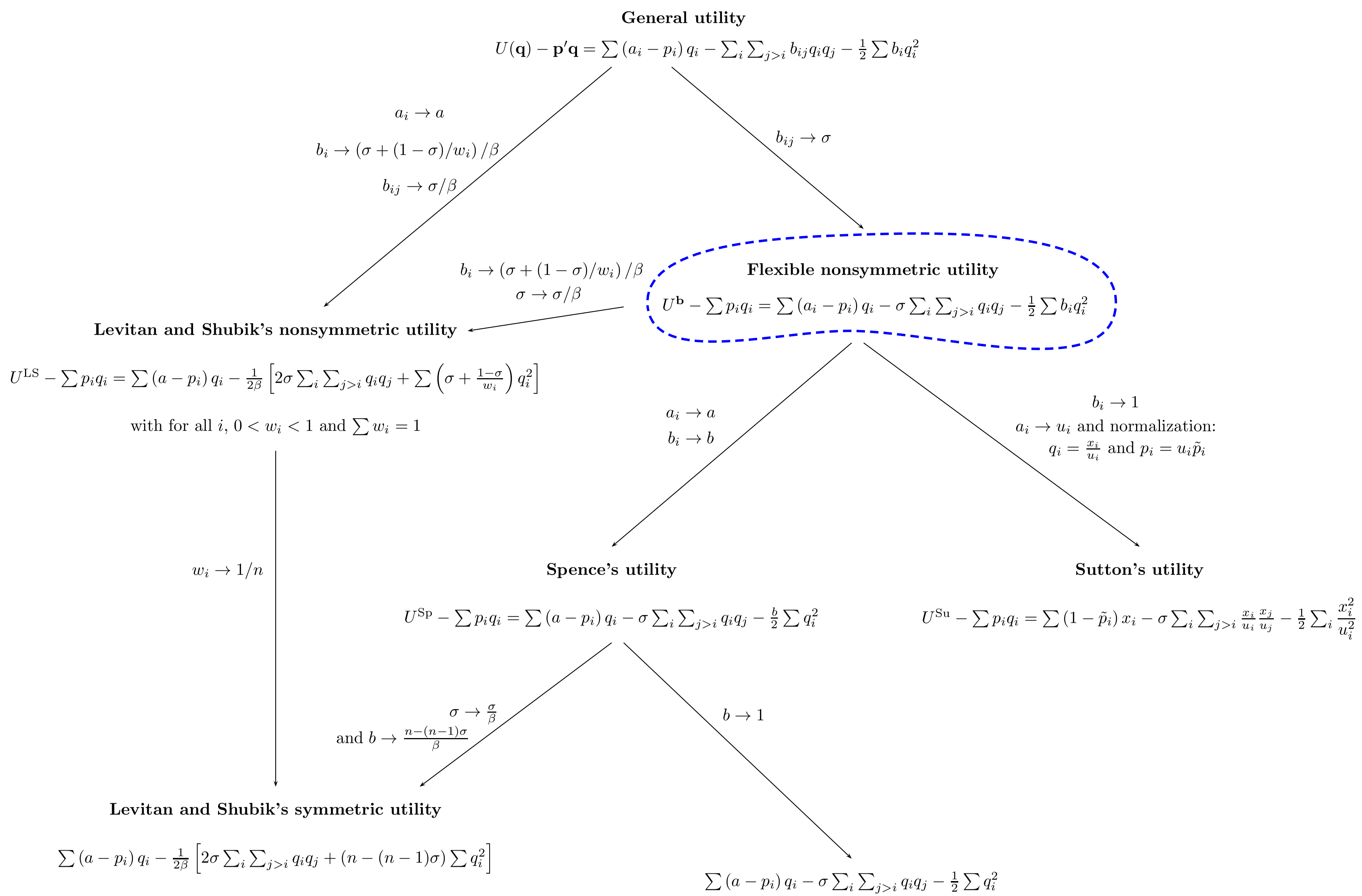

Figure 1: Links between the various formulations

Because of this property, the formulae are more intuitive with heterogenous v_i than with a constant one. This mean-variance property was first noticed for the Cournot model, where it holds even for a general demand, by Linnemer (2003) and Valletti (2003). Valletti also gives the result for QQUM, in a variant close to symmetric formulation (using $b-\sigma$ instead of b). It could be used to study the effects of shocks on v_i in the spirit of Zhao (2001) and Février and Linnemer (2004).

As discussed briefly in the previous section, Spence's formulation (12) is more natural as the coefficient of the q_i^2 terms does not, a priori, vary with the coefficient of the q_iq_j terms. That is b is, a priori, independent of σ nor n. Also, as shown in Figure 1, one can derive the Levitan-Shubik's symmetric formulation from Spence's one by a change of parameters. For this reason, we recommend to use the more general Spence's formulation.

