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## Abstract

We consider a version of the Stigler model of fixed sample price search, where consumer utility depends on whether or not at least one sampled price fulfils a pre-set target.

JEL-Codes: D830.

Keywords: fixed sample search, constrained search.

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# Constrained Fixed Sample Search

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**Abstract**

We consider a version of the Stigler model of fixed sample price search, where consumer utility depends on whether or not at least one sampled price fulfils a pre-set target.

Keywords: fixed sample search, constrained search
JEL classification: D83

## 1 Introduction

In the classic Stigler (1961) model of fixed sample (or pre-determined) price search, a risk-neutral consumer ends up paying the best (lowest) encountered price. In this paper, we change this setup: here, the consumer obtains additional utility if at least one sampled price (and hence the price paid) is lower than a pre-set threshold (constraint).

The concept of constrained search was introduced in Bonilla et al. (2019), in the context of sequential search. Here we ask: what is the effect of a price target (constraint) on the optimal fixed sample search strategy? On the one hand, although the analysis is completely different, we establish that the non-monotonicity property of search intensity (here captured by sample size) is robust across the two standard search methods. Here, for relatively high (easy) price targets the searcher will increase the optimal sample size as this target decreases, but will reduce it for relatively low price targets as even harder targets are increasingly difficult to fulfil. On the other hand, our results in this paper also point to a hitherto unexamined disadvantage of the fixed sample search method compared to sequential search.

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## 2 Analysis

A consumer aims to buy one unit of some good, but does not know the price that any particular shop is charging. Assuming a continuum of shops, from the perspective of the consumer, prices are distributed according to the cumulative probability distribution function $F(.)$ with support $[\underline{p}, \overline{p}]$. At cost $c$ per shop (same for all shops), the consumer can observe the exact price that a shop is charging. The consumer uses the fixed sample search strategy, choosing the optimal number of shops to visit given that the objective is to buy at the best possible price.

Crucially, we assume that the consumer receives additional utility (here, a payment reduction of $y$) if at least one sampled price is no higher than an ex-ante set price target $\widehat{p}$. This is a highly stylised setup, but one that can relatively easily be embedded into a richer framework. For example, one could think of the multiple-good demand problem of a consumer who has a budget constraint and can only afford one good if the price of the other (obtained through search) is low enough.[1]

Let $N$ denote the number of sampled shops. Following Hey (1979) and others, assume that $N$ is a continuous choice variable - it simplifies the analysis of what is of course an integer problem. Let $p(N)$ denote the expected best (lowest) price observed if the sample is of size $N$. The random variable $p(N)$ is characterised by the cumulative probability distribution function $G(.)$ with support $[\underline{p}, \overline{p}]$.

In our version of the story, if the consumer commits to sampling $N$ shops, the expected overall payment is $p(N) + cN$ if all sampled prices above $\widehat{p}$, and it is $p(N) + cN - y$ if at least one observed price is no higher than $\widehat{p}$. The consumer's problem is:

$$\min_N L \equiv [1-G(\hat{p})] \left[ \int_{\widehat{p}}^{\overline{p}} pdG(p)/[1-G(\widehat{p}] + cN \right] + G(\hat{p}) \left[ \int_{\underline{p}}^{\widehat{p}} pdG(p)/G(\widehat{p}) + cN - y \right],$$

where

$$G(\hat{p}) = 1 - [1 - F(\hat{p})]^N.$$

The expected best price if $N$ shops are sampled is:

$$p(N) = \int_{\underline{p}}^{\overline{p}} pdG(p) = \underline{p} + \int_{\underline{p}}^{\overline{p}} [1 - F(p)]^N \, dp,$$

[1] For the pioneering work on this, see Burdett and Malueg (1981) and Manning and Morgan (1982).

where the latter equality is derived using integration by parts.

The optimisation problem simplifies to:

$$\min_N L \equiv p(N) + cN - G(\widehat{p})y \tag{1}$$

One can think of the optimally chosen sample size $N^*$ as a measure of search intensity. We show that the optimal search intensity is non-monotonic in the target price $\widehat{p}$. As soon as the constraint starts to bite, the consumer increases the sample size. However, the incentive to do so decreases after a certain threshold price, when the additional costs exceed the increasingly unlikely benefits of increased sampling.

