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Abstract

Our novel approach to modeling monopolistic competition with heterogeneous consumers involves a space of characteristics of a differentiated good (consumers' ideal points), alike Hotelling (1929). Firms have heterogeneous costs à la Melitz (2003). In addition to price setting, each firm also chooses its optimal location/niche in this space. We formulate conditions for positive sorting: more efficient firms serve larger market segments and face tougher competition in the equilibrium. Our framework entails rich equilibrium patterns displaying non-monotonic markups, high in the most and least populated niches, and the unequal gains from trade across different consumers.

JEL-Codes: F100, L110, L130.

Keywords: firm heterogeneity, product space, positive sorting, product niches.

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... The Fox seemed perplexed, and very curious. ... "Are there hunters on that planet?" "No." "Ah, that is interesting! Are there chickens?" "No." "Nothing is perfect," sighed the fox.

"Le Petit Prince". Antoine de Saint-Exupéry.

1 Introduction

Ever since Dixit and Stiglitz (1977), monopolistic competition has been a workhorse model in international trade, New Economic Geography, growth, and macroeconomics. A large literature on monopolistic competition[1] demonstrates the important role of firm heterogeneity in determining general equilibrium outcomes and in explaining a broad array of empirically observed phenomena (Melitz 2003; Chaney 2008; Zhelobodko et al. 2012; Mrazova and Neary 2017; Dhingra and Morrow 2019). At the same time, little attention has been paid to the role of consumer heterogeneity and to the interplay between heterogeneous demand and heterogeneous supply under monopolistic competition (which can be, for instance, crucial for policy analysis). We seek to narrow this gap in the literature and to make one more step towards understanding the implications of this two-sided heterogeneity in a general equilibrium framework.

In this paper, we develop a novel theory of monopolistic competition with bilateral heterogeneity: (*i*) *horizontal heterogeneity* of consumers in tastes; (*ii*) *vertical heterogeneity* of firms in productivities. Consumer tastes are represented as locations in a space of product types (or product space, for short), and follow a one-dimensional, symmetric, and unimodal distribution with a compact support. These assumptions capture the idea of "popularity": the product type located at the origin is the most popular among consumers, while the endpoint locations are the least popular. In modeling firm behavior, our major departure from traditional Melitz-type models of monopolistic competition with variable elasticity of substitution is that, apart from setting the profit-maximizing price, each active firm chooses its location in the product space.[2] This new dimension of firm behavior can be considered as choosing a product niche: i.e., which group of consumers to serve.

Each firm's location choice entails the following trade-off. On the one hand, a more popular niche results in a higher demand for the firm's product and, thereby, in a potentially higher profit. On the other hand, assume that all active firms choose to serve the most popular niche. Then, the local competitive pressure there becomes so high that incentives arise to switch to less popular but less competitive niches. To sum up, each firm compromises between access to a *larger local market* and *softer local competition*. Or, as in our epigraph, a firm (a fox) wishes to "hunt" for numerous consumers (chickens), but tries to avoid fierce competitors (hunters). Such a setup

[1] See Thisse and Ushchev (2018) for a recent survey.

[2] Recent work on monopolistic competition with variable elasticity of substitution (see, for instance, Behrens and Murata 2007) has pointed out that not only this model is tractable but also flexible and capable of explaining a broad array of empirically observed phenomena, e.g. variable markups (Bellone et al. 2014) and incomplete pass-through (De Loecker et al. 2016).

provides new insights on general equilibrium outcomes of monopolistic competition models (for instance, the distribution of firm sales, prices, markups, etc.), which standard representative-consumer-based models fail to deliver. Moreover, it enables us to explore the interaction between two very different aspects of product differentiation: (i) the *hedonic* aspect (see Rosen 1974) and (ii) the *market power* aspect.

We then ask what patterns of equilibria may arise in this new setting. First, we consider the most general case by allowing consumers purchasing product types different from their most preferred ones that comes at a cost: given other things equal, the utility derived from consuming product types different from the most preferred one is lower and negatively related to the distance between the product types (as in the Hotelling model). In other words, besides monopolistic competition we consider direct spatial competition among firms (Hotelling 1929; Kaldor 1935; Lancaster 1966; Beckmann 1972; Rosen 1974; Salop 1979). We also do not specify functional forms of the consumer utility and distance decay. Hence, the shape of the market demand is affected by the exogenous spatial distribution of consumer tastes, the endogenous spatial distribution of local competitive toughness, and the exogenous decay rate of utility (derived from consuming several product types) with distance.

It appears that such a setup makes a complete analytical characterization of equilibria a prohibitively complex task. However, we are still able to describe some properties of the equilibrium in the most general case (conditional on that the equilibrium exists). In particular, we find that more productive firms charge lower prices and produce larger volumes. More importantly, we show that if the firm's profit function (as a function of firm's productivity, location, and price) is supermodular in location and price, then each equilibrium displays positive assortative matching: i.e., more productive firms locate in denser niches. This finding has important implications for the distribution of firm's sales, prices, and markups and may result in a deeper understanding of data: a firm may be 10 times smaller than another, not because it has 10 times higher costs, but because it is slightly less efficient and forced to take a much narrower market niche. Moreover, this outcome is important for understanding the distribution and relative size of the gains from international trade across consumers. Compared to standard models of trade with monopolistic competition, in our framework trade has an additional effect on consumer's well-being. Namely, it affects the matching between firms and locations, which in turn creates a variation in the magnitude of the gains from trade across consumers.

To get sharper results, we explore in more detail a special case of our model, in which the distance decay rate of the consumer utility is infinitely high. In this case, a firm serves only those consumers, for whom the firm's product type is the most preferred one. Though, this simplification assumes away direct spatial competition among firms, there is still *indirect* spatial competition channeled through the general equilibrium mechanism. Moreover, it is in line with recent evidence that households tend to concentrate their spending on a few preferred products that vary across households (see Neiman and Vavra 2019). We find that, if the price elasticity of demand is decreasing with consumption[3] (the Marshall's Second Law of Demand), then (*i*) an equilibrium always exists, and (*ii*) all equilibria exhibit positive assortative matching. If, in addition to that, the taste distribution is log-concave, then the equilibrium is always unique.

Another implication of our theory is that markups vary non-monotonically across the product space. Specifically, we prove that the markups are highest in the most populated locations (where the most productive firms are located) and in the least populated ones (where the least productive firms are located). This result on markups differs from that in models of spaceless monopolistic competition (see, for instance, Zhelobodko et al. 2012), where firms' markups increase with their productivity. Our non-monotonicity result is driven by the interplay of two forces: firm heterogeneity and consumer heterogeneity. If firms were homogeneous, then the markup distribution would follow the spatial distribution of local competitive toughness. Since less popular niches exhibit lower competitive pressure, markups there are higher. In other words, to compensate lower demand in more "remote" locations, homogeneous firms would charge higher prices there. However, because firms are actually heterogeneous, positive assortative matching drives less productive firms further away from denser locations. Since less productive firms charge, ceteris paribus, lower markups, positive assortative matching creates another component in the markup distribution, which decreases with the distance from the densely populated but extremely competitive niche – the origin. As a result, the markup distribution appears to be non-monotonic on the product space.

We then consider the effects of a uniform increase in the population density that can be interpreted as the effects of frictionless trade with a similar country. Naturally, more firms enter the market, which increases the toughness of competition within each product type. This in turn changes the matching pattern: firms relocate to less popular niches, and the range of served niches expands. This finding is in line with patterns in the data. In particular, Fieler and Harrison (2019) find that one of the implications of tariff reductions on manufacturing in China in 1998-2007 was the introduction of new products. Also, our theory is potentially in line with findings in Holmes and Stevens (2014) who show that in the US smaller firms are less affected by competition with China as they produce custom or specialty goods. As foreign exporting firms are typically more productive, in our framework they choose more populated niches with a weaker impact on firms located in less populated niches (that can be interpreted as custom or specialty product types).

We also find that, while prices decrease for all products in response to a uniform population increase, markups set by unproductive smaller firms can rise. This outcome also finds some empirical support. Fieler and Harrison (2019) find that the tariff cuts in China resulted in an increase in firm's revenue to cost ratios or, equivalently, in firm's markups. Moreover, this rise in markups was higher for small firms than for large firms. Thus, our model provides theoretical underpinnings for some recent empirical findings, which the standard Dixit-Stiglitz-Melitz paradigm fails to explain.

[3]This is case is often viewed as the most relevant one in monopolistic competition with variable elasticity of substitution. See, e.g., Zhelobodko et al. 2012, Dhingra and Morrow (2019).

Finally, we numerically analyze the welfare implications of a uniform change in the population density.[4] We find that, as in traditional models, all consumers gain from a rise in the population size. However, these gains are quantitatively not the same across consumers. Our simulations show that two patterns of the distribution of these gains prevail. The first pattern corresponds to the case when the gains are decreasing in location: i.e., consumers located closer to the origin (that is, consumers with more popular tastes) gain disproportionally more than all other consumers. In the second pattern, the distribution of the gains has a hump shape with consumers located in the middle or closer to the endpoint locations in the product space gaining more. The former pattern is likely to take place when the level of firm heterogeneity is low, while the latter pattern arises when the level of firm heterogeneity is high.[5] Our numerical analysis also shows that a proportional rise in the population density increases the level of inequality in the economy: on average, a 5% increase in the population density raises the level of inequality (measured by the Theil index) by 2.5%. These findings provide new insights on the distributional consequences of international trade and on how international trade can affect welfare inequality.

Literature review

Our paper contributes to at least three important strands of literature. First, it connects with papers that analyze markets with spatially distributed consumers (see Lancaster (1966, 1975), Salop (1979), Chen and Riordan (2007), and Vogel (2008) among others). Regarding this literature, it is important to stress fundamental differences between our framework and standard spatial competition. Indeed, although the product space is described as a one-dimensional interval, which is akin to Hotelling (1929), we assume that consumers (i) buy in volume, and (ii) exhibit love for variety. This leads to a very different demand structure compared to Hotelling-type setups. Another distinctive feature of our approach is that monopolistically competitive firms make decisions on entry, production, price, and location. To the best of our knowledge, no existing market competition model captures a similarly rich pattern of firm behavior. The closest paper in this literature is Goryunov et al. (2017) who consider a monopolistic competition framework with spatially distributed consumers. However, in contrast to the present paper, this work focuses on the case with homogeneous firms and uniformly distributed consumers. Another paper related to ours is Ushchev and Zenou (2018), who develop a model of price competition in product-variety networks. Both consumers and suppliers of a differentiated product are embedded into a network which captures proximity between product varieties: two varieties are linked to each other if they are close substitutes, otherwise no link exists. Each consumer's location is her most preferred variety, while her willingness to pay for other varieties decays exponentially with their geodesic distance (induced by the network) from her most preferred variety. Like in most of the network

[4]The theoretical analysis of the welfare changes appears to be too complicated.

[5]In our simulations, we assume that the distribution of firm productivity is Pareto.

literature, the network structure of the economy is assumed *fixed*. Therefore, Ushchev and Zenou (2018) abstract from niche choices of firms and spatial sorting.

Second, our paper is related to the literature on spatial selection/sorting of heterogeneous firms. One of the most related papers is Nocke (2006) who considers sorting of heterogeneous firms across imperfectly competitive markets of different size. He finds a similar outcome - more productive firms choose to locate in larger markets. However, our paper differs in at least two aspects. We tackle sorting between firms and product niches in a continuous fashion, somewhat similar to continuous economic geography in Allen and Arkolakis (2014). More importantly, Nocke (2006) mainly focuses on sorting per se, while we consider a general equilibrium framework with monopolistic competition analyzing its existence and uniqueness and explore its implications for markups and consumer welfare. Among other studies, Okubo et al. (2010) explores how trade liberalization affects sorting across location in a two-country model with linear demand. Behrens et al. (2014) construct a model of selection of talented individuals across ex-ante homogeneous cities.[6] Gaubert (2018) develops a quantitative model of sorting of heterogeneous firms across cities where firm's choice depends on local input prices and agglomeration externalities. Our paper complements this strand of the literature by focusing in more detail on selection of firms across product niches in a quite general setup with continuous product space. Carballo et al. (2018) empirically study self-selecting of firms into specific foreign market niches, but their space of products is very different from ours. There is some similarity of our approach with Eckel and Neary (2010) who develop a model of flexible manufacturing with core competence of every firm. However, sorting of firms is not addressed in this paper.

Finally, the present paper also contributes to the literature on the role of consumer heterogeneity in monopolistic competition and its implications for the distribution of the gains from trade. Among this large literature, Fajgelbaum et al. (2011) and Tarasov (2012) develop models of international trade with income heterogeneity and non-homothetic preferences. Osharin et al. (2014) consider a model of monopolistic competition where the elasticity of substitution between any pair of varieties is consumer-specific. Nigai (2016) considers a quantitative trade model with heterogeneous (in income and preferences) consumers and shows that the assumption of a representative consumer may overestimate (underestimate) the welfare gains from trade of the poor (rich). In contrast to these studies and many others, our paper focuses on horizontal consumer heterogeneity assuming away income effects and, therefore, provides a new rationale for the unequal distribution of the gains from trade.