Fact 13 (Main property of the Spence's formulation)**.** *When* $\mathbf{B}$ *is the Spence's matrix, for any vectors* $\mathbf{x}$ *and* $\mathbf{y}$*:*

$$\mathbf{x}'\mathbf{B}^{-1}\mathbf{y} = \frac{n}{(b-\sigma)}\mathrm{Cov}(\mathbf{x},\mathbf{y}) + \frac{n}{(b+(n-1)\sigma)}\overline{\mathbf{x}}\,\overline{\mathbf{y}} \tag{14}$$

therefore, using (A.1) *and* (A.2)*, for any* $\mathbf{p}$ *such that* $\mathbf{q}(\mathbf{p}) > 0$*,*

- $2V(\mathbf{p}) = \frac{n}{(b-\sigma)}\mathrm{Var}(\mathbf{a}-\mathbf{p}) + \frac{n}{(b+(n-1)\sigma)}\overline{\mathbf{a}-\mathbf{p}}^2$
- $\Pi(\mathbf{p}) = \frac{n}{(b-\sigma)}\mathrm{Cov}(\mathbf{p}-\mathbf{c},\mathbf{a}-\mathbf{p}) + \frac{n}{(b+(n-1)\sigma)}\left(\overline{\mathbf{p}-\mathbf{c}}\right)\left(\overline{\mathbf{a}-\mathbf{p}}\right)$.

and therefore the First-best and Monopoly values are

- $q_i^* = 2q_i^m = \frac{v_i-\overline{\mathbf{v}}}{(b-\sigma)} + \frac{\overline{\mathbf{v}}}{(b+(n-1)\sigma)}$
- $2W^* = \frac{8}{3}W^m = 4\Pi^m = \frac{n}{(b-\sigma)}\mathrm{Var}(\mathbf{v}) + \frac{n}{(b+(n-1)\sigma)}\overline{\mathbf{v}}^2$.

Dealing with a variation of n. This is an important research question and we just touch on the subject here. When one more variety is added, there are two main effects. First, there is a variety effect: the utility tends to increase with one more product. Second, there is a competition effect: prices tend to decrease with the arrival of one more competitor. A priori, it is not easy to disentangle them. To study the variety effect, one can look at the variation of surplus/welfare for first-best prices (or monopoly prices) as they do not vary with n. In that case there are three sources of variations: i) the average marginal surplus, $\overline{\mathbf{v}}$, could increase of decrease, ii) the variance of these surplus, $\mathrm{Var}(\mathbf{v})$, could also increase of decrease, and finally iii) there is a direct positive effect of n. For example, the introduction of a new variety such that $v_{n+1} = \overline{\mathbf{v}}$ implies that the mean does not change, and that $n\mathrm{Var}(\mathbf{v}_n) = (n+1)\mathrm{Var}(\mathbf{v}_{n+1})$, but W^* increases.

Fact 14 (Additional property of Levitan-Shubik's formulation)**.** *In the LS-formulation, the choice of* $b \equiv n-(n-1)\sigma$ *implies* $\frac{n}{(b-\sigma)} \equiv \frac{1}{(1-\sigma)}$ *and* $\frac{n}{(b+(n-1)\sigma)} \equiv 1$ *(and* $\sigma < 1$*). Therefore, for any* $\mathbf{p}$ *such that* $\mathbf{q}(\mathbf{p}) > 0$*,*

- $2V(\mathbf{p}) = \frac{1}{(1-\sigma)}\mathrm{Var}(\mathbf{a}-\mathbf{p}) + \overline{\mathbf{a}-\mathbf{p}}^2$
- $\Pi(\mathbf{p}) = \frac{1}{(1-\sigma)}\mathrm{Cov}(\mathbf{p}-\mathbf{c},\mathbf{a}-\mathbf{p}) + \left(\overline{\mathbf{p}-\mathbf{c}}\right)\left(\overline{\mathbf{a}-\mathbf{p}}\right)$.

The parameter b increases with n, which decreases the utility, and cancels out the positive effect of an additional variety. However, the introduction of a $n+1$th variety cannot leave both the variance and the mean constant. The competition effect would be the only one only in the (very) symmetric case where for all i, $v_i = \overline{\mathbf{v}}$ (no variance term). So, indeed, one could want to switch to this special case of the LS-formulation to focus on the competition effect. Yet, notice a last drawback of LS-formulation: when $n \to +\infty$ (and $v_i = \overline{\mathbf{v}}$) the equilibrium prices do not converge to marginal costs.

Table 2: The two commonly used models with heterogenous $\mathbf{a}$ and $\mathbf{c}$