To see this, first observe that for $\hat{p} \leq \underline{p}$ the threshold price constraint cannot possibly be fulfilled and, since $F(\hat{p}) = F(\underline{p}) = 0$, the optimal sample size $N^*$ is obtained by solving:

$$\min_N L \equiv \underline{p} + \int_{\underline{p}}^{\bar{p}} [1 - F(p)]^N \, dp + cN$$

In turn, for $\hat{p} \geq \bar{p}$ the threshold price constraint is always fulfilled and, since $F(\hat{p}) = F(\bar{p}) = 1$, now $N^*$ is the solution to:

$$\min_N L = \underline{p} + \int_{\underline{p}}^{\bar{p}} [1 - F(p)]^N \, dp + cN - y$$

As $y$ is just a constant, the optimal sample size is clearly the same in both cases (denote it by $N_0$), and is obtained from the first-order condition $dL/dN = 0$:

$$\int_{\underline{p}}^{\bar{p}} [1 - F(p)]^N ln(1 - F(p))dp + c = 0$$

For $\widehat{p} \in (\underline{p}, \overline{p})$, the first-order condition for an optimum in problem (1) simply equates the expected marginal benefit from an increased sample, and the marginal cost $c$:

$$-\int_{\underline{p}}^{\bar{p}} [1 - F(p)]^N ln(1 - F(p))dp - [1 - F(\widehat{p})]^N \ln(1 - F(\widehat{p})y = c \tag{2}$$

The expected marginal benefit $MB(N, \widehat{p})$ has two components, as an increase in $N$ has two effects. Ceteris paribus (in particular, for a given $\widehat{p}$), higher search intensity results in a lower expected best price as well as a higher probability of obtaining the additional utility $y$ (the second term being simply $\partial G(\widehat{p})/\partial N$).

Since $\ln(1-F(p))<0$, the expected marginal benefit is positive. It can also be shown to be decreasing in sample size:

$$\frac{\partial MB}{\partial N}=-\int\limits_{\underline{p}}^{\overline{p}}[1-F(\widehat{p}]^{N}\ln(1-F(\widehat{p}))^{2}dp-[1-F(\widehat{p}]^{N}\ln(1-F(\widehat{p}))^{2}y<0$$

Then, the optimal sample size $N^*$ balances these expected benefits with the additional cost of increasing the sample size (and technically it will of course be one of the two integers nearest to the $N$ that satisfies (2)).

Furthermore, when graphed as a function of $N$, the downward sloping $MB(N,\widehat{p})$ curve is identical for both $\widehat{p}=\underline{p}$ and $\widehat{p}=\overline{p}$, and it moves as $\hat{p}$ changes. To see this, please note that:

$$\frac{\partial MB}{\partial\widehat{p}}=[1-F(\widehat{p})]^{N-1}f(\widehat{p})y\left[N\ln(1-F(\widehat{p}))+1\right]$$

As $\widehat{p}$ approaches $\underline{p}$ from the right, the above derivative is positive; in turn, as $\widehat{p}$ approaches $\overline{p}$ from the left, the derivative approaches $-\infty$. Since $\ln(1-F(\widehat{p}))$ is strictly monotonic and decreasing, $\partial MB/\partial\widehat{p}$ is zero for a unique target price $\widehat{p}\in(\underline{p},\overline{p})$, which we denote by $\widetilde{p}$. Figure 1(a) captures the above:

FIGURE 1

For $\widehat{p}\in(\widetilde{p},\overline{p})$ the marginal benefit curve tilts and shifts to the right as $\widehat{p}$ decreases. In turn, for $\widehat{p}\in(\underline{p},\widetilde{p})$ it tilts and it shifts to the left as $\widehat{p}$ decreases. A decrease of $\widehat{p}$ from $\overline{p}$ creates incentives to increase the sample size, so as to decrease expected best price and improve the chances of fulfilling a relatively easy target price. However, the effect of this on both $p(N)$ and $G(\widehat{p})$ is decreasing, so at some point it is not worth it. It is this second negative effect that is novel here, and it stems from the fact that it is of course increasingly difficult to hit relatively low pre-set price targets.