The rest of the paper is organized as follows. In Section 2 we develop a general framework with unspecified functional forms of the consumer utility and distance decay rate. Section 3 considers the special case of our model with an infinitely high distance decay rate. In Section 4 we perform numerical analysis of our model. Section 5 concludes.

[6] See also Behrens and Robert-Nicoud (2015) for a survey.

2 The general model

In this section, we develop a model of a closed economy which blends the features of monopolistic competition à la Melitz (2003) with the characteristics approach to product differentiation developed by Lancaster (1966). Such a model allows us to study the role of interactions between two very different facets of product differentiation: (i) the *hedonic* aspect: price of a certain type of product depends on its type-specific characteristics (Rosen 1974); and (ii) the *market-power* aspect: because varieties are differentiated, pricing above marginal cost need not result in losing all the customers. Thus, the demand for a certain type of product is not only affected by its price, but also by the "location" of the product in the space of product characteristics. As a result, each firm chooses both price and location. In this context, a firm's location choice means targeting a certain market segment (taking into account its size and the level of competition).

2.1 Product space and demand

Spatial structure. Product space X is one-dimensional and is represented as a real line: $X \equiv \mathbb{R}$.[1] Each point of the product space corresponds to a certain type of product. A consumer's location $x \in X$ represents her most preferred product type.[2] Let $l(x) \geq 0$ stand for the population density at $x \in X$, and denote by $L \equiv \int_X l(x)dx$ the total population in the economy. To simplify the analysis, we assume that the population density is symmetric w.r.t. the origin $x = 0$, decreases with the distance from the origin, and has compact support $[-S, S]$, where $S > 0$. We also assume that $l(\cdot)$ is continuously differentiable and strictly decreasing over $(-S,\, S)$. In other words, product types are ordered by "popularity" in the descending order: product type $x \in X$ is preferred by more consumers than product type $y \in X$ if and only if $|x| < |y|$. Hence, we find it natural to refer to $l(\cdot)$ as the *spatial distribution of consumer tastes*, which we treat as interchangeable with "population density" in what follows. This is done both for brevity and for the sake of exploiting the intuitive appeal of Hotelling's spatial metaphor.

The utility function of a consumer located at $x \in X$ is given by

$$\mathcal{U}_x = V\left(\int_X k_\tau(x, y) \int_{\Omega_y} u(q(\omega))\mathrm{d}\omega \mathrm{d}y\right) + q_0, \tag{2.1}$$

where $V : \mathbb{R}_+ \to \mathbb{R}$ is an upper-tier utility function, $u : \mathbb{R}_+ \to \mathbb{R}$ is a lower-tier utility function, Ω_y is the set of varieties of type $y \in X$ available at the market, $k_\tau(x, y)$ is a spatial discount factor, $q(\cdot)$ is an individual consumption pattern defined over the whole set $\Omega \equiv \bigcup_{y\in X} \Omega_y$ of varieties

[1] Alternatively, X could be represented by a compact interval, in which case it can be viewed as a geographical space in a linear-city model.

[2] This bears some resemblance with the *ideal variety* concept introduced by Hotelling (1929). We refrain from using the term "ideal variety" to avoid gibberish: in our model, a variety is something different from a product type, as each type of product available on the market is represented by a continuum of varieties.

available on the market, and q_0 is the consumption of the outside good, which is chosen as a numéraire. We impose the following assumptions:

Assumption 1. The upper-tier utility $V(\cdot)$ is sufficiently differentiable, satisfies $V'(\upsilon) > 0$ and $V''(\upsilon) < 0$ for all $\upsilon > 0$, and satisfies the choke-price property: $V'(0) < \infty$.

Assumption 2. The lower-tier utility $u(\cdot)$ is sufficiently differentiable, satisfies $u'(q) > 0$, $u'(0) < \infty$, and $u''(q) < 0$ for all $q > 0$, and is normalized to zero at zero: $u(0) = 0$.

Assumption 3. The kernel $k_\tau : X \times X \to \mathbb{R}_+$ representing the spatial discount factor in (2.1) has the following structure:

$$k_\tau(x, y) = \tau\psi(\tau|x - y|), \tag{2.2}$$

where $\tau > 0$ is a "transport cost" parameter which captures the decay rate of utility with distance from the most preferred product type, while $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is the distance decay function, which is (i) decreasing: $\psi'(\cdot) < 0$, and (ii) such that $2\int_{\mathbb{R}_+} \psi(z)\mathrm{d}z = 1$. In other words, the family $\{k_\tau\}_{0<\tau<\infty}$ of decay kernels constitutes a *standard mollifier* (see, e.g., Evans, 2010, p. 713). To give a few examples, the distance decay function $\psi(\cdot)$ may be (i) negative exponential: $\psi_1(z) \equiv \exp\{-z\}$; (ii) Gaussian: $\psi_2(z) \equiv (2\pi)^{-1/2}\exp\{-z^2/2\}$.

A consumer located at $x \in X$ seeks to maximize her utility (*2.1*) subject to the budget constraint given by

$$\int_X \int_{\omega\in\Omega_y} p(\omega)\, q(\omega)\, \mathrm{d}\omega\mathrm{d}y + q_0 \leq I, \tag{2.3}$$

where $p(\omega)$ is the market price for variety ω of the y-type product, while I is consumer's income. Assuming I is sufficiently high and using the standard monotonicity argument, the consumer's utility maximization problem can be restated as follows:

$$\max_{q(\cdot)} \left[V\left(\int_X k_\tau(x, y) \int_{\Omega_y} u(q_x(\omega))\mathrm{d}\omega\mathrm{d}y \right) - \int_X \int_{\omega\in\Omega_y} p(\omega)q(\omega)\mathrm{d}\omega\mathrm{d}y \right]. \tag{2.4}$$

The individual demand $q_x(\omega)$ of a consumer located at $x \in X$ for a variety $\omega \in \Omega_y$ can be obtained from the consumer's FOCs, which are as follows:

$$\frac{p(\omega)}{k_\tau(x, y)} = \frac{u'(q_x(\omega))}{\lambda(x)}, \tag{2.5}$$

where y is such that $\omega \in \Omega_y$, while $\lambda(x)$ is a product-type specific demand shifter defined by

$$\lambda(x) \equiv \frac{1}{V'\left(\int_X k_\tau(x, y) \int_{\Omega_y} u(q_x(\omega))\mathrm{d}\omega\mathrm{d}y\right)}. \tag{2.6}$$

The local aggregator $\lambda(x)$ can be viewed as a measure of *local competitive toughness* associated with the market segment $x \in X$: a higher $\lambda(x)$ means a downward shift of the demand schedule for each particular variety $\omega \in \Omega_x$.

Solving (2.5) for $q_x(\omega)$, we obtain the individual Marshallian demand of an x-type consumer — i.e. a consumer whose preferred product type is x — for variety ω:

$$q_x(\omega) = D\left(\lambda(x)\frac{p(\omega)}{k_\tau(x,y)}\right), \tag{2.7}$$

where y is the product type variety ω belongs to ($\omega \in \Omega_y$), while $D(\cdot)$ is the downward-sloping demand schedule defined by $D(z) \equiv \max\{0,\ (u')^{-1}(z)\}$ for all $z \in \left(0,\ (u')^{-1}(0)\right)$.

To obtain the market demand $Q_x(\omega)$ for variety $\omega \in \Omega_x$, we integrate (2.7) across the product space X with respect to the distribution of consumer tastes. Doing so, we obtain:

$$Q_x(\omega) = \int_X D\left(\lambda(y)\frac{p(\omega)}{k_\tau(x,y)}\right) l(y)\mathrm{d}y. \tag{2.8}$$

As can be seen from equation (2.8), the shape of market demands is affected by (i) the exogenous spatial distribution $l(\cdot)$ of consumer tastes, (ii) the endogenous spatial distribution $\lambda(\cdot)$ of local competitive toughness, and (iii) the spatial discount factor.

2.2 Firm behavior

The supply side in the model follows Melitz (2003). Each firm is single-product, i.e. it can produce, at most, one variety.[3] The only factor of production is labor, one unit of which is inelastically supplied by each individual. To enter the market, firms need to pay a sunk entry cost, $f_e > 0$. After paying the cost, they draw their marginal cost $c > 0$ from an absolutely continuous univariate distribution described by a differentiable cdf $G : [c_{\min}, \infty) \to [0, 1]$, or, alternatively, by a pdf $g(\cdot)$ defined by $g(c) \equiv G'(c)$ for any $c > c_{\min}$. Here $c_{\min} \geq 0$ is the marginal cost of the most efficient firm which can ever exist. In what follows, we call a firm whose draw is c a *c-type* firm.

In addition to the variable cost, starting production requires a fixed cost $f > 0$. Firms whose operating profits are not sufficient to cover the fixed cost, choose not to produce. The remaining firms treat the pattern $\lambda(\cdot)$ of local competitiveness as given and choose their optimal location and price (the price is determined by both the marginal cost of production and the location of a firm).

Using equation (2.8) for the market demand yields a c-type firm's operating profit $\Pi(c, p, x)$, which is a function of the firm's price and the firm's location choice:

$$\Pi(c, p, x) = (p - c)Q(p, x), \tag{2.9}$$

[3] In this paper, we do not consider multi-product firms.

where $Q(p,x)$ is the demand schedule faced by a firm producing a variety of type x (see the previous section):

$$Q(p,x) = \int_X D\left(\frac{\lambda(y)p}{k_\tau(x,y)}\right) l(y)\mathrm{d}y. \tag{2.10}$$

Note that, since $l(x)$ is symmetric around zero, it is natural to consider a symmetric equilibrium allocation of firms: i.e., firms are indifferent between locating at $x > 0$ and $-x$ (this in turn implies that $\lambda(x)$ is symmetric around zero as well). In other words, the situation on $(-\infty, 0)$ is a mirror image of that on $(0, \infty)$. Therefore, without loss of generality, hereafter we only consider locations $x \geq 0$.

Let $\pi(c)$ stand for the profit a c-type firm earns under its profit-maximizing choices:

$$\pi(c) \equiv \max\{\,\Pi(c,p,x)\,|\,p \geq c,\ x \geq 0\,\}.$$

By the envelope theorem, $\pi(c)$ is continuously differentiable in c and more productive firms earn higher profits*:* $\pi'(c) = \Pi_c < 0$, where $(\cdot)_c$ stands for the partial derivative w.r.t. c. A firm chooses to produce if and only if $\pi(c) > f$. If, in addition, we can guarantee that $\pi(c_{\min}) > f > \pi(\infty)$, then the equation $\pi(c) = f$ has the unique solution $\bar{c} > c_{\min}$. We call $\bar{c}$ the cutoff cost. In other words, $\bar{c}$ is the marginal cost of the least productive active firm, which is indifferent between producing and non-producing.

For each $c \in [c_{\min}, \bar{c}]$, let $(p(c),\ x(c)) \in \mathbb{R}^2$ stand for the c-type firm's profit-maximizing choice of price and location. Since the two equations, $p = p(c)$ and $x = x(c)$, define parametrically a curve in the (p,x)-plane, it is natural to call the set $\{(p(c),\ x(c)) \in \mathbb{R}^2\,|\,c \in [c_{\min}, \bar{c}]\}$ the *price-location curve.* Consider firms whose optimal locations $x(c) \in [0, S)$.[4] For these firms, $(p(c),\ x(c))$ solve the following FOCs (see (2.9)):

$$\int_X D\left(\frac{\lambda(y)p}{k_\tau(x,y)}\right) l(y)\mathrm{d}y + (p-c)\int_X D'\left(\frac{\lambda(y)p}{k_\tau(x,y)}\right)\frac{\lambda(y)}{k_\tau(x,y)} l(y)\mathrm{d}y = 0, \tag{2.11}$$

$$\int_X D'\left(\frac{\lambda(y)p}{k_\tau(x,y)}\right)\left[\frac{\partial k_\tau/\partial x}{k_\tau(x,y)^2}\right]\lambda(y)l(y)\mathrm{d}y = 0. \tag{2.12}$$

Equations (2.11) - (2.12) imply the following result.

Proposition 1. *(i) More productive firms produce at larger scales and charge lower prices:*

$$\frac{\mathrm{d}p(c)}{\mathrm{d}c} > 0, \qquad \frac{\mathrm{d}}{\mathrm{d}c}Q(p(c),\ x(c)) < 0. \tag{2.13}$$

(ii) More productive firms choose more competitive locations on $[0, S)$ *if and only if the profit*

[4]Note that, since $l(x) = 0$ if $x \notin (-S, S)$, for some firms, the optimal choice of location can be at the corners $-S$ and S. Clearly, locations outside $[-S, S]$ cannot be optimal, as demand there is equal to zero.

is supermodular along the price-location curve:

$$\Pi_{px}(c, p(c), x(c)) > 0. \tag{2.14}$$

Proof. (i) Totally differentiating both sides of the FOCs (2.11) - (2.12) w.r.t. c yields

$$\begin{pmatrix} \mathrm{d}p(c)/\mathrm{d}c \\ \mathrm{d}x(c)/\mathrm{d}c \end{pmatrix} = -\begin{pmatrix} \Pi_{pp} & \Pi_{px} \\ \Pi_{px} & \Pi_{xx} \end{pmatrix}^{-1} \begin{pmatrix} \Pi_{cp} \\ \Pi_{cx} \end{pmatrix}, \tag{2.15}$$

where the right-hand side is evaluated at $(p, x) = (p(c), x(c))$. As implied by (2.9) - (2.10), we have: $\Pi_{cp} = -Q_p > 0$, $\Pi_{cx} = -Q_x = 0$, plugging which back to (2.15) yields

$$\begin{pmatrix} \mathrm{d}p(c)/\mathrm{d}c \\ \mathrm{d}x(c)/\mathrm{d}c \end{pmatrix} = \frac{1}{\Pi_{pp}\Pi_{xx} - \Pi_{px}^2} \begin{pmatrix} \Pi_{xx}Q_p \\ -\Pi_{px}Q_p \end{pmatrix}. \tag{2.16}$$

Using (2.16) and the chain rule, and taking into account that $Q_x = 0$, we obtain:

$$\frac{\mathrm{d}p(c)}{\mathrm{d}c} = \frac{\Pi_{xx}}{\Pi_{pp}\Pi_{xx} - \Pi_{px}^2} Q_p > 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}c} Q(p(c), x(c)) = \frac{\Pi_{xx}}{\Pi_{pp}\Pi_{xx} - \Pi_{px}^2} Q_p^2 < 0,$$

where both inequalities hold due to the SOC. This proves (2.13).