Levitan and Shubik	**Spence**
Indirect demands:	
$\begin{aligned} p_i &= a_i - \frac{n}{\beta}\left((1-\sigma)q_i + \sigma\overline{\mathbf{q}}\right) \\ &= a_i - \frac{n}{\beta}\left(q_i + \sigma\left(\overline{\mathbf{q}} - q_i\right)\right) \\ &= a_i - \frac{n}{\beta}\left(\overline{\mathbf{q}} + (1-\sigma)\left(q_i - \overline{\mathbf{q}}\right)\right) \end{aligned}$	$\begin{aligned} p_i &= a_i - (b-\sigma)q_i - n\sigma\overline{\mathbf{q}} \\ &= a_i - (b+(n-1)\sigma)\,q_i - n\sigma\left(\overline{\mathbf{q}} - q_i\right) \\ &= a_i - (b+(n-1)\sigma)\,\overline{\mathbf{q}} - (b-\sigma)\left(q_i - \overline{\mathbf{q}}\right) \end{aligned}$
Direct demands:	
$\begin{aligned} q_i &= \frac{\beta}{n(1-\sigma)}\left[(a_i-p_i) - \sigma\left(\overline{\mathbf{a}-\mathbf{p}}\right)\right] \\ &= \frac{\beta}{n}\left[(a_i-p_i) + \frac{\sigma}{1-\sigma}\left((a_i-p_i) - (\overline{\mathbf{a}-\mathbf{p}})\right)\right] \\ &= \frac{\beta}{n}\left[\overline{\mathbf{a}-\mathbf{p}} + \frac{\left((a_i-p_i)-(\overline{\mathbf{a}-\mathbf{p}})\right)}{1-\sigma}\right] \end{aligned}$	$\begin{aligned} q_i &= \frac{1}{b-\sigma}\left[(a_i-p_i) - \frac{n\sigma}{b+(n-1)\sigma}\left(\overline{\mathbf{a}-\mathbf{p}}\right)\right] \\ &= \frac{1}{b+(n-1)\sigma}\left[(a_i-p_i) + \frac{n\sigma}{b-\sigma}\left((a_i-p_i) - (\overline{\mathbf{a}-\mathbf{p}})\right)\right] \\ &= \frac{\overline{\mathbf{a}-\mathbf{p}}}{b+(n-1)\sigma} + \frac{\left((a_i-p_i)-(\overline{\mathbf{a}-\mathbf{p}})\right)}{b-\sigma} \end{aligned}$
Equilibrium quantities and prices under Cournot competition:	
$\begin{aligned} q_i^C &= \frac{\beta}{2n-(2n-1)\sigma}\left[v_i - \frac{n\sigma}{2n-(n-1)\sigma}\overline{\mathbf{v}}\right] \\ &= \frac{\beta}{2n-(2n-1)\sigma}\left[(v_i - \overline{\mathbf{v}}) + \frac{2n-(2n-1)\sigma}{2n-(n-1)\sigma}\overline{\mathbf{v}}\right] \end{aligned}$	$\begin{aligned} q_i^C &= \frac{1}{2b-\sigma}\left[v_i - \frac{n\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\right] \\ &= \frac{1}{2b-\sigma}\left[(v_i - \overline{\mathbf{v}}) + \frac{2b-\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\right] \end{aligned}$
$\begin{aligned} p_i^C - c_i &= \frac{n-(n-1)\sigma}{2n-(2n-1)\sigma}\left[v_i - \frac{n\sigma}{2n-(n-1)\sigma}\overline{\mathbf{v}}\right] \\ &= \frac{n-(n-1)\sigma}{2n-(2n-1)\sigma}\left[(v_i - \overline{\mathbf{v}}) + \frac{2n-(2n-1)\sigma}{2n-(n-1)\sigma}\overline{\mathbf{v}}\right] \end{aligned}$	$\begin{aligned} p_i^C - c_i &= \frac{b}{2b-\sigma}\left[v_i - \frac{n\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\right] \\ &= \frac{b}{2b-\sigma}\left[(v_i - \overline{\mathbf{v}}) + \frac{2b-\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\right] \end{aligned}$
Equilibrium quantities and prices under Bertrand competition:	
$\begin{aligned} q_i^B &= \frac{\beta}{2n-(2n-1)\sigma-\Psi}\left[v_i - \frac{n\sigma}{2n-(n-1)\sigma-\Psi}\overline{\mathbf{v}}\right] \\ &= \frac{\beta}{2n-(2n-1)\sigma-\Psi}\left[(v_i - \overline{\mathbf{v}}) + \frac{2n-(2n-1)\sigma-\Psi}{2n-(n-1)\sigma-\Psi}\overline{\mathbf{v}}\right] \end{aligned}$	$\begin{aligned} q_i^B &= \frac{1}{2b-\sigma-\Phi}\left[v_i - \frac{n\sigma}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\right] \\ &= \frac{1}{2b-\sigma-\Phi}\left[(v_i - \overline{\mathbf{v}}) + \frac{2b-\sigma-\Phi}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\right] \end{aligned}$