We can further characterise the optimal strategy (also see Figure 1 (b)). Equation (2) determines the optimal sample size $N^*$ as a function of $\widehat{p}$. Using implicit differentiation one then obtains:

$$\frac{\partial N^*}{\partial\hat{p}}=\frac{-[1-F(\hat{p})]^{N^*-1}f(\hat{p})[N^*ln(1-F(\hat{p}))+1]y}{[1-F(\hat{p})]^{N^*}\left[\ln(1-F(\hat{p})\right]^2y+\int\limits_{\underline{p}}^{\overline{p}}(1-F(p))^{N^*}\left[ln(1-F(p))\right]^2dp}\quad(3)$$

The $N^*(\widehat{p})$ function has two kinks, one at each extreme of the price range on the market. For $\widehat{p}=\overline{p}$, the left-hand limit of the above derivative is zero, while for $\hat{p}=\underline{p}$ the right-hand limit is positive, given by:

$$\lim_{\hat{p}\to\underline{p}} \frac{\partial N^*}{\partial \hat{p}} = \frac{f(\hat{p})y}{\int_{\underline{p}}^{\bar{p}} [1-F(p)]^N [ln(1-F(p)]^2 dp}$$

One can already conclude that $(i)$ the optimal sample size $N^*$ is continuous in price threshold $\widehat{p}$, $(ii)$ $N^* = N_0$ in both cases when the constraint is ignored - either because it is impossible or because it is spurious, and $(iii)$ search intensity picks up ($N^* > N_0$) as soon as a meaningful price target is in place. In other words, the optimal sample size function $N^*(\widehat{p})$ is indeed non-monotonic for $\widehat{p} \in (\underline{p}, \overline{p})$ and has one maximal stationary point.[2]

To further investigate the uniqueness of this peak search intensity, please note from (3) that $\partial N^*/\partial\widehat{p} = 0$ for $N^* ln(1-F(\hat{p}))+1 = 0$. There is only one interior maximum $N^*$ if this equation has a unique solution, which is indeed the case provided $N^* ln(1-F(\hat{p}))$ is monotonic in $\widehat{p}$. To check this, first observe that:

$$\frac{\partial[N^* ln(1-F(\hat{p}))]}{\partial\hat{p}} = \frac{\partial N^*}{\partial\hat{p}} ln(1-F(\widehat{p})) - \frac{N^* f(\hat{p}))}{1-F(\hat{p})}$$

Substitute $\partial N^*/\partial\widehat{p}$ from (3), re-arrange and factorise, to obtain:

$$\frac{\partial[N^* ln(1-F(\hat{p}))]}{\partial\hat{p}} = \frac{A-B}{C} < 0,$$

where

$$A \equiv N^* f(\hat{p}) \int_{\underline{p}}^{\bar{p}} (1-F(p))^{N^*} \left[ln(1-F(p))\right]^2 dp \quad (>0)$$

$$B \equiv [1-F(\widehat{p})]^{N^*} f(\widehat{p}) ln(1-F(\widehat{p}))y \quad (<0)$$

$$C \equiv [F(\hat{p})-1]\{[(1-F(\hat{p})]^{N^*} \left[ln(1-F(\hat{p}))\right]^2 y + \int_{\underline{p}}^{\bar{p}} [1-F(p)]^{N^*} \left[ln(1-F(p))\right]^2 dp\} \ (<0)$$

This proves that the search intensity function $N^*(\widehat{p})$ has a unique interior maximum, given by:

$$N^*(\widetilde{p}) = -\frac{1}{\ln(1-F(\widetilde{p}))}$$

What is the best price the consumer can expect to obtain for any optimally chosen sample size given a price target $\widehat{p}$? Denote this expected best price by

[2] The *rate* at which the optimal search intensity changes with $\widehat{p}$ depends on the properties of $f$ and $F$. In particular, the effect of $\widehat{p}$ on the change in hazard rate $f(\widehat{p})/[1-F(\widehat{p})]$ depends on log-concavity. This is true also for constrained sequential search and the shape of the reservation strategy function there.

$p^* \equiv p(N^*)$. Given that - ceteris paribus - we have $\partial p(N)/\partial N < 0$, it follows that $p^*$ has a minimum for $\widehat{p} = \widetilde{p}$. Since $N^* = N_0$ for both $\widehat{p} = \underline{p}$ and $\widehat{p} = \overline{p}$, the expected best price $p^*$ is also the same (denote it by $p_0$) in both extreme cases. Overall therefore, $p^*$ is higher than $\widehat{p}$ for low values of $\widehat{p}$. Crucially, this implies that for any parameter values there exists a range of price targets $\widehat{p}$ such that $p^*(\widehat{p}) > \widehat{p}$. Or, putting it differently: there are no parameter values for which $p^*(\widehat{p}) \leq \widehat{p}$ for *all* $\widehat{p} \in (\underline{p}, \overline{p})$. Figure 2 illustrates this, and depicts the case where $p^* = \widehat{p}$ once.

FIGURE 2

# 3 Discussion

How do the results for constrained fixed sample search compare to those for constrained sequential search?