(ii) The equivalence of (2.14) to $\mathrm{d}x(c)/\mathrm{d}c > 0$ follows immediately from (2.16) and the SOC. □

We assume that the appeal of a product type y to a x-type consumer decays with the distance $|x - y|$ between x and y. This is reminiscent to the model proposed by Ushchev and Zenou (2018), where consumer's willingness to pay for a variety decreases with the geodesic distance from a consumer to a firm in a product-variety network. However, unlike these authors, we neither use specific functional forms for preferences nor for the distance decay patterns. Therefore, providing full analytical characterization of equilibria and a clear-cut comparative statics in the most general case is problematic. In particular, as firms compete both within and across locations, price competition among firms cannot be described as an aggregative game (Anderson et al. 2013), since the whole schedule $\lambda(\cdot)$ of competitive toughness matters for individual pricing behavior of each firm. To obviate this difficulty, we pay special attention to the case when the distance decay rate is very high (formally, infinitely high), which can be viewed as the extreme case of *fully localized* competition: firms compete within but not across locations in the product space. In this case, price competition among firms can be described by an aggregative game, which radically simplifies the analysis of firm behavior and equilibrium characterization.

3 The special case

In this section, we consider the special case of the above model where the distance decay rate is infinitely high. Formally, we take the limit in (2.1) under $\tau \to \infty$. This is where Assumption 3 plays its role: because the distance decay kernel $k_\tau(x, \cdot)$ given by (2.2) is a standard mollifier, it converges to the *Dirac's delta* concentrated at x when $\tau \to \infty$.[1] Consequently, the utility function (2.1) of an x-type consumer becomes

$$\mathcal{U}_x = V\left(\int_{\Omega_x} u(q(\omega))\mathrm{d}\omega\right) + q_0. \tag{3.1}$$

A consumer located at $x \in X$ seeks to maximize her utility (3.1) subject to the budget constraint (2.3), which becomes

$$\int_{\omega\in\Omega_x} p(\omega)\, q(\omega)\, \mathrm{d}\omega + q_0 \leq I. \tag{3.2}$$

Along the same lines as in the general case, one can show that the market demand $Q_x(\omega)$ for a variety $\omega \in \Omega_x$ becomes

$$Q_x(\omega) = l(x)D(\lambda(x)p(\omega)), \tag{3.3}$$

where the product-type specific demand shifter $\lambda(x)$ is now defined by

$$\lambda(x) \equiv \frac{1}{V'\left(\int_{\omega\in\Omega_x} u\left(q\left(\omega\right)\right) d\omega\right)}. \tag{3.4}$$

The operating profit function is then given by

$$\Pi(c, p, x) = (p - c)l(x)D(\lambda(x)p).$$

3.1 Firm behavior: sorting between firms and locations

In this section, we show that under quite general assumption about the lower-tier utility $u(\cdot)$, firms that choose internal locations, $S > x(c) > 0$, are completely sorted across the locations: less productive firms choose to locate further from zero. In other words, $x(c)$ is increasing in c.

When $\tau \to \infty$, the first order condition in (2.11) can be written as follows:

$$\frac{\partial\Pi\left(p, x, c\right)}{\partial p} = 0 \Leftrightarrow 1 + \frac{(p-c)}{D\left(\lambda(x)p\right)} \cdot \frac{\partial}{\partial p} D\left(\lambda(x)p\right) = 0. \tag{3.5}$$

[1] More precisely, we have: $m_\tau \rightharpoonup \delta_x$ as $\tau \to \infty$ were m_τ is the linear functional defined by $m_\tau(\varphi) \equiv \int_X k_\tau(x, y)\varphi(y)\mathrm{d}y$ for any $\varphi \in C(X)$, while $\rightharpoonup$ stands for convergence in the weak topology. The Dirac's delta δ_x concentrated at $x \in X$ is a linear functional over $C(X)$ defined as follows: $\delta_x(\varphi) \equiv \varphi(x)$ for all $\varphi \in C(X)$. See Evans (2010) for details.

The latter is equivalent to

$$\frac{p-c}{p} = -\frac{D\left(\lambda(x)p\right)}{\lambda(x)pD'\left(\lambda(x)p\right)}, \tag{3.6}$$

which is the usual monopolistic condition of unit elasticity. The above equation gives the relationship between the price and firm's location in the taste space, which we define as $p\left(x,c\right)$.

Given this relationship, the firm then chooses its optimal location by solving (2.12) (when $\tau \to \infty$), which implies[2]

$$\frac{l(x)}{l'(x)} \cdot \frac{\lambda'(x)}{\lambda(x)} = -\frac{D\left(\lambda(x)p\left(x,c\right)\right)}{\lambda(x)p\left(x,c\right)D'\left(\lambda(x)p\left(x,c\right)\right)}. \tag{3.7}$$

Combining (3.6) and (3.7), we derive a neat expression for a markup $\mathcal{M}(x,c)$:

$$\mathcal{M}(x,c) \equiv \frac{p\left(x,c\right)-c}{p\left(x,c\right)} = \frac{\lambda'(x)}{\lambda(x)}\frac{l(x)}{l'(x)} = \frac{\hat{\lambda}(x)}{\hat{l}(x)}, \tag{3.8}$$

where $\hat{\lambda}(x) \equiv \lambda'(x)/\lambda(x)$ and $\hat{l}(x) \equiv l'(x)/l(x)$. The above expression for markups implies the following lemma.

Lemma 1. *If $l(x)$ is strictly decreasing in x on $(0,S)$, then for any interval $(a,b) \subseteq (0,S)$ such that Ω_x is not empty for any $x \in (a,b)$ in the equilibrium, $\lambda(x)$ is strictly decreasing on (a,b) in the equilibrium.*

Proof. If Ω_x is not empty for any $x \in (a,b)$ in the equilibrium, then any point x on (a,b) is an optimal location for some firms that stay in the market. The markups set by these firms are strictly positive (since there is the fixed cost of production). From (3.8), positive markups imply that $\lambda'(x) < 0$ on (a,b) (as $l'(x) < 0$ on (a,b)). □

The result in the lemma is explained by a simple trade-off. Choosing an optimal location, firms face a trade-off between the size of the location and the level of competition there. Decreasing $l(x)$ means that, all else equal, the further is firm's location from zero, the lower is the demand for its product. Hence, if firms find it profitable to locate further from zero, lower demand must be compensated by a lower level of competition at this location, which in turn means lower $\lambda(x)$. The expression in (3.8) also implies that, depending on the behavior of the fraction $\hat{\lambda}(x)/\hat{l}(x)$ (which is, in fact, the ratio of the elasticities of the population and competition measures), markups can, in general, grow or decline with a rise in the distance from the zero location.

Next, we explore how the marginal cost of production affects locations firms choose. It turns out that necessary and sufficient conditions for spatial equilibria to exhibit positive (or negative) spatial sorting of firms can be expressed in terms of the demand schedule properties. More precisely, the following proposition holds.

[2]Due to the envelope theorem, $\partial p\left(x,c\right)/\partial x = 0$. In what follows, we assume that $\lambda(x)$ is differentiable.

Proposition 2. *Assume that $l(x)$ is strictly decreasing in x for all $x \in (0, S)$. If, in addition, $\mathcal{E}_D(\lambda p) \equiv -(\lambda p) D'(\lambda p) / D(\lambda p)$ is strictly increasing (decreasing) in its argument λp (in other words, the demand elasticity is strictly increasing (decreasing) in price), then, in equilibrium, for all c such that $S > x(c) > 0$, we have: $dx(c)/dc > 0$ (< 0).*

Proof. The proof is based on the log-supermodularity property of the variable profit function. Specifically, we have

$$\log \Pi(p, x, c) = \log(p - c) + \log l(x) + \log D(\lambda(x)p).$$

Thus,

$$\frac{\partial^2 \log \Pi}{\partial p \partial c} = \frac{1}{(p-c)^2} > 0,$$

$$\frac{\partial^2 \log \Pi}{\partial x \partial c} = 0,$$

$$\frac{\partial^2 \log \Pi}{\partial p \partial x} = -\lambda'(x) \frac{\partial \mathcal{E}_D(\lambda p)}{\partial \lambda p} > 0 \Longleftrightarrow \frac{\partial \mathcal{E}_D(\lambda p)}{\partial \lambda p} > 0,$$

since $-\lambda'(x) > 0$. The above log-supermodularity properties of the profit function result in the statements of the proposition. □

Recall that, in the general case with finite τ, the necessary and sufficient condition for $dx(c)/dc > 0$ is that $\Pi_{px}(c, p(c), x(c)) > 0$. The issue is that, in the general case, this condition cannot be expressed in terms of the primitives of the model. However, when $\tau = \infty$, a natural condition on the elasticity of demand (the Marshall's Second Law of Demand) appears to be sufficient for perfect sorting among firms located on $(0, S)$ in the equilibrium - less productive firms locate further from the zero location.

One can readily verify that linear demand has an increasing demand elasticity. Most specifications well established in the literature[3] also satisfy this property. It is worth noting that CES demand has a constant elasticity of demand. In particular, the variable profit of a firm can be written as follows:

$$\Pi(c, p(x,c), x) = (p(x,c) - c)\, l(x) D(\lambda(x) p(x,c)) = \frac{(\sigma-1)^{\sigma-1}}{(\sigma)^{\sigma}} c^{1-\sigma} \frac{l(x)}{(\lambda(x))^{\sigma}}.$$

Such a profit function implies that, given $\lambda(x)$, all firms (irrespective of their marginal cost) choose the location(s) where $l(x)/(\lambda(x))^{\sigma}$ achieves its maximum on $[0, S]$. In the context of the general equilibrium, this outcome may result in multiple equilibria. Indeed, if there exists a general equilibrium with a certain equilibrium schedule of λ, then any reallocation of firms across the

[3]Other examples include the CARA demand system (Behrens and Murata, 2007) and Stone-Geary demnd system (Simonovska, 2015). See Zhelobodko et al. (2012) and Arkolakis et al. (2018) for more examples.

locations that keeps $\lambda(x)$ the same is also an equilibrium (see more on the concept of the general equilibrium in the model in the next section).

3.2 The general equilibrium

In this section, we describe the general equilibrium in the model. We assume that $l(S)$ is sufficiently low. This assumption together with the presence of the fixed cost of production imply that the location of the firm with marginal cost $\bar{c}$, $x(\bar{c})$, always belongs to $[0,\, S)$. That is, there are some locations (close to S) that are not served by firms (consumers there purchase only the numéraire). This case is of a particular interest as it implies one more endogenous margin of production - the set of niches served by firms in the market.

We showed that, when the demand elasticity is strictly increasing (see Proposition 2), firms are positively sorted on $(0, S)$: $dx(c)/dc > 0$. This implies that the most productive firms choose zero as the optimal location: $x(c_{min}) = 0$. The mass of firms at location $x \geq 0$ is then given by

$$\mu(x) = M_e\, g\left(c(x)\right)\, c'(x),$$

where $c(x)$ is the inverse function of $x(c)$ and represents the productivity of firms located at x.

The general equilibrium is then **a bundle** $(M_e, \bar{c}, \{\lambda(x),\, p(x,c),\, x(c)\}_{x\in\Omega, c\in[c_{min},\bar{c}]})$, such that the following conditions hold:

C1 The measure of competition intensity satisfies:

$$\lambda(x) = \frac{1}{V'\left(\mu(x)u(q_x)\right)}, \tag{3.9}$$

where $q_x = D\left(\lambda(x)p(x,c)\right)$ is the per capita consumption of one variety produced by a firm located at x. As there are no firms located at $x > x(\bar{c}) \equiv \bar{x}$, $\lambda(x) = 1/V'\left(0\right)$ for all $x \in (\bar{x},\; S]$. To hold the continuity of the problem, the value of $\lambda(x)$ defined in (3.9) at the rightmost location $\bar{x}$ must be equal to $1/V'\left(0\right)$. Equivalently, $c'(\bar{x})$ must be equal to zero.