where $\Psi = \frac{(n-1)\sigma^2}{n-\sigma}$	**where** $\Phi = \frac{(n-1)\sigma^2}{b+(n-2)\sigma}$
$\begin{aligned} p_i^B - c_i &= \frac{n-(n-1)\sigma-\Psi}{2n-(2n-1)\sigma-\Psi}\left[v_i - \frac{n\sigma}{2n-(n-1)\sigma-\Psi}\overline{\mathbf{v}}\right] \\ &= \frac{n-(n-1)\sigma-\Psi}{2n-(2n-1)\sigma-\Psi}\left[(v_i - \overline{\mathbf{v}}) + \frac{2n-(2n-1)\sigma-\Psi}{2n-(n-1)\sigma-\Psi}\overline{\mathbf{v}}\right] \end{aligned}$	$\begin{aligned} p_i^B - c_i &= \frac{b-\Phi}{2b-\sigma-\Phi}\left[v_i - \frac{n\sigma}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\right] \\ &= \frac{b-\Phi}{2b-\sigma-\Phi}\left[(v_i - \overline{\mathbf{v}}) + \frac{2b-\sigma-\Phi}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\right] \end{aligned}$
Equilibrium surplus under Cournot and Bertrand competition:	
$2V_C = \frac{n^2(1-\sigma)\beta}{(2n-(2n-1)\sigma)^2}\mathrm{Var}(\mathbf{v}) + \frac{n^2}{(2n-(n-1)\sigma)^2}\overline{\mathbf{v}}^2$	$2V_C = \frac{n(b-\sigma)}{(2b-\sigma)^2}\mathrm{Var}(\mathbf{v}) + \frac{n(b+(n-1)\sigma)}{(2b+(n-1)\sigma)^2}\overline{\mathbf{v}}^2$
$2V_B = \frac{n^2(1-\sigma)\beta}{(2n-(2n-1)\sigma-\Psi)^2}\mathrm{Var}(\mathbf{v}) + \frac{n^2}{(2n-(n-1)\sigma-\Psi)^2}\overline{\mathbf{v}}^2$	$2V_B = \frac{n(b-\sigma)}{(2b-\sigma-\Phi)^2}\mathrm{Var}(\mathbf{v}) + \frac{n(b+(n-1)\sigma)}{(2b+(n-1)\sigma-\Phi)^2}\overline{\mathbf{v}}^2$
Equilibrium aggregate profit under Cournot and Bertrand competition:	
$\Pi_C = \frac{n(n-(n-1)\sigma)\beta}{(2n-(2n-1)\sigma)^2}\left[\mathrm{Var}(\mathbf{v}) + \frac{(2n-(2n-1)\sigma)^2\overline{\mathbf{v}}^2}{(2n-(n-1)\sigma)^2}\right]$	$\Pi_C = \frac{nb}{(2b-\sigma)^2}\left[\mathrm{Var}(\mathbf{v}) + \frac{(2b-\sigma)^2\overline{\mathbf{v}}^2}{(2b+(n-1)\sigma)^2}\right]$
$\Pi_B = \frac{n(n-(n-1)\sigma-\Psi)\beta}{(2n-(2n-1)\sigma-\Psi)^2}\left[\mathrm{Var}(\mathbf{v}) + \frac{(2n-(2n-1)\sigma-\Psi)^2\overline{\mathbf{v}}^2}{(2n-(n-1)\sigma-\Psi)^2}\right]$	$\Pi_B = \frac{n(b-\Phi)}{(2b-\sigma-\Phi)^2}\left[\mathrm{Var}(\mathbf{v}) + \frac{(2b-\sigma-\Phi)^2\overline{\mathbf{v}}^2}{(2b+(n-1)\sigma-\Phi)^2}\right]$
Equilibrium welfare under Cournot and Bertrand competition:	
$2\mathrm{W}_C = \frac{n\beta(3n-(3n-2)\sigma)}{(2n-(2n-1)\sigma)^2}\,\mathrm{Var}(\mathbf{v}) + \frac{n\beta(3n-2(n-1)\sigma)}{(2n-(n-1)\sigma)^2}\,\overline{\mathbf{v}}^2$	$2\mathrm{W}_C = \frac{n(3b-\sigma)}{(2b-\sigma)^2}\,\mathrm{Var}(\mathbf{v}) + \frac{n(3b+(n-1)\sigma)}{(2b+(n-1)\sigma)^2}\,\overline{\mathbf{v}}^2$
$2\mathrm{W}_B = \frac{n\beta(3n-(3n-2)\sigma-2\Psi)}{(2n-(2n-1)\sigma-\Psi)^2}\,\mathrm{Var}(\mathbf{v}) + \frac{n\beta(3b+(n-1)\sigma-2\Psi)}{(2n-(n-1)\sigma-\Psi)^2}\,\overline{\mathbf{v}}^2$	$2\mathrm{W}_B = \frac{n(3b-\sigma-2\Phi)}{(2b-\sigma-\Phi)^2}\,\mathrm{Var}(\mathbf{v}) + \frac{n(3b+(n-1)\sigma-2\Phi)}{(2b+(n-1)\sigma-\Phi)^2}\,\overline{\mathbf{v}}^2$