First, our analysis here and a comparison with Bonilla et al. (2019) reveal a robust result: in both cases, search intensity is a non-monotonic function of a constraint (target) whose attainment guarantees extra utility.

Second, it is well-known that in many settings, sequential search performs better than fixed sample search, the main disadvantage of the latter method being that it implies an ex-ante commitment to a rigid sample size. In this paper, we point out a further apparent drawback of fixed sample search. The underlying reason for this drawback is that with constrained fixed sample search, there is no search intensity that would *ex post* guarantee the fulfilling of the target. With pre-determined search and an active (i.e. not impossible or spurious) constraint (target), there is always a positive probability that the *realised* outcome does not fulfil this target, regardless of search intensity.

In sharp contrast, this is not the case with constrained sequential search. Firstly, very easy active targets are automatically fulfilled - they have no impact on search intensity as captured there by the reservation strategy. Secondly, with constrained sequential search, there is *always* a range of (still relatively easy) targets that the searcher is able to and does in fact *commit to* fulfilling.

We believe that constrained (fixed sample and sequential) search methods could fruitfully be integrated into richer economic models. In the context of price search, we already mentioned the potential application to consumer demand theory. Bonilla et al.(2019) and Bonilla and Kiraly (2013) obtain interesting results in models where constrained job search provides the link between two frictional markets. More generally, constrained search is relevant whenever access to another market (frictional or not) or further options are conditional on securing an appropriate "ticket" first, through search.

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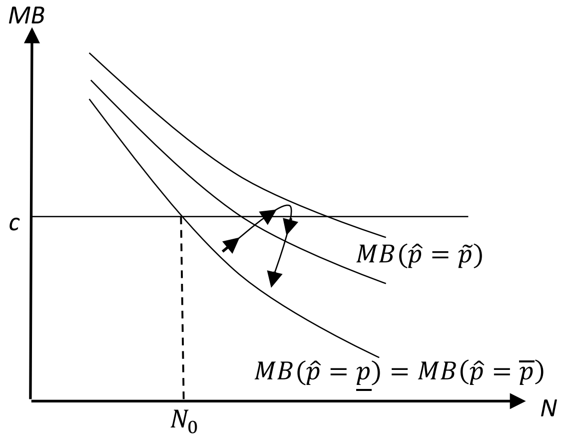
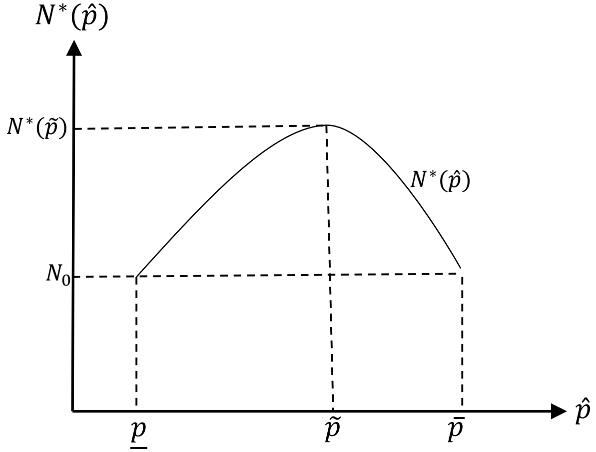
*Figure 1*

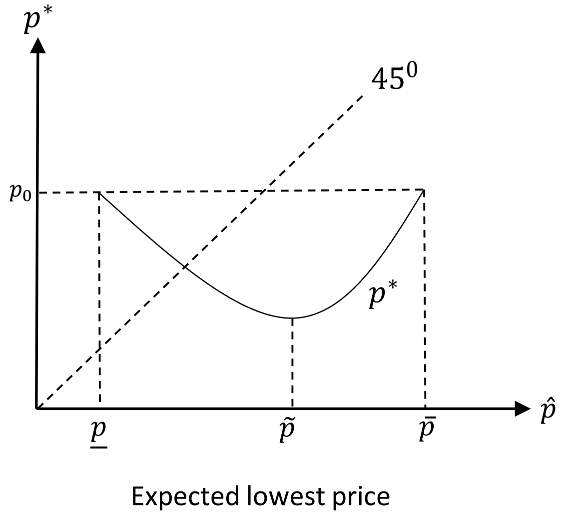



(a) Determination of optimal sample size ($N^*$)




(b) Optimal sample size

*Figure 2*



$p^*$
$45^0$
$p_0$
$p^*$
$\underline{p}$
$\tilde{p}$
$\bar{p}$
$\hat{p}$
Expected lowest price