C2 The schedule of prices, $p(x,c)$, solves with respect to p

$$\left(p - c\right)\lambda(x) = -\frac{D\left(\lambda(x)p\right)}{D'\left(\lambda(x)p\right)}. \tag{3.10}$$

C3 The optimal location of a firm with c, $x(c)$, solves with respect to x

$$\frac{p(x,c) - c}{p(x,c)} = \frac{\lambda'(x)}{\lambda(x)}\frac{l(x)}{l'(x)}, \tag{3.11}$$

with $x(c_{min}) = 0$.

C4 The cutoff $\bar{c}$ is determined by the zero-profit condition:

$$\Pi\left(\bar{c}, p(\bar{c}), x(\bar{c})\right) = f. \tag{3.12}$$

C5 The mass of entrants is determined by the free entry condition:

$$\int_{c_{min}}^{\bar{c}} \left(\Pi\left(c, p(c), x(c)\right) - f\right) \cdot g\left(c\right) dc = f_e. \tag{3.13}$$

Next, we explore the existence and uniqueness of the equilibrium defined above. Note that the above definition of the general equilibrium implies that the spatial pattern $\{c(x), \lambda(x)\}_{x\in[0,\overline{x}]}$ is described by the following system of differential equations

$$\frac{\mathrm{d}\lambda}{\mathrm{d}x} = -a(x)\lambda\mathcal{M}(x, c),$$

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{1}{M_e}\frac{\left(V'\right)^{-1}(1/\lambda)}{g(c)\,u(q_x)},$$

where $a(x) \equiv -l'(x)/l(x) > 0$ is the rate at which population decreases with the distance $|x|$ from the origin. It is straightforward to show (see Section 3.3) that $\mathcal{M}(x, c)$ and q_x are functions of $\lambda(x)c$. Thus, the system can be rewritten as follows:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}x} = -a(x)\lambda\mathcal{M}(\lambda c), \tag{3.14}$$

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{1}{M_e}\frac{\left(V'\right)^{-1}(1/\lambda)}{g(c)\,u(q(\lambda c))}. \tag{3.15}$$

Hence, the existence of the equilibrium is in fact determined by the existence of the solution of the above system with the following boundary conditions: $c(0) = c_{min}$ and $\lambda(\overline{x}) = 1/V'(0) \equiv \lambda_{min}$. In particular, the following proposition holds.

Proposition 3. *If $l(S)$ is sufficiently low and $l(0)$ is sufficiently high, then there exists a general equilibrium in the model described by the conditions in C1-C5.*

Proof. In the Appendix. □

Sufficiently low $l(S)$ is implies that $\bar{x} < S$, while sufficiently high $l(0)$ is necessary to guarantee the positive mass of entrants, M_e, into the market. In the Appendix, we formulate the exact conditions on $l(S)$ and $l(0)$ in terms of the primitives in the model. We also show that, under quite a general condition on $l(x)$, the general equilibrium is unique. Specifically, the following proposition holds.

Proposition 4. *Assume that, in addition to the conditions in Proposition 3, $a'(x) \geq 0$. Then, the general equilibrium is unique.*

Proof. In the Appendix. □

Notice that $a'(x) \geq 0$ if and only if $l'(x)^2 - l''(x)l(x) \geq 0$, which, for instance, holds for $l(x) = a - bx^\gamma$ where $\gamma \geq 1$.[4] Note that the condition is sufficient meaning that the equilibrium can be unique even when $a'(x) < 0$.

3.3 The distribution of markups

In this section, we explore how firm's markup depends on that firm location and marginal cost of production. To do so, we first express firm's markups in terms of quantities sold. Specifically, the firm's profit maximization problem can be reformulated in the following way. Given the inverse demand function, a firm maximizes its profit with respect to its location and the quantity per consumer sold at this location, q. Taking into an account (2.5), the inverse demand function is given by

$$p(q,\, x) = u'(q)\,/\lambda(x).$$

Hence, a firm's variable profit function can be written as follows:

$$\Pi(c, q, x) = (u'(q)\,/\lambda(x) - c)\, ql(x).$$

This implies that given firm's location x, the quantity per consumer supplied by the firm solves

$$\frac{\partial \Pi(c, q, x)}{\partial q} = 0 \Leftrightarrow u'(q) + q\, u''(q) = \lambda(x)c. \tag{3.16}$$

Let us define the solution of the above expression as $q(x, c)$: a quantity per consumer sold at x by a firm with cost c. Note that $q(x, c)$ is completely determined by $\lambda(x)c$ and is a decreasing function of $\lambda(x)c$.

Given $q(x, c)$, the firm then chooses its optimal location (in the case, when the optimal location is internal: $x \in (0,\, S)$) by solving:

$$\frac{\partial \Pi(q, x, c)}{\partial x} = 0 \Leftrightarrow \frac{\lambda'(x)}{\lambda(x)} \frac{l(x)}{l'(x)} = 1 - \frac{\lambda(x)c}{u'(q(x, c))} = -\frac{q(x, c)\, u''(q(x, c))}{u'(q(x, c))}.$$

The latter implies that a firm's markup, $\mathcal{M}(x, c)$, is equal to $\mathcal{E}_{u'}(q(x, c))$. Since, $q(x, c)$ is a function of $\lambda(x)c$, $\mathcal{M}(x, c)$ is a function of $\lambda(x)c$. Moreover, if $\mathcal{E}_D$ is increasing in price, $\mathcal{E}_{u'}$ is increasing in quantity. This in turn implies that $\mathcal{M}(x, c)$ is a decreasing function of $\lambda(x)c$.

In the general equilibrium, less productive firms choose locations that are further from zero: $c(x)$ is increasing in x for all $x > 0$. At the same time, $\lambda(x)$ is decreasing in x. As a result, $\lambda(x)c(x)$ and, therefore, the markup function can be non-monotonic in x. In fact, the behavior of the markup function in the equilibrium is determined by the interplay of two forces: firm heterogeneity and consumer heterogeneity. In particular, when firms are homogeneous in terms

[4] We need this condition on $l(x)$ to guarantee the uniqueness of the cutoff $\overline{c}$, which is not straightforward in our framework.

of their productivity and consumers have different tastes, the behavior of the markup function is solely determined by $\lambda(x)$, which is decreasing in x. This implies that the markup function is increasing in x: firms located further from zero set higher markups. Indeed, to compensate lower demand in more "remote" locations, homogeneous firms charge higher prices there. When firms are heterogeneous, less productive firms choose more remote locations to avoid tougher competition in denser locations. Since less productive firms charge lower markups, the presence of firm heterogeneity adds a decreasing trend in the behavior of the markup function. As a result, the markup function can be non-monotonic.

In particular, we can prove the following proposition.

Proposition 5. *1)* $(\lambda(x)c(x))'_{x=\bar{x}} < 0$. *2) If* $| l'(0) | < \infty$ *and* c_{min} *is sufficiently close to zero, then* $(\lambda(x)c(x))'_{x=0} > 0$. *3) If, in addition,* $g'(c) \geq 0$ *and* $(l'(x)/l(x))'_x \leq 0$*, then the markup function,* $\mathcal{M}(\lambda(x)c(x))$*, has a U-shape on* $[0, \bar{x}]$.

Proof. In the Appendix. □

The first two statements in the proposition mean that the markup function is decreasing around zero (under some restrictions on the parameters) and increasing around $\bar{x}$. The intuition behind is as follows. Given other things equal, lower c_{min} implies a higher level of firm heterogeneity in the neighborhood of 0 in the equilibrium. When this level is high enough, we have the decreasing markup function in the neighborhood of 0 as discussed before. In the neighborhood of $\bar{x}$, $c'(x)$ is close to zero, implying a low level of firm heterogeneity there. As a result, the markup function is increasing. Finally, under some additional assumptions on $g(c)$ and $l(x)$, the markup function is globally U-shaped. Note that the assumption on $g(c)$ seems to be natural: it is more likely to get a bad productivity draw than a good one. For instance, a Pareto distribution satisfies this property.

An important implication of the above findings is that, due to the positive sorting in the equilibrium, the relationship between firm's marginal costs and markups has a U-shape as well. In other words, in the equilibrium, the most and least productive firms set the highest markups, while in traditional models of monopolistic competition with firm heterogeneity, the highest markups are set by the most productive firms only - the relationship between firm's marginal costs and markups is negative.

3.4 Comparative static: A proportional rise in the population density

In this section, we analyze the implications of a proportional change in $l(x)$ in all locations: $l^{new}(x) = \delta l^{old}(x)$. Without loss of generality, we assume that $\delta > 1$ meaning that the population density rises that can be interpreted as the outcome of free trade with an identical economy. To explore the effects of the change in $l(x)$, we distinguish between the short-run and long-run effects. This also simplifies understanding of the intuition behind. By the short-run effects we mean the implications of the change in $l(x)$ when the mass of entrants, M_e, does not react to changes in $l(x)$. The following lemma holds.

Lemma 2. *Under fixed M_e, a proportional rise in $l(x)$ increases the cutoffs $\bar{x}$ and $\bar{c}$. Given this change in $l(x)$, the values of the functions $\lambda(x)$ and $c(x)$ rise in all locations (only $c(0) = c_{min}$ does not change).*

Proof. In the Appendix. □

The intuition of the findings above is as follows. All else equal, a rise in the population size implies higher firm's profits. As a result, some inefficient firms that did not produce before find it profitable to produce now under a higher value of the population size: $\bar{c}$ rises. In the similar way, some product niches that were not "attractive" to firms before become now larger in terms of their size and, therefore, generate positive profits: $\bar{x}$ rises. Finally, a rise in the number of firms in the neighborhood of $\bar{x}$ leads to a higher level of competition in this region (increasing $\lambda(x)$). As a result, tougher competition forces firms to relocate closer to the origin, implying that $c(x)$ rises in all locations, except for $x = 0$.

To analyze the long-run effects, one needs to take into account the corresponding change in M_e and its effects on the equilibrium outcomes. We expect that a uniform rise in the population density leads to a higher value of M_e. Though this outcome is very intuitive and, moreover, confirmed by our simulations, under the presence of sorting between firms and product niches we cannot provide a strict proof for this statement. Nevertheless, in the below considerations we assume that M_e increases. In the proof of the uniqueness of the equilibrium (see Step 4), we show that a rise in M_e implies that $\lambda(x)$ increases at all locations. Combining this with the results in Lemma 2, we can formulate the following lemma.

Lemma 3. *Given a proportional rise in $l(x)$, if the number of entrants in the equilibrium,M_e, increases under this change in $l(x)$, then the function $\lambda(x)$ shifts upwards implying that the cutoff $\bar{x}$ increases.*

The above lemma implies that a uniform rise in the population size makes some firms choose product niches that were not served before. This is because the short-run and long-run forces work in the same direction with respective to $\lambda(x)$ and $\bar{x}$. In the long-run, new entrants induce tougher competition at each location. As a result, less productive firms are forced to move to less populated niches to avoid competition, which in turn increases $\bar{x}$.

Regarding $c(x)$ and $\bar{c}$, the short-run and long-run effects seem to be different. On the one hand, a uniform rise in $l(x)$ shifts $c(x)$ upwards and increases $\bar{c}$ (as stated in Lemma 2 and discussed after). On the other hand, in the long-run there are new entrants that force less productive firms to choose less populated niches and least productive firms to exit: $c(x)$ shifts downwards and $\bar{c}$ decreases. It appears that it is very complicated to show which effect is stronger in our model.

However, we run numerous simulations and in all of them the long-run effect is stronger meaning that a uniform rise in the population density shifts $c(x)$ downwards and decreases the productivity cutoff $\bar{c}$. The latter outcome is in line with results in standard models of monopolistic competition with variable markups: a rise in the market size makes least productive firms leave the market.

4 Welfare implications: Numerical analysis

In this section, we explore the distributional consequences of a uniform rise in the population density. Note that the explicit analysis of the welfare changes in the model is prohibitively complicated. Therefore, we perform detailed numerical analysis.

In the equilibrium, the welfare at location x can be written as follows:

$$\mathcal{U}_x = I + V\left(\mu(x)u(q(x))\right) - \mu(x)p(x)q(x),$$

where $q(x)$ is the consumed quantity of each variety at x and $p(x)$ is its price.

To explore in more detail, how changes in δ can affect consumer welfare, we employ the following decomposition (that works for small changes in δ):

$$\frac{\triangle\mathcal{U}_x}{\mathcal{U}_x} \approx \mathcal{E}_\mu^U(x)\frac{\triangle\mu(x)}{\mu(x)} + \mathcal{E}_q^U(x)\frac{\triangle q(x)}{q(x)} + \mathcal{E}_p^U(x)\frac{\triangle p(x)}{p(x)},$$

where $\mathcal{E}_\mu^U(x)$, $\mathcal{E}_q^U(x)$, and $\mathcal{E}_p^U(x)$ are the corresponding elasticities at location x. In the equilibrium,

$$\mathcal{E}_q^U(x) = \mu(x)\left(V'\left(\mu(x)u(q(x))\right)u'(q(x)) - p(x)\right) = 0.$$

Hence, we have

$$\frac{\triangle\mathcal{U}_x}{\mathcal{U}_x} \approx \mathcal{E}_\mu^U(x)\frac{\triangle\mu(x)}{\mu(x)} + \mathcal{E}_p^U(x)\frac{\triangle p(x)}{p(x)}. \tag{4.1}$$

We call the first term in the right-hand side of the above the variety effect on welfare, while the second term is the price effect on welfare.