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Appendix

A Surplus, profit, welfare

Consumers' surplus, Aggregate profits, and Welfare Using (6), the indirect utility function as a function of prices.

$$V(\mathbf{p}) = \max_{\mathbf{q}} U(\mathbf{q}) - \mathbf{p}'\mathbf{q} = \frac{1}{2}(\mathbf{a}-\mathbf{p})'\mathbf{B}^{-1}(\mathbf{a}-\mathbf{p}) = \frac{1}{2}(\mathbf{q}(\mathbf{p}))'\mathbf{B}(\mathbf{q}(\mathbf{p})) \tag{A.1}$$

As $\mathbf{B}^{-1}$ is also a $n \times n$ positive definite matrix, $V(\mathbf{p})$ is a quadratic form in $\mathbf{a}-\mathbf{p}$.

Aggregate profit (i.e. the sum of all profits) is:

$$\Pi(\mathbf{p}) = (\mathbf{p}-\mathbf{c})'\mathbf{B}^{-1}(\mathbf{a}-\mathbf{p}) = (\mathbf{a}-\mathbf{c})'\mathbf{q}(\mathbf{p}) - \mathbf{q}(\mathbf{p})'\mathbf{B}\,\mathbf{q}(\mathbf{p}) \tag{A.2}$$

and therefore welfare writes:

$$W(\mathbf{p}) = \frac{1}{2}(\mathbf{a}-\mathbf{c}+\mathbf{p}-\mathbf{c})'\mathbf{B}^{-1}(\mathbf{a}-\mathbf{p}) = (\mathbf{a}-\mathbf{c})'\mathbf{q}(\mathbf{p}) - \frac{1}{2}\mathbf{q}(\mathbf{p})'\mathbf{B}\,\mathbf{q}(\mathbf{p}) \tag{A.3}$$

These expressions (A.1), (A.2), and (A.3) hold independently of the type of competition, or the ownership structure of firms, for any positive semi-definite matrix $\mathbf{B}$.

Consumers' surplus, Aggregate profits, and Welfare in equilibrium First, combining (A.1) with (9) and (10) leads to (using the fact that for a symmetric matrix $\mathbf{M}' = \mathbf{M}$)

$$\begin{aligned} 2V^{\mathrm{C}} &= \mathbf{v}' \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{B} \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{v} && \text{(A.4)} \\ 2V^{\mathrm{B}} &= \mathbf{v}' \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} \mathbf{B} \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} \mathbf{v} && \text{(A.5)} \end{aligned}$$

Next, combining (A.1) with (9) and (10) leads to

$$\begin{aligned} \Pi^{\mathrm{C}} &= \mathbf{v}' \left[\left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} - \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{B} \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1}\right] \mathbf{v} \\ &= \mathbf{v}' \left[\left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{diag}(\mathbf{b}) \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1}\right] \mathbf{v} \end{aligned} \tag{A.6}$$

$$\begin{aligned} \Pi^{\mathrm{B}} &= \mathbf{v}' \left[\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} - \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} \mathbf{B} \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\right] \mathbf{v} \\ &= \mathbf{v}' \left[\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} \mathbf{diag}(\beta)^{-1} \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\right] \mathbf{v} \end{aligned} \tag{A.7}$$

Finally, combining (A.1) with (9) and (10) leads to

$$\begin{aligned} W^{\mathrm{C}} &= \mathbf{v}' \left[\left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} - \frac{1}{2} \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{B} \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1}\right] \mathbf{v} \\ 2W^{\mathrm{C}} &= \mathbf{v}' \left[\left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \left(\mathbf{B} + 2\,\mathbf{diag}(\mathbf{b})\right) \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1}\right] \mathbf{v} \end{aligned} \tag{A.8}$$

$$\begin{aligned} W^{\mathrm{B}} &= \mathbf{v}' \left[\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} - \frac{1}{2} \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} \mathbf{B} \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\right] \mathbf{v} \\ 2W^{\mathrm{B}} &= \mathbf{v}' \left[\left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1} \left(\mathbf{B} + 2\,\mathbf{diag}(\beta)^{-1}\right) \left(\mathbf{B} + \mathbf{diag}(\beta)^{-1}\right)^{-1}\right] \mathbf{v} \end{aligned} \tag{A.9}$$

B F.o.c. with multi-product firms

Cournot competition Recall that I_k is the set of product under the control of firm k. We introduce the following r matrices. For all k, $1 \leq k \leq r$, let $\mathbf{B_k} = (b_{ij})_{i,j \in I_k}$. That is, $\mathbf{B_k}$ is a square matrix of size $\#I_k$ the number of varieties under the control of firm k. If each firm owns only one good, $\mathbf{B_k}$ is simply $b_{kk} = b_k$. Next, let $\mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r})$ denote the block diagonal matrix, whose blocks are $\mathbf{B_k}$, $k = 1, \cdots, r$. The matrix $\mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r})$ is positive definite. Computations similar to the ones leading to (9) now gives:

$$\mathbf{p} - \mathbf{c} = \mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r})\mathbf{q}$$

and therefore, using $\mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} - \mathbf{Bq}$,

$$\left(\mathbf{B} + \mathbf{diag}(\mathbf{B_1}, \ldots, \mathbf{B_r})\right) \mathbf{q} = \mathbf{a} - \mathbf{c} = \mathbf{v} \tag{B.1}$$

Bertrand competition Recall that I_k is the set of product under the control of firm k. We introduce the following r matrices. For all k, $1 \leq k \leq r$, let $\mathcal{B}_{\mathbf{k}} = (\beta_{ij})_{i,j \in I_k}$ where the β_{ij} are the elements of matrix $\mathbf{B^{-1}}$. That is, $\mathcal{B}_{\mathbf{k}}$ is a square matrix of size $\# I_k$ the number of varieties under the control of firm k. If each firm owns only one good, $\mathcal{B}_{\mathbf{k}}$ is simply $\beta_{kk} = \beta_k$. Next, let $\mathbf{diag}(\mathcal{B}_{\mathbf{1}}, \dots, \mathcal{B}_{\mathbf{r}})$ denote the block diagonal matrix, whose blocks are $\mathcal{B}_{\mathbf{k}}$, $k = 1, \cdots, r$. The matrix $\mathbf{diag}(\mathcal{B}_{\mathbf{1}}, \dots, \mathcal{B}_{\mathbf{r}})$ is positive definite. Computations similar to the ones leading to (9) now gives:

$$\mathbf{q} = \mathbf{diag}(\mathcal{B}_{\mathbf{1}}, \dots, \mathcal{B}_{\mathbf{r}})(\mathbf{p} - \mathbf{c})$$

and therefore, using $\mathbf{p} - \mathbf{c} = \mathbf{a} - \mathbf{c} - \mathbf{Bq}$,

$$\left(\mathbf{B} + \mathbf{diag}(\mathcal{B}_{\mathbf{1}}, \dots, \mathcal{B}_{\mathbf{r}})^{-1}\right) \mathbf{q} = \mathbf{a} - \mathbf{c} = \mathbf{v} \tag{B.2}$$

where, as in the text, prices have been eliminated in order to show the similarity of the Bertrand and Cournot characterization. The only difference from the Cournot f.o.c. is that $\mathbf{diag}(\mathbf{B_1}, \dots, \mathbf{B_r})$ as been replaced by $\mathbf{diag}(\mathcal{B}_{\mathbf{1}}, \dots, \mathcal{B}_{\mathbf{r}})^{-1}$.

C Common symmetric model

In section 3, we have shown how to find the Cournot-Nash or Bertrand-Nash equilibrium of QQUM. So here we only have to compute the relevant matrices.

First, notice that

$$\mathbf{B} = \begin{pmatrix} b & \sigma & \cdots & \cdots & \sigma \\ \sigma & \ddots & \ddots & & \vdots \\ \vdots & \ddots & b & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \sigma \\ \sigma & \cdots & \cdots & \sigma & b \end{pmatrix} \text{ using } \mathbf{I} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \text{ and } \mathbf{J} = \begin{pmatrix} 1 & 1 & \cdots & \cdots & 1 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1 & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 \\ 1 & \cdots & \cdots & 1 & 1 \end{pmatrix}$$

one can write

$$\mathbf{B} = (b - \sigma)\mathbf{I} + \sigma \mathbf{J}$$

It is important to emphasize, here, a convenient property of the matrix $\mathbf{J}$, namely that it is a sum or, if one divides by n, a mean operator. This property plays a key role in the analysis. Indeed, let $\mathbf{u}' = (1, \cdots, 1)$, then for any vector $\mathbf{x}$,

$$\frac{1}{n}\mathbf{Jx} = \overline{\mathbf{x}}\,\mathbf{u} \text{ and } \frac{1}{n}\mathbf{u}'\,\mathbf{Jx} = \overline{\mathbf{x}} \text{ and } \mathbf{x}'\mathbf{Jy} = n^2 \overline{\mathbf{x}}\,\overline{\mathbf{y}}$$

Direct and inverse demands The expression of the inverse demand functions given at the top of Table 2:

$$\mathbf{p} = \mathbf{a} - \mathbf{Bq} = \mathbf{a} - ((b - \sigma)\mathbf{I} + \sigma\mathbf{J})\,\mathbf{q} = \mathbf{a} - (b - \sigma)\mathbf{q} + n\sigma\overline{\mathbf{x}}\,\mathbf{u}$$

It is readily confirmed (one can simply verify it by computing $\mathbf{BB}^{-1}$, using $\mathbf{J}^2 = n\mathbf{J}$) that

$$\mathbf{B}^{-1} = \frac{1}{b - \sigma}\left(\mathbf{I} - \frac{\sigma}{b + (n-1)\sigma}\mathbf{J}\right)$$

this gives the demand functions given at the top of Table 2:

$$\mathbf{q} = \mathbf{B}^{-1}(\mathbf{a} - \mathbf{p}) = \frac{1}{b - \sigma}\left(\mathbf{I} - \frac{\sigma}{b + (n-1)\sigma}\mathbf{J}\right)(\mathbf{a} - \mathbf{p}) = \frac{1}{b - \sigma}\left((\mathbf{a} - \mathbf{p}) - \frac{n\sigma}{b + (n-1)\sigma}(\overline{\mathbf{a}} - \overline{\mathbf{p}})\right)$$

We can also compute $\mathbf{x}'\mathbf{B}^{-1}\mathbf{y}$ to prove Fact 13. Using $\mathbf{x}'\mathbf{y} = n\text{Cov}(\mathbf{x},\mathbf{y}) + n\overline{\mathbf{x}}\,\overline{\mathbf{y}}$.

$$\begin{aligned}
\mathbf{x}'\mathbf{B}^{-1}\mathbf{y} &= \frac{1}{b-\sigma}\left(\mathbf{x}'\mathbf{y} - \frac{\sigma}{b+(n-1)\sigma}\mathbf{x}'\mathbf{J}\mathbf{y}\right) \\
&= \frac{1}{b-\sigma}\left(n\text{Cov}(\mathbf{x},\mathbf{y}) + n\overline{\mathbf{x}}\,\overline{\mathbf{y}} - \frac{n^2\sigma}{b+(n-1)\sigma}\overline{\mathbf{x}}\,\overline{\mathbf{y}}\right) \\
&= \frac{1}{b-\sigma}\left(n\text{Cov}(\mathbf{x},\mathbf{y}) + \left(1 - \frac{n\sigma}{b+(n-1)\sigma}\right)n\overline{\mathbf{x}}\,\overline{\mathbf{y}}\right) \\
&= \frac{n}{b-\sigma}\text{Cov}(\mathbf{x},\mathbf{y}) + \frac{n}{b+(n-1)\sigma}\overline{\mathbf{x}}\,\overline{\mathbf{y}}
\end{aligned}$$

Cournot equilibrium prices and quantities As shown by (9), one only needs to compute the inverse of $\mathbf{B} + \mathbf{diag}(\mathbf{b})$ in order to compute the equilibrium quantities in the Cournot game. This is straightforward,

$$(\mathbf{B} + \mathbf{diag}(\mathbf{b}))^{-1} = ((2b-\sigma)\mathbf{I} + \sigma\mathbf{J})^{-1} = \frac{1}{2b-\sigma}\left(\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right)$$

hence

$$\mathbf{q}^C = \frac{1}{2b-\sigma}\left(\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right)\mathbf{v} = \frac{1}{2b-\sigma}\left(\mathbf{v} - \frac{n\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\mathbf{u}\right)$$

and

$$\mathbf{p}^C - \mathbf{c} = \mathbf{diag}(\mathbf{b})\mathbf{q}^C = \frac{b}{2b-\sigma}\left(\mathbf{v} - \frac{n\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\mathbf{u}\right)$$

Bertrand equilibrium prices and quantities As shown by (10), one only needs to compute the inverse of $\mathbf{B} + \mathbf{diag}(\boldsymbol{\beta})^{-1}$ in order to compute the equilibrium quantities in the Bertrand game. This is again straightforward (although a little bit cumbersome),

$$\begin{aligned}
\left(\mathbf{B} + \mathbf{diag}(\boldsymbol{\beta})^{-1}\right)^{-1} &= \left(\frac{(b-\sigma)(2b+(2n-3)\sigma)}{b+(n-2)\sigma}\mathbf{I} + \sigma\mathbf{J}\right)^{-1} \\
&= ((2b-\sigma-\Phi)\,\mathbf{I} + \sigma\mathbf{J})^{-1} \\
&= \frac{1}{2b-\sigma-\Phi}\left(\mathbf{I} - \frac{\sigma}{2b-(n-1)\sigma-\Phi}\mathbf{J}\right)
\end{aligned}$$

where

$$\Phi = \frac{(n-1)\sigma^2}{b+(n-2)\sigma}$$

hence

$$\mathbf{q}^B = \frac{1}{2b-\sigma-\Phi}\left(\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma-\Phi}\mathbf{J}\right)\mathbf{v} = \frac{1}{2b-\sigma-\Phi}\left(\mathbf{v} - \frac{n\sigma}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\mathbf{u}\right)$$

and

$$\mathbf{p}^B - \mathbf{c} = \mathbf{diag}(\mathbf{B}^{-1})^{-1}\mathbf{q}^B = \frac{b-\Phi}{2b-\sigma-\Phi}\left(\mathbf{v} - \frac{n\sigma}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\mathbf{u}\right)$$

Cournot and Bertrand equilibrium consumers' surplus One can either use (A.1) and replace q by $\mathbf{q}^C$ and $\mathbf{q}^B$ respectively, using the expressions

$$\begin{aligned}
\mathbf{q}^C &= \frac{1}{2b-\sigma}\left[(\mathbf{v} - \overline{\mathbf{v}}\mathbf{u}) - \frac{2b-\sigma}{2b+(n-1)\sigma}\overline{\mathbf{v}}\mathbf{u}\right] \\
\mathbf{q}^B &= \frac{1}{2b-\sigma-\Phi}\left[(\mathbf{v} - \overline{\mathbf{v}}\mathbf{u}) - \frac{2b-\sigma-\Phi}{2b+(n-1)\sigma-\Phi}\overline{\mathbf{v}}\mathbf{u}\right]
\end{aligned}$$

which are the most convenient to see the variance terms. Or, more directly (but with slightly more matrix computations), one can use (A.4). For example, in the case of Cournot (the computations for Bertrand are almost exactly the same), we start from

$$2V^{\mathrm{C}} = \mathbf{v}' \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{B} \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{v}$$

so we have to compute the matrix which is between $\mathbf{v}'$ and $\mathbf{v}$, we do it in two steps. First (using again $\mathbf{J}^2 = n\mathbf{J}$),

$$\begin{aligned}
\left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{B} &= \frac{1}{2b-\sigma}\left[\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right]\left[(b-\sigma)\mathbf{I} + \sigma\mathbf{J}\right] \\
&= \frac{b-\sigma}{2b-\sigma}\left[\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right]\left[\mathbf{I} + \frac{\sigma}{b-\sigma}\mathbf{J}\right] \\
&= \frac{b-\sigma}{2b-\sigma}\left[\mathbf{I} + \left(\frac{\sigma}{b-\sigma} - \frac{\sigma}{2b+(n-1)\sigma}\right)\mathbf{J} - \frac{\sigma^2}{(2b+(n-1)\sigma)(b-\sigma)}\mathbf{J}^2\right] \\
&= \frac{b-\sigma}{2b-\sigma}\left[\mathbf{I} + \frac{\sigma b}{(2b+(n-1)\sigma)(b-\sigma)}\mathbf{J}\right]
\end{aligned}$$