4.1 Parameterization of the model

We consider the following parameterization of our model. The upper-tier utility function is given by $\ln\left(\zeta + \mu\left(\cdot\right)u\left(\cdot\right)\right)$, where ζ is normalized to unity in the benchmark scenario (which implies that $\lambda_{min} = 1$). For the lower-tier utility we consider a quadratic function that generates linear demand: $u\left(q\right) = aq - \frac{b}{2}q^2$. Following Hepenstrick and Tarasov (2015), we set a to unity and b to 0.22 in the benchmark.[1]

[1] Hepenstrick and Tarasov (2015) calibrate this utility function by targeting US consumers' income elasticity of the extensive margin of consumption.

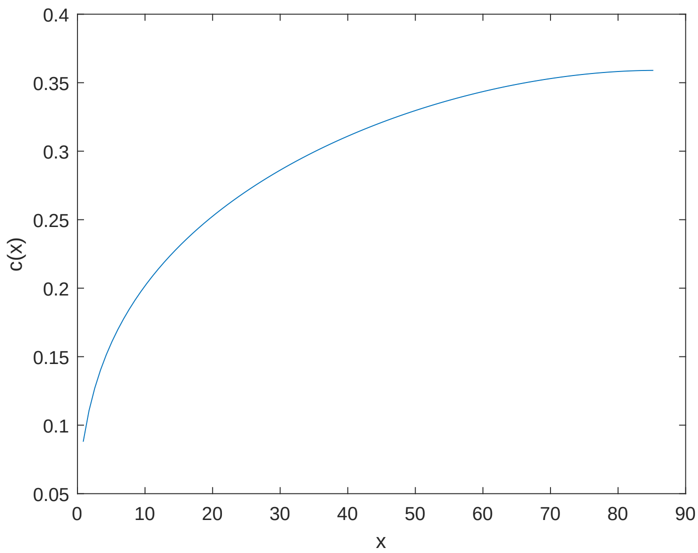
Table 1: Parameterization and Meta-analysis

Function	Parameterization	Benchmark values	Meta-analysis
$V(\cdot)$:	$\ln(\zeta + \mu(\cdot) u(\cdot))$	$\zeta = 1$	
$u(q)$:	$aq - \frac{b}{2}q^2$	$a = 1,\ b = 0.22$	
$l(x)$:	$A\left(1 - \left(\frac{x}{S}\right)^{\gamma}\right)$	$A = 10,\ \gamma = 1,\ S = 100$	$A \sim U(10, 15),\ \gamma \sim U(1, 2.2)$
$g(c)$:	$\frac{\alpha c^{\alpha-1}}{c_{max}^{\alpha} - c_{min}^{\alpha}}$	$\alpha = 3.4,\ c_{min} = 0.05,\ c_{max} = 1.0$	$\alpha \sim U(1, 5)$
		$I = 2, f = 0.68, f_e = 0.02$	

For the distribution of consumer tastes, we take $l(x) = A(1 - (x/S)^{\gamma})$ where $\gamma \geq 1$ (which guarantees a unique equilibrium). In the benchmark case, we set A to 10, S is normalized to 100, and the curvature parameter γ is equal to unity. Finally, the distribution of firm productivities is truncated Pareto: the density function is $g(c) = \alpha c^{\alpha-1}/(c_{max}^{\alpha} - c_{min}^{\alpha})$. In the benchmark scenario, the shape parameter α is set to 3.4 (Bernard et al. 2007), c_{min} is equal to 0.05, and c_{max} is normalized to unity. The rest of the parameters is as follows: the consumer income I is set to 2, the fixed cost of production f is 0.68, and the cost of entry f_e is 0.02. Table 1 summarizes the above parameterization.

In the benchmark case, the equilibrium values of some key variables are the following: $\bar{x} \approx 84.4$, $\bar{c} \approx 0.359$, and $\lambda(0) \approx 2.03$ (recall that λ_{min} is normalized to unity). In this equilibrium, the product niches on $(84.4, 100]$ remain unserved. Figure 1 depicts the matching function $c(x)$ that appears to be concave.

Figure 1: $c(x)$



We also look at the distribution of prices and markups on the product space in the benchmark equilibrium (see Figure 2 and 3, respectively). It is not surprising that the price function is increasing, as there is less competition in more remote locations and, moreover, there are less productive firms. At the same time, as proved before, the markup schedule has a U-shape with the maximum of about 0.7 at the origin. The average markup (weighted by the share of firms at each location) is around 0.3726.

Figure 2: $p(x)$

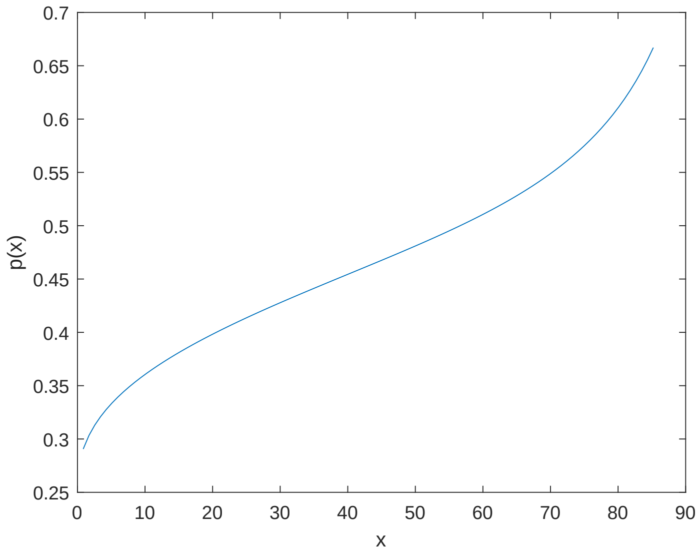
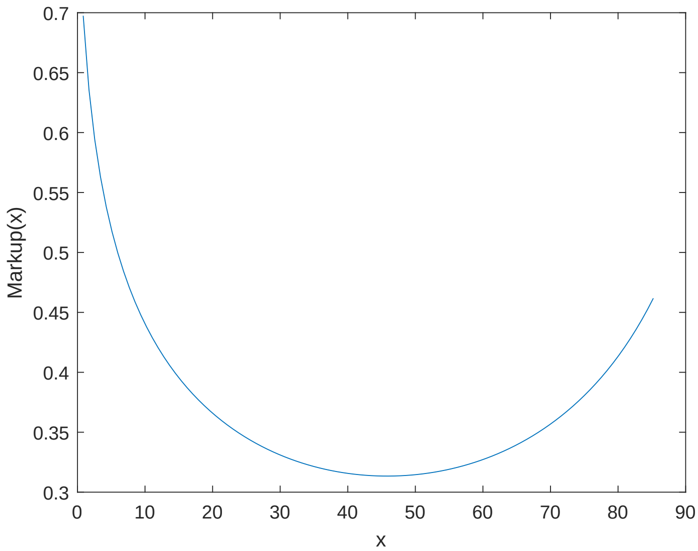
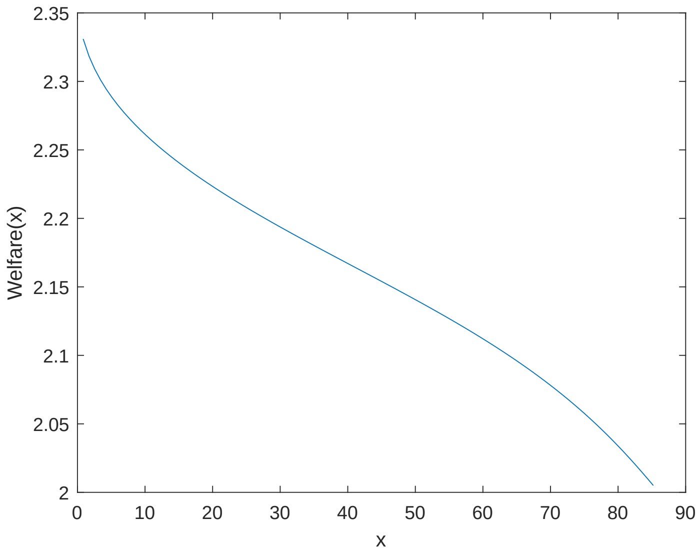


Figure 3: $Markup(x)$



Finally, we look at the distribution of (per capita) welfare across locations (see Figure 4). As it can be inferred from the figure, the welfare function is decreasing. In particular, the welfare of consumers located at zero is by around 17% higher than the welfare of consumers at the most remote locations. In words, our model predicts more benefits from living in more densely populated areas. This is not only due to a higher number of firms in larger product niches, but also due to selection of more productive firms into these niches. The level of inequality (we use the Theil index) is 1.687.

Figure 4: $\mathcal{U}_x$



4.2 The effects of increasing population density

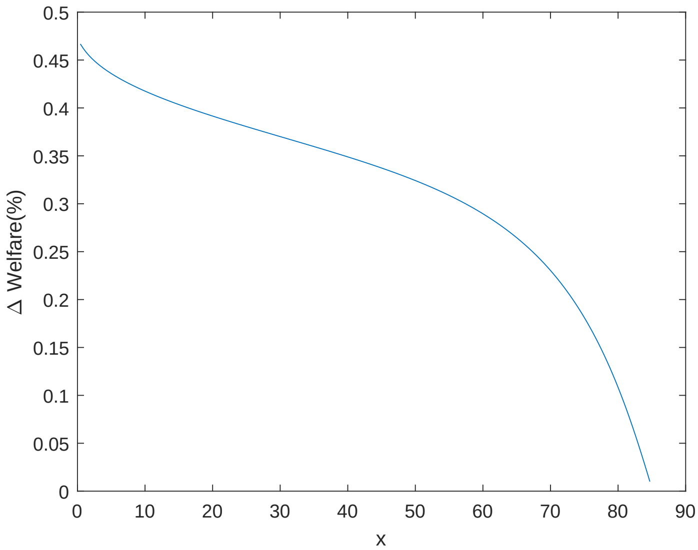
In this section, we numerically explore how a 5% uniform increase in the population density (which allows us to use the welfare decomposition in (4.1)) affects the equilibrium and, particularly, consumer welfare. Under the benchmark specification, the change in the population density increases $\bar{x}$ from 84.4 to 85.2 and decreases $\bar{c}$ from 0.359 to 0.356. As predicted by the theory, the matching function shifts downwards – at each location (besides the origin) there are now more productive firms located.

We also find that at all locations the number of varieties increases, while the prices fall. This implies that all consumers gain from the increase in population, which is in line with traditional models of monopolistic competition. However, these gains are quantitatively not the same across consumers. Figure 5 depicts the percentage change in consumer welfare at each location. As can be seen, the highest gains are at the origin, the lowest are around $\bar{x}$, and the difference between them is quite substantial.[2] The average welfare (weighted by the share of consumers at each location) rises by 0.39 (from 2.1923 to 2.2008). The inequality level increases as well, by 3.26 (from 1.687 to 1.742). This finding provides one more insight on how international trade can affect the level

[2]**Note** that in Figure 5 we consider welfare changes for locations that were served by firms before the change in the population density (this is the interval $[0, 84.4]$). Taking into account that there are no changes in welfare for locations from $[85.2, 100]$ (as they remain unserved), the curve in the figure goes down very steeply to zero, if we continue it.

of inequality in the economy that channeled through the matching mechanism between firms and product niches.

Figure 5: $\triangle\mathcal{U}_x$ total



We then consider the decomposition of welfare changes into the variety and price effects. Figures 6 and 7 depict changes in welfare across locations due to these effects, correspondingly. As can be seen, both curves are decreasing with respect to location x (except for the price effect which is slightly increasing around zero). In other words, consumers at more remote locations gain less in terms of both variety and price effects - the number of varieties (the prices) goes up (go down) by less at more remote locations. In the next subsection, we explore how this pattern of relative welfare changes depends on some parameters in the model.

Figure 6: $\triangle\mathcal{U}_x$ decomposition ("variety effect")

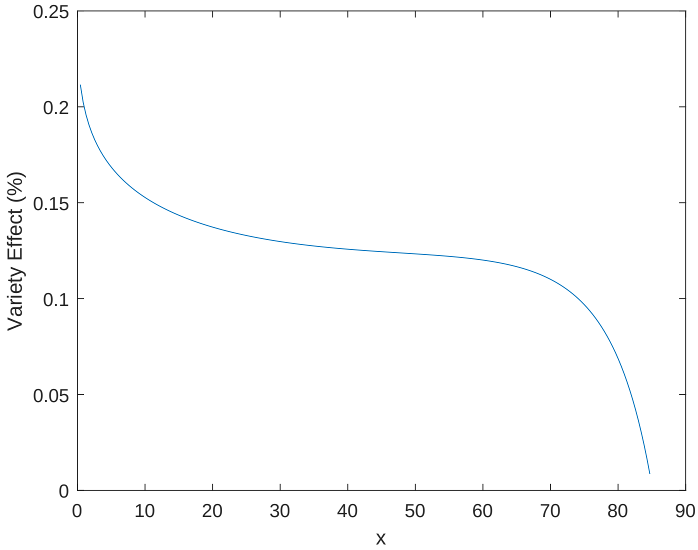
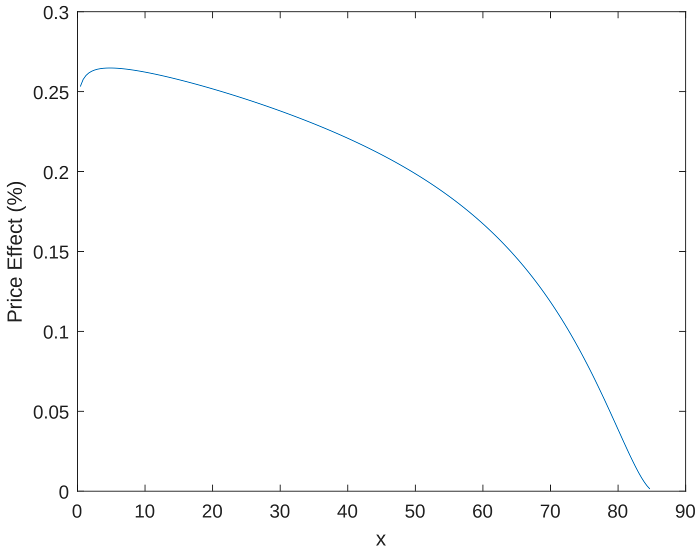
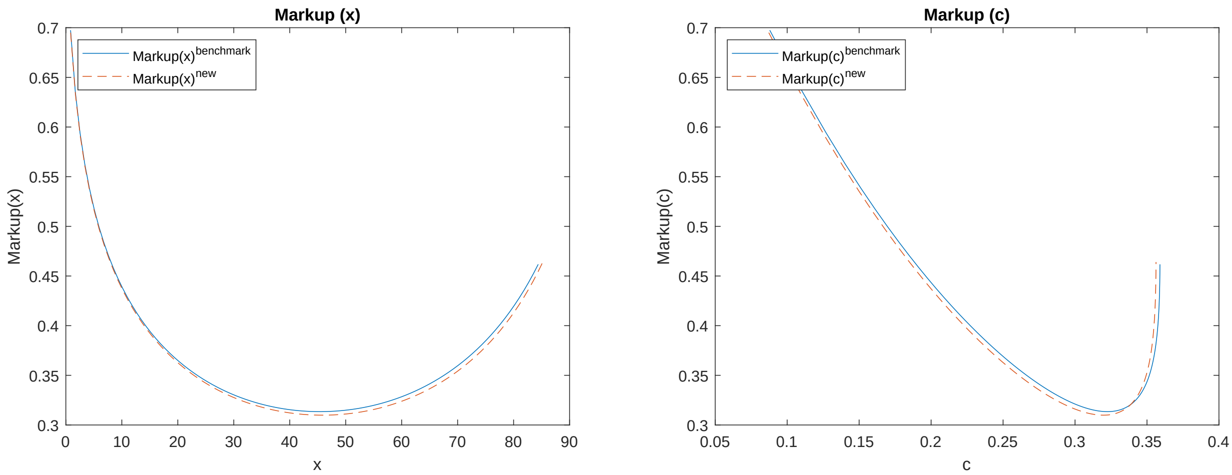


Figure 7: $\triangle\mathcal{U}_x$ decomposition ("price effect")



Finally, we look at changes in markups. As can be seen the left panel of Figure 8, markups fall at all locations. Specifically, the average markup falls by 0.9% (from 0.3726 to 0.3692). This finding is similar to that in spaceless models of monopolistic competition with variable markups (see, e.g., Zhelobodko et al. 2012). However, if we consider the distribution of markups with respect to productivity (the right panel of the figure), we will see that some relatively unproductive smaller firms increase their prices after the increase in the population density. This outcome takes place, as the uniform increase in $l(x)$ forces firms to choose more remote niches where the level of competition is lower. As a result, some firms find it profitable to increase their prices at new locations. As mentioned in the introduction, this firm behavior seems to be in line with some empirical evidence and cannot be captured by standard monopolistic competition frameworks.

Figure 8: Markup (x) (left) and Markup (c) (right)



4.2.1 Numerical meta-analysis

In this subsection, we numerically explore how the parameters characterizing the distribution of consumer taste, A and γ, and the shape parameter of the firm productivity distribution, α, affect the implications of the uniform increase in population considered in the previous section. Specifically, we set this triple to some draws from uniform distributions: $A \sim U(10,15)$, $\gamma \sim U(1,2.2)$, and $\alpha \sim U(1,5)$; in total, we consider 600 independent draws for the triple (A,γ,α). Then, for each draw, we compute the equilibrium and examine how a 5% uniform increase in the population density affects this equilibrium (all other parameters remain the same) with a particular interest in the size and distribution of the gains from the change and the inequality in the economy.

First, we examine how the above parameters numerically affect the level of inequality. Specifically, we run the following regression:

$$TH_i = \underset{(0.007)}{0.73} + \underset{(0.0005)}{0.09} A_i + \underset{(0.002)}{0.23} \gamma_i - \underset{(0.0006)}{0.02} \alpha_i + \xi_i,$$

where TH_i is the Theil index corresponding to the triple $(A_i, \gamma_i, \alpha_i)$ and ξ_i is the error term. The standard errors are in the brackets. As can be seen, higher γ (that makes the distribution of consumer tastes more dispersed) increases the level of inequality in the economy. A rise in the level of firm heterogeneity (lower α) raises inequality as well, though the impact is not too strong: reducing α by one adds 0.02 to the Theil index (note that the average Theil index in the sample is 2.11).

We then look at the changes in welfare and inequality caused by the increase in the population density. We find that the average change (across all 600 draws) of the average welfare is 0.46%, while the average change in the inequality is around 2.47%. The role of the parameters in explaining the changes in welfare and inequality can be represented by the following regressions:

$$\triangle\mathcal{U}_i = \underset{(0.007)}{0.84} - \underset{(0.0005)}{0.003}\,A_i + \underset{(0.002)}{0.003}\gamma_i - \underset{(0.0006)}{0.12}\,\alpha_i + \varepsilon_i,$$

$$\triangle TH_i = \underset{(0.02)}{4.32} - \underset{(0.002)}{0.12}\,A_i - \underset{(0.006)}{0.38}\,\gamma_i + \underset{(0.002)}{0.08}\,\alpha_i + \epsilon_i,$$

where $\triangle\mathcal{U}_i$ ($\triangle TH_i$) is the percentage change in the average welfare (inequality) corresponding to the triple $(A_i, \gamma_i, \alpha_i)$, ε_i and ϵ_i are the error terms.

The first thing to notice is that the effect of γ on the average welfare change is positive, but not statistically significant. At the same time, the effect on the change in the level of inequality is highly significant and negative: a rise of γ by one reduces the change in the inequality by 0.38. That is, in economies with more dispersed preferences, a uniform increase in the population density has a smaller impact on the level of inequality. We also find that a higher level of firm heterogeneity increases the welfare gains, which is in line with findings in Melitz and Redding 2015. Specifically, a decrease of α by one raises the change in the welfare by 0.12 (recall that the average change is 0.46%). The effect of firm heterogeneity on the change in inequality appears to be relatively less substantial and negative.

Finally, we examine how the parameters affect the location of consumers who have gains most (in the benchmark case it is consumers located around the origin). Specifically, we consider the following regression:

$$ARGMAX_i = \underset{(8.46)}{58.7} + \underset{(0.58)}{2.4}\,A_i + \underset{(2.39)}{0.29}\gamma_i - \underset{(0.74)}{22.8}\alpha_i + \zeta_i,$$

where $ARGMAX_i$ is the location of consumers who gain most from the 5% increase in the population density given $(A_i, \gamma_i, \alpha_i)$. As can be inferred, $ARGMAX$ is mostly affected by A and α: a uniformly higher number of consumers or a higher level of firm heterogeneity moves the location of consumers with the highest gains further from the origin; the impact of γ is positive, but insignificant.

Indeed, setting α to unity in the benchmark specification (all other parameters remain the same), we observe a quite different distribution of the welfare gains under a 5% increase in the population density (see Figure 9). The gains are relatively equal across the most of the locations with a sharp decrease closer to $\bar{x}$. The location of the highest gains is around 60. Decomposing the welfare changes into the variety and price effects, one can see that both effects have a hump shape (see Figures 10 and 11). In general, running a number of simulations, we find that the discussed two patterns of the distribution of the welfare gains (see Figures 5 and 9) prevail. It is either that consumers at the origin gain most (Figure 5) or that the gains are (slightly) increasing in x at the beginning of the interval $[0, \bar{x}]$ and then steeply go down (Figure 9).

Figure 9: $\triangle\mathcal{U}_x$ total

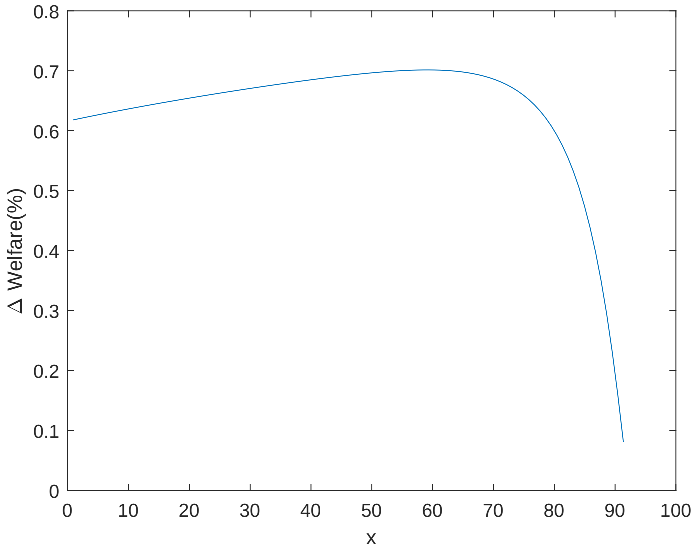


Figure 10: $\triangle\mathcal{U}_x$ decomposition ("variety effect")

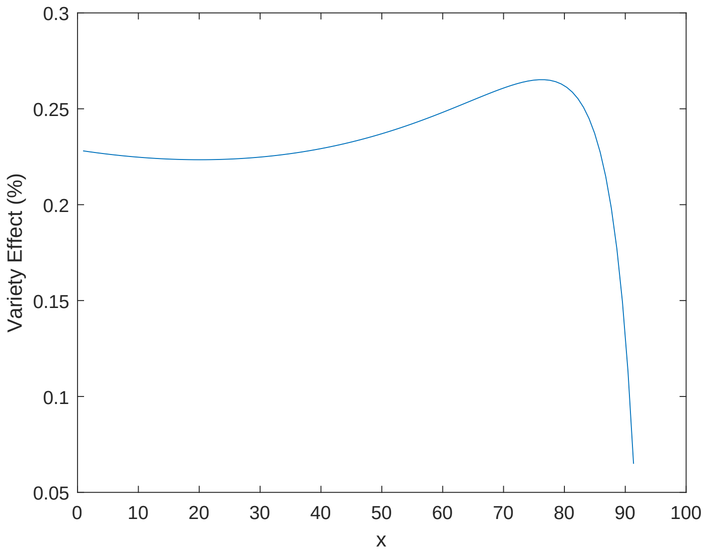
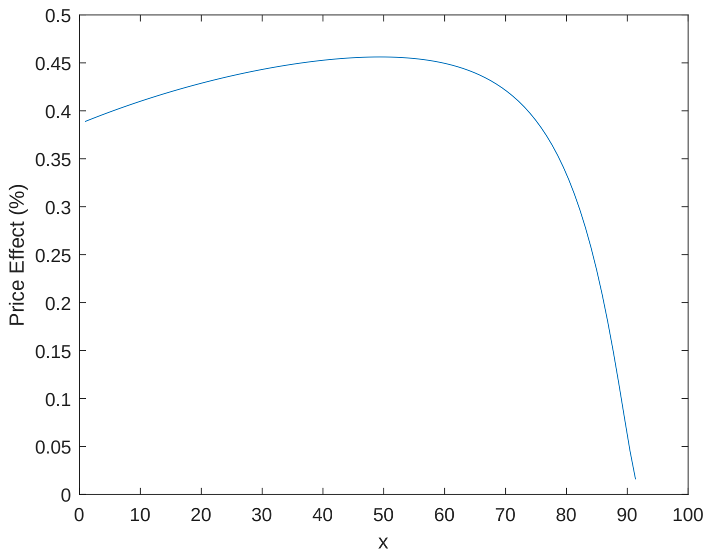


Figure 11: $\triangle \mathcal{U}_x$ decomposition ("price effect")



When α is equal to unity, the average welfare rises by 0.69 (from 2.7289 to 2.7478). The inequality level increases by 2.96 (from 1.757 to 1.809). We also observe a similar pattern of changes in the markup schedule with least productive firms charging higher prices.

5 Conclusion

This paper develops a monopolistic competition model that features matching between heterogeneous firms and product niches. Specifically, we formulate a sufficient condition for positive sorting between firms and product niches: more productive firms choose more populated product niches; while less productive firms move to smaller niches to avoid competition with the leaders. This outcome provides new insights on the equilibrium distribution of firm sales, prices, and markups that are now explained not only by comparative costs of these firms, but also by the distribution and size of available market niches. Moreover, the positive sorting of firms on the product space implies a new channel through which market shocks can affect the distribution of welfare across consumers. This channel is absent in standard spaceless models of monopolistic competition. The framework we develop seems to be quite rich in implications. A natural extension is to calibrate the model on data, analogously to recent exercises provided by Melitz and Redding (2015) and Arkolakis et al. 2018.