Now, we compute $\mathbf{M} = \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1} \mathbf{B} \left(\mathbf{B} + \mathbf{diag}(\mathbf{b})\right)^{-1}$

$$\begin{aligned}
\mathbf{M} &= \frac{(b-\sigma)}{(2b-\sigma)^2}\left[\mathbf{I} + \frac{\sigma b}{(2b+(n-1)\sigma)(b-\sigma)}\mathbf{J}\right]\left[\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right] \\
&= \frac{(b-\sigma)}{(2b-\sigma)^2}\left[\mathbf{I} + \left(\frac{b(2b+(n-1)\sigma) - (2b+(n-1)\sigma)(b-\sigma) - n\sigma b)}{(2b+(n-1)\sigma)(b-\sigma)}\right)\frac{\sigma}{2b+(n-1)\sigma}\mathbf{J}\right] \\
&= \frac{(b-\sigma)}{(2b-\sigma)^2}\left[\mathbf{I} - \left(\frac{\sigma^2\left((n-2)b - (n-1)\sigma\right)}{(2b+(n-1)\sigma)^2(b-\sigma)}\right)\mathbf{J}\right]
\end{aligned}$$

Finally, using

$$\mathrm{Var}\left(\mathbf{v}\right) = \frac{1}{n}\mathbf{v}'\mathbf{v} - \overline{\mathbf{v}}^2 \quad \text{and } \mathbf{v}'\,\mathbf{J}\mathbf{v} = n^2\overline{\mathbf{v}}^2$$

$$\begin{aligned}
2V^{\mathrm{C}} &= \mathbf{v}'\mathbf{M}\mathbf{v} \\
&= \frac{(b-\sigma)}{(2b-\sigma)^2}\mathbf{v}'\left[\mathbf{I} - \left(\frac{\sigma^2\left((n-2)b-(n-1)\sigma\right)}{(2b+(n-1)\sigma)^2(b-\sigma)}\right)\mathbf{J}\right]\mathbf{v} \\
&= \frac{n(b-\sigma)}{(2b-\sigma)^2}\left[\mathrm{Var}\left(\mathbf{v}\right) + \left(1 - \frac{n\sigma^2\left((n-2)b-(n-1)\sigma\right)}{(2b+(n-1)\sigma)^2(b-\sigma)}\right)\overline{\mathbf{v}}^2\right] \\
&= \frac{n(b-\sigma)}{(2b-\sigma)^2}\mathrm{Var}\left(\mathbf{v}\right) + \frac{n}{(2b-\sigma)^2}\left((b-\sigma) - \frac{n\sigma^2\left((n-2)b-(n-1)\sigma\right)}{(2b+(n-1)\sigma)^2}\right)\overline{\mathbf{v}}^2 \\
&= \frac{n(b-\sigma)}{(2b-\sigma)^2}\mathrm{Var}\left(\mathbf{v}\right) + \frac{n(b+(n-1)\sigma)}{(2b+(n-1)\sigma)^2}\overline{\mathbf{v}}^2
\end{aligned}$$

which is the formula given in Table 2. One can check that if $n = 1$ it is the consumers' surplus of the linear demand monopoly, i.e. $2V = \frac{(a-c)^2}{4b}$. If $\sigma = 0$ the formula becomes $2V = \frac{n}{4b}\left(\mathrm{Var}\left(\mathbf{v}\right) + \overline{\mathbf{v}}^2\right)$. More generally, it increases with the variance of $\mathbf{v}$ because consumers enjoy diversity and it increases with the average marginal surplus $\overline{\mathbf{v}}$.

Cournot and Bertrand firms' profits Individual profits (again we show the computation for Cournot as the Bertrand ones are similar) are immediately given deduced from the expressions of q_i^C and $p_i^C - c_i$. One can either sum these individual profits or directly use the equilibrium expression for total profit given by (A.6). That is,

$$\Pi^C = \mathbf{v}' \left[(\mathbf{B} + \mathbf{diag}(\mathbf{b}))^{-1} - (\mathbf{B} + \mathbf{diag}(\mathbf{b}))^{-1} \mathbf{B} (\mathbf{B} + \mathbf{diag}(\mathbf{b}))^{-1} \right] \mathbf{v}$$

we need to compute the matrix between $\mathbf{v}'$ and $\mathbf{v}$ which is

$$\frac{1}{2b-\sigma} \left[\mathbf{I} - \frac{\sigma}{2b+(n-1)\sigma} \mathbf{J} \right] - \frac{(b-\sigma)}{(2b-\sigma)^2} \left[\mathbf{I} - \left(\frac{\sigma^2 \left((n-2)b - (n-1)\sigma \right)}{(2b+(n-1)\sigma)^2 (b-\sigma)} \right) \mathbf{J} \right]$$

and simplifies into

$$\frac{b}{(2b-\sigma)^2} \left[\mathbf{I} - \frac{\sigma(4b+(n-2)\sigma)}{(2b+(n-1)\sigma)^2} \mathbf{J} \right]$$

and finally

$$\Pi^C = \frac{nb}{(2b-\sigma)^2} \left[\mathrm{Var}(\mathbf{v}) + \frac{(2b-\sigma)^2}{(2b+(n-1)\sigma)^2} \overline{\mathbf{v}}^2 \right]$$