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Proofs

In this Appendix, we provide the proofs of some lemmas and propositions.

The Proof of Proposition 3

We proceed in four steps.

Step 1. We start with a series of definitions. First, we define the following function:

$$\pi(\lambda c) \equiv \max_{z \geq 0}[(u'(z) - \lambda c)z].$$

In fact, this is the rescaled profit of a c-type firm under local competitive toughness λ. We define

$$x_{max} \equiv l^{-1}\left(\frac{\lambda_{min} f}{\pi(\lambda_{min} c_{min})}\right). \tag{.1}$$

We assume that $x_{max} < S \Longleftrightarrow l(S) < \lambda_{min} f/\pi(\lambda_{min} c_{min})$ (that is, $l(S)$ is sufficiently low). We also define

$$c_{max} \equiv \frac{1}{\lambda_{min}}\pi^{-1}\left(\frac{\lambda_{min} f}{l(0)}\right). \tag{.2}$$

We assume that $c_{max} > c_{min} \Longleftrightarrow l(0) > \lambda_{min} f/\pi(\lambda_{min} c_{min})$ (that is, $l(0)$ is sufficiently high). Note that, if the latter condition fails to hold, there clearly exists no equilibrium. Indeed, in this case, the most productive firm would not break at $x = 0$, even if the competitive toughness λ is at its minimum possible level: $\lambda = \lambda_{min} > 0$. Therefore, $l(0) > \lambda_{min} f/\pi(\lambda_{min} c_{min})$ is an absolutely necessary condition for the set of active firms to be non-empty.

Next, we define the *cutoff curve* $C \subset \mathbb{R}^2_+$ as follows:

$$C \equiv \left\{(x, c) \in \mathbb{R}^2_+ \, : \, l(x)\pi(\lambda_{min} c) = \lambda_{min} f, \; 0 \leq x \leq x_{max}, \; c_{min} \leq c \leq c_{max}\right\}.$$

Clearly, C is the set of all a priory feasible solutions $(\overline{x}, \overline{c})$ of the zero-profit condition. Geometrically, C is a downward sloping curve on the (x, c)-plane connecting the points $(0, c_{max})$ and (x_{max}, c_{min}), where x_{max} and c_{max} are defined, respectively, by (.1) and (.2). Note that, from the definition of c_{max}, it follows that $\lambda_{min} c_{max} < u'(0)$ (since $\pi(\lambda_{min} c_{max}) = \lambda_{min} f/l(0) > 0$).

Since $x_{max} < S$, the population decay rate $a(x) \equiv -l'(x)/l(x)$ is a bounded continuous function over $[0, x_{max}]$.[1] Therefore, using the Weierstrass theorem, we can define:

$$A \equiv \max_{0 \leq x \leq x_{max}} a(x) < \infty. \tag{.3}$$

Step 2. Consider any $\overline{x} \in (0, x_{max}]$. Because the cutoff curve C is downward sloping, there

[1]Observe that $a(x)$ need not be bounded and continuous over the whole range $[0, S]$. To see this, set $S = 1$ and consider a linear symmetric population density: $l(x) = 1 - |x|$ for $x \in (-S, S)$. Then, we have $a(x) = 1/(1 - x)$, which is clearly unbounded over $(0, 1)$.

exists a unique $\overline{c} \in [c_{min}, c_{max})$ such that $(\overline{x}, \overline{c}) \in C$. By Picard's theorem (see, e.g., Pontryagin, 1962), there exists $\varepsilon > 0$ such that, for any $x \in (\overline{x} - \varepsilon, \overline{x}]$, there exists a unique solution $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ to (3.14) – (3.15) satisfying the boundary conditions: $\lambda_{\overline{x}}(\overline{x}) = \lambda_{min}$, $c_{\overline{x}}(\overline{x}) = \overline{c}$. Picard's theorem applies here, since the right-hand sides of (3.14) – (3.15) are well-defined and continuously differentiable and, thereby, locally Lipshitz in (λ, c) in the vicinity of $(\lambda_{min}, \overline{c})$. In particular, the denominator of the right-hand side of (3.15) never equals zero. Indeed, because $(\overline{x}, \overline{c}) \in C$, we have: $\lambda_{min}\overline{c} < \lambda_{min}c_{max} < u'(0)$ (see Step 1).

Next, we show that the above local solution $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be extended backwards either on $[x_0, \overline{x}]$, where $x_0 \in [0, \overline{x})$ and $c_{\overline{x}}(x_0) = c_{min}$, or on $[0, \overline{x}]$. In intuitive geometric terms, it means the following: the solution $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be extended backwards either until it hits the plane $\{(x, \lambda, c) \in \mathbb{R}^3 : x = 0\}$ or up to the plane$\{(x, \lambda, c) \in \mathbb{R}^3 : c = c_{min}\}$. Note that the case when $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ hits the intersection line of these two planes, i.e. the straight line $\{(x, \lambda, c) \in \mathbb{R}^3 : x = 0, c = c_{min}\}$, is not ruled out.

Assume the opposite: $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be **only** extended backwards on $(x_0, \overline{x}]$, where $x_0 \in (0, \overline{x})$ and $\lim_{x \downarrow x_0} c_{\overline{x}}(x) > c_{min}$. By the continuation theorem for ODE solutions (Pontryagin, 1962), this may only hold true in two cases:

Case 1: an "explosion in finite time" occurs, i.e.

$$\limsup_{x \downarrow x_0} \|(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))\| = \infty, \tag{.4}$$

where $|| \cdot ||$ stands for the standard Euclidean norm in $\mathbb{R}^2$.

Case 2: the right-hand side of the system (3.14)–(3.15) is not well defined at (x_0, λ, c), where $(\lambda, c) = \lim_{x \downarrow x_0} (\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$.

Let us first explore the possibility of Case 1. One can show that $\lambda_{\overline{x}}(x)$ is bounded on $(x_0, \overline{x}]$. Indeed, we have on $(x_0, \overline{x}]$ (recall that $\mathcal{M}(\lambda c)$ is decreasing in λc, as the price elasticity of demand is increasing)

$$0 > \frac{\mathrm{d}\lambda_{\overline{x}}(x)}{\mathrm{d}x} > -A\mathcal{M}(\lambda_{min}c_{min})\,\lambda_{\overline{x}}(x).$$

This implies that $\mathrm{d}\ln \lambda_{\overline{x}}(x)/\mathrm{d}x$ is uniformly bounded from above in the absolute value, which in turn means that $\lambda_{\overline{x}}(x)$ is bounded from above on $(x_0, \overline{x}]$. Clearly, $c_{\overline{x}}(x)$ is also bounded, as it increases in x and satisfies:

$$0 \leq c_{min} < \lim_{x \downarrow x_0} c_{\overline{x}}(x) \leq c_{\overline{x}}(x) \leq c_{\overline{x}}(\overline{x}) = \overline{c} < \infty,$$

for all $x \in (x_0, \overline{x}]$. As a result, (.4) cannot hold, meaning that Case 1 is not possible.

Let us now explore the possibility of Case 2. When $u'(0) = \infty$, this clearly cannot be the case, as the right-hand side of (3.14)–(3.15) is well defined for all $c > c_{min}$, for all $\lambda > \lambda_{min}$, and for all $x \geq 0$. Thus, it remains to explore the case when $u'(0) < \infty$. In this case, the

ODE system (3.14)–(3.15) is not well defined, when $\lim_{x\downarrow x_0} \lambda_{\overline{x}}(x)c_{\overline{x}}(x) = u'(0)$ (in this case, the denominator of the right-hand side in (3.15) is equal to zero). Assume that this is the case. Then, $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))_{x\in(x_0,\overline{x}]}$ and $\lambda c = u'(0)$ define each a curve in the (λ, c)-plane. Note that $u'(0) > \lambda_{\overline{x}}(x)c_{\overline{x}}(x)$ for any $x \in (x_0, \overline{x}]$, otherwise $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ could not be extended backwards on $(x_0, \overline{x}]$. Hence, the curve $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))_{x\in(x_0,\overline{x}]}$ lies strictly below the curve $\lambda c = u'(0)$ in the (λ, c)-plane and intersects it at $(\lim_{x\downarrow x_0} \lambda_{\overline{x}}(x), \lim_{x\downarrow x_0} c_{\overline{x}}(x))$ (the limits exist, as $\lambda_{\overline{x}}(x)$ and $c_{\overline{x}}(x)$ are monotone and bounded). This in turn implies that

$$\lim_{x\downarrow x_0} \left| \frac{\mathrm{d}c_{\overline{x}}(x)/\mathrm{d}x}{\mathrm{d}\lambda_{\overline{x}}(x)/\mathrm{d}x} \right| \le \frac{u'(0)}{\lim_{x\downarrow x_0} \lambda^2_{\overline{x}}(x)}. \tag{.5}$$

However, using (3.14)–(3.15), we have:

$$0 > \lim_{x\downarrow x_0} \frac{\mathrm{d}\lambda_{\overline{x}}(x)}{\mathrm{d}x} > -\infty, \qquad \lim_{x\downarrow x_0} \frac{\mathrm{d}c_{\overline{x}}(x)}{\mathrm{d}x} = +\infty,$$

which contradicts the inequality (.5) when $u'(0) < \infty$. That is, Case 2 is not possible as well. Hence, we observe a contradiction to that $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be only extended backwards on $(x_0, \overline{x}]$, where $x_0 \in (0, \overline{x})$ and $\lim_{x\downarrow x_0} c_{\overline{x}}(x) > c_{min}$.

As a result, the solution $(\lambda_{\overline{x}}(x), c_{\overline{x}}(x))$ can be extended backwards either up to the plane $\{(x, \lambda, c) \in \mathbb{R}^3 : x = 0\}$ or up to the plane$\{(x, \lambda, c) \in \mathbb{R}^3 : c = c_{min}\}$, or both options hold simultaneously.

Step 3. We now construct an equilibrium without taking into account free entry into the market: i.e., we assume that M_e is given. To do this, we define the following function over $[0, x_{max}]$:

$$\varphi(\overline{x}) = \begin{cases} c_{\overline{x}}(0) - c_{min}, & \text{if } (\lambda_{\overline{x}}(x), c_{\overline{x}}(x)) \text{ can be extended up to } \{x = 0\}, \\ -c^{-1}_{\overline{x}}(c_{min}), & \text{if } (\lambda_{\overline{x}}(x), c_{\overline{x}}(x)) \text{ can be extended up to } \{c = c_{min}\}. \end{cases} \tag{.6}$$

By continuity of solutions to ODE w.r.t. initial values (Pontryagin, 1962), $\varphi(\overline{x})$ is a continuous function of $\overline{x}$. Furthermore, it is readily verified that the following inequalities hold:

$$\varphi(0) = c_{max} - c_{min} > 0, \qquad \varphi(x_{max}) = -x_{max} < 0.$$

Hence, by the intermediate value theorem, there exists $\overline{x}^* \in (0, x_{max})$, such that $\varphi(\overline{x}^*) = 0$. Setting $(\lambda^*(x), c^*(x)) \equiv (\lambda_{\overline{x}^*}(x), c_{\overline{x}^*}(x))$ and $\overline{c}^* \equiv c_{\overline{x}^*}(\overline{x}^*)$, derive a candidate equilibrium:

$$\left\{ \overline{x}^*, \overline{c}^*, (\lambda^*(x), c^*(x))_{x\in[0,\overline{x}^*]} \right\}. \tag{.7}$$

We now verify that the candidate equilibrium (.7) is indeed an equilibrium when M_e is given. That $(\lambda^*(x), c^*(x))$ is a solution to (3.14) – (3.15) follows by construction. The equality $\varphi(\overline{x}^*) = 0$

means that $(\lambda^*(x), c^*(x))$ can be extended simultaneously up to both planes: $\{x = 0\}$ and $\{c = c_{min}\}$. This, in turn, is equivalent to $c^*(0) = c_{min}$, i.e. $(\lambda^*(x), c^*(x))$ satisfies one of the boundary conditions. The other boundary condition, $\lambda^*(\overline{x}^*) = \lambda_{min}$, is satisfied by construction. Finally, $(\overline{x}^*, \overline{c}^*) \in C$ means that $(\overline{x}^*, \overline{c}^*)$ satisfy the zero-profit condition (3.12).

Step 4. So far, we have been proceeding as if M_e were a constant. However, M_e is endogenous, and is determined by the free entry condition given by:

$$\Pi_e(M_e) \equiv \int_{c_{min}}^{\overline{c}^*(M_e)} \left[\frac{l(x^*(c, M_e))}{\lambda^*(c, M_e)} \pi\left(\lambda^*(c, M_e)c\right) - f \right] g(c)\mathrm{d}c = f_e, \tag{.8}$$

where $\lambda^*(c, M_e)$ is a decreasing function parametrically described by the downwards-sloping curve $(\lambda^*(x, M_e),\, c^*(x, M_e))|_{x\in[0,\overline{x}^*]}$, while $x^*(\cdot, M_e)$ is the inverse to $c^*(\cdot, M_e)$. We assume that $l(0)$ is such that

$$f_e < \int_{c_{min}}^{c_{max}} \left[\frac{l(0)}{\lambda_{min}} \pi\left(\lambda_{min}c\right) - f \right] g(c)\mathrm{d}c. \tag{.9}$$

Further, we show that this condition is sufficient for equation (.8) to have a solution $M_e^* > 0$.

First, we show that $\Pi_e(\infty) = 0$. Observe that, when $M_e \to \infty$, equation (3.15) implies that $\mathrm{d}c^*/\mathrm{d}x$ becomes uniformly small. Taking into account that $c^*(0) = c_{min}$, we have that

$$\lim_{M_e\to\infty} \overline{c}^*(M_e) = c_{min}, \qquad \lim_{M_e\to\infty} \overline{x}^*(M_e) = x_{max}.$$

It is straightforward to see that the above implies that $\Pi_e(\infty) = 0$.

Next, we consider $\Pi_e(0)$. Observe that, when $M_e \to 0$, equation (3.15) implies that $\mathrm{d}c^*/\mathrm{d}x$ becomes uniformly large or, equivalently, $\mathrm{d}x^*/\mathrm{d}c$ becomes uniformly small. This implies that

$$\lim_{M_e\to 0} \overline{x}^*(M_e) = 0, \qquad \lim_{M_e\to 0} \overline{c}^*(M_e) = c_{max}.$$

Hence,

$$\Pi_e(0) = \int_{c_{min}}^{c_{max}} \left[\frac{l(0)}{\lambda_{min}} \pi\left(\lambda_{min}c\right) - f \right] g(c)\mathrm{d}c.$$

According to our assumption, $\Pi_e(0) > f_e > 0 = \Pi_e(\infty)$. This means that equation (.8) has a solution $M_e^* > 0$. This completes the proof.

The Proof of Proposition 4

We proceed in four steps. Until Step 4, we ignore the free-entry condition and treat the mass $M_e > 0$ of entrants as exogenous. At Step 4, we take (.8) into account and show that it uniquely determines M_e.

Step 1. Assume there are at least two equilibrium outcomes corresponding to the same value of M_e:

$$\left\{\overline{x}^*, \overline{c}^*, (\lambda^*(x), c^*(x))_{x\in[0,\overline{x}^*]}\right\} \quad \text{and} \quad \left\{\overline{x}^{**}, \overline{c}^{**}, (\lambda^{**}(x), c^{**}(x))_{x\in[0,\overline{x}^{**}]}\right\}.$$

Note that $\overline{x}^* \neq \overline{x}^{**}$. Indeed, if $\overline{x}^* = \overline{x}^{**}$, then $\overline{c}^* = \overline{c}^{**}$ (since the cutoff curve C is downward-sloping). Hence, $(\lambda^*(x),\, c^*(x))$ and $(\lambda^{**}(x),\, c^{**}(x))$ are solutions to the same system of ODE satisfying the same boundary conditions. By Picard's theorem, this implies that $(\lambda^*(x),\, c^*(x)) = (\lambda^{**}(x),\, c^{**}(x))$ pointwise.

Let us assume without loss of generality that $\overline{x}^* < \overline{x}^{**}$. Because $(\overline{x}^*,\, \overline{c}^*) \in C$ and $(\overline{x}^{**},\, \overline{c}^{**}) \in C$, $\overline{x}^* < \overline{x}^{**}$ implies that $\overline{c}^* > \overline{c}^{**}$. Since $\left\{\overline{x}^{**}, \overline{c}^{**}, (\lambda^{**}(x), c^{**}(x))_{x\in[0,\overline{x}^{**}]}\right\}$ is an equilibrium for given M_e, we have that $c^{**}(0) = c_{min}$. Furthermore, $(c^{**})'_x(x) > 0$. Combining this with $\overline{x}^* < \overline{x}^{**}$, we derive the following inequalities:

$$c^{**}(\overline{x}^{**} - \overline{x}^*) > c^{**}(0) = c_{min} = c^*(0) = c^*(\overline{x}^* - \overline{x}^*). \tag{.10}$$

For each $z \in [0, \overline{x}^*]$, define $\Delta(z)$ as follows:

$$\Delta(z) \equiv c^{**}(\overline{x}^{**} - z) - c^*(\overline{x}^* - z). \tag{.11}$$

As has been shown, $\Delta(\overline{x}^*) > 0$. Taking into account that $\overline{c}^* > \overline{c}^{**}$, $\Delta(0) < 0$. By the intermediate value theorem, there exists $\xi \in (0, \overline{x}^*)$, such that $\Delta(\xi) = 0$. Let ξ_0 be the smallest of such ξs. Clearly, we have: $c^{**}(\overline{x}^{**} - \xi_0) = c^*(\overline{x}^* - \xi_0)$ and $c^{**}(\overline{x}^{**} - z) < c^*(\overline{x}^* - z)$ for all $z < \xi_0$.

Step 2. Next, we show that

$$\lambda^{**}(\overline{x}^{**} - \xi_0) > \lambda^*(\overline{x}^* - \xi_0). \tag{.12}$$

Using (3.14) yields (recall that $\lambda^{**}(\overline{x}^{**}) = \lambda_{min} = \lambda^*(\overline{x}^*)$)

$$(\lambda^{**}(\overline{x}^{**} - z))'_z\big|_{z=0} = a\left(\overline{x}^{**}\right)\lambda_{min}\mathcal{M}\left(\lambda_{min}\overline{c}^{**}\right) > a\left(\overline{x}^*\right)\lambda_{min}\mathcal{M}\left(\lambda_{min}\overline{c}^*\right) = (\lambda^*(\overline{x}^* - z))'_z\big|_{z=0},$$

which holds true because $a'(x) \geq 0$, $\overline{c}^* > \overline{c}^{**}$, and the markup function $\mathcal{M}(\cdot)$ is strictly decreasing. Furthermore, we have:

$$(\lambda^{**}(\overline{x}^{**} - z))'_z\big|_{z=0} > (\lambda^*(\overline{x}^* - z))'_z\big|_{z=0} > 0.$$

Thus, $\lambda^{**}\left(\overline{x}^{**} - z\right) > \lambda^*\left(\overline{x}^* - z\right)$ holds true for sufficiently small values of z.

Assume that there is some $\xi_1 \in (0, \xi_0)$, such that $\lambda^{**}(\overline{x}^{**} - \xi_1) = \lambda^*(\overline{x}^* - \xi_1)$, while $\lambda^{**}(\overline{x}^{**} - z) > \lambda^*(\overline{x}^* - z)$ for all $z < \xi_1$. Denote $\lambda_1 \equiv \lambda^*(\overline{x}^* - \xi_1)$. Differentiating the log of the ratio

$\lambda^{**}(\overline{x}^{**}-z)/\lambda^{*}(\overline{x}^{*}-z)$ w.r.t. z at $z=\xi_1$ yields (recall that, from the previous step, $c^{**}(\overline{x}^{**}-z) < c^{*}(\overline{x}^{*}-z)$ for all $z<\xi_0$):

$$\left[\ln\left(\frac{\lambda^{**}(\overline{x}^{**}-z)}{\lambda^{*}(\overline{x}^{*}-z)}\right)\right]'_z\Bigg|_{z=\xi_1} = a\left(\overline{x}^{**}-\xi_1\right)\mathcal{M}\left(\lambda_1 c^{**}\left(\overline{x}^{**}-\xi_1\right)\right) - a\left(\overline{x}^{*}-\xi_1\right)\mathcal{M}\left(\lambda_1 c^{*}(\overline{x}^{*}-\xi_1)\right) > 0.$$

By continuity, $\left[\ln\left(\frac{\lambda^{**}(\overline{x}^{**}-z)}{\lambda^{*}(\overline{x}^{*}-z)}\right)\right]'_z > 0$ must hold for any $z\in(\xi_1-\varepsilon,\,\xi_1)$, where $\varepsilon>0$ is sufficiently small. Hence, the ratio $\lambda^{**}(\overline{x}^{**}-z)/\lambda^{*}(\overline{x}^{*}-z)$ increases over $(\xi_1-\varepsilon,\,\xi_1)$ and strictly exceeds 1 at $z=\xi_1-\varepsilon$. Thus, $\lambda^{**}(\overline{x}^{**}-\xi_1)/\lambda^{*}(\overline{x}^{*}-\xi_1)$ also strictly exceeds 1, i.e. $\lambda^{**}(\overline{x}^{**}-\xi_1) > \lambda^{*}(\overline{x}^{*}-\xi_1)$. Based on that, we conclude that ξ_1 does not exist. This proves (.12).

Step 3. Differentiating the function $\Delta(z)$ defined by (.11) at $z=\xi_0$, we obtain:

$$\Delta'_z(\xi_0) = -\frac{1}{M_e g\left(c_0^*\right)}\left[\frac{\left(V'\right)^{-1}\left(1/\lambda_0^{**}\right)}{u\left(q\left(\lambda_0^{**}c_0^*\right)\right)} - \frac{\left(V'\right)^{-1}\left(1/\lambda_0^{*}\right)}{u\left(q\left(\lambda_0^{*}c_0^*\right)\right)}\right] < 0. \tag{.13}$$

where $c_0^* \equiv c^*(\overline{x}^*-\xi_0) = c^{**}(\overline{x}^{**}-\xi_0)$, $\lambda_0^* \equiv \lambda^*(\overline{x}^*-\xi_0)$, and $\lambda_0^{**} \equiv \lambda^{**}(\overline{x}^{**}-\xi_0)$. The inequality (.13) holds true because, by (.12), we have $\lambda_0^{**} > \lambda_0^*$, while the function $\left(V'\right)^{-1}(1/\lambda)/u\left(q\left(\lambda c\right)\right)$ increases in λ for any given $c > c_{min}$. However, by definition of ξ_0, $\Delta(z)$ must change sign from negative to positive at $z=\xi_0$. Hence, it must be true that $\Delta'_z(\xi_0)\geq 0$. This contradicts (.13) and implies that, for any fixed value of M_e, there is a unique equilibrium outcome corresponding to this value of M_e.

Step 4. To finish the proof of uniqueness, it remains to show that $\mathrm{d}\Pi_e(M_e)/\mathrm{d}M_e < 0$ for any $M_e > 0$. Let us define

$$\mathfrak{N}(c, M_e) \equiv \frac{l(x^*(c, M_e))}{\lambda^*(c, M_e)}\pi\left(\lambda^*(c, M_e)c\right).$$

Then, we have:

$$\frac{\mathrm{d}\Pi_e(M_e)}{\mathrm{d}M_e} = \int\limits_{c_{min}}^{\overline{c}^*(M_e)} \frac{\partial\mathfrak{N}(c, M_e)}{\partial M_e} g(c)\mathrm{d}c + \left[\mathfrak{N}(\overline{c}^*(M_e), M_e) - f\right]\frac{\mathrm{d}\overline{c}^*(M_e)}{\mathrm{d}M_e},$$

where the last term equals zero due to the cutoff condition. Hence,

$$\frac{\mathrm{d}\Pi_e(M_e)}{\mathrm{d}M_e} = \int\limits_{c_{min}}^{\overline{c}^*(M_e)} \frac{\partial\mathfrak{N}(c, M_e)}{\partial M_e}\mathrm{d}G(c).$$

Thus, a sufficient condition for $\mathrm{d}\Pi_e(M_e)/\mathrm{d}M_e < 0$ for any $M_e > 0$ is given by

$$\frac{\partial\mathfrak{N}(c, M_e)}{\partial M_e} < 0 \text{ for any } M_e > 0 \text{ and any } c \in \left[c_{min}, \overline{c}^*(M_e)\right].$$

It is straightforward to see that, due to the envelope theorem, the latter is hold when

$$\frac{\partial \lambda^*(x, M_e)}{\partial M_e} > 0 \text{ for any } M_e > 0 \text{ and any } x \in [0, \overline{x}^*(M_e)]\,.$$

In fact, it is sufficient to show that

$$\frac{\partial \lambda^*(x, M_e)}{\partial M_e} \geq 0 \text{ for any } M_e > 0 \text{ and any } x \in [0, \overline{x}^*(M_e)]$$

and $\partial \lambda^*(x, M_e)/\partial M_e > 0$ on some non-zero measure subset of $[0, \overline{x}^*(M_e)]$. The rest of the proof amounts to establishing the latter statement.

Assume that, on the contrary, for some $M_e > 0$, there exists a compact interval $[x_1, x_2] \subseteq [0, \overline{x}^*(M_e)]$, such that $\partial \lambda^*(x, M_e)/\partial M_e \leq 0$ for all $x \in [x_1, x_2]$. Without loss of generality, let us also assume that $[x_1, x_2]$ cannot be extended further without violating the condition $\partial \lambda^*(x, M_e)/\partial M_e \leq 0$ (otherwise, we can replace it with a larger one). We will therefore refer to $[x_1, x_2]$ as a *non-extendable* interval. We consider several possible cases.

Case 1: Assume that $x_1 = 0$. In this case, we have: $c^*(x_1, M_e) = c_{\min}$, hence $\partial c^*(x_1, M_e)/\partial M_e = 0$. Recall that

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{1}{M_e} \frac{(V')^{-1}(1/\lambda)}{g(c)\, u(q_x)}.$$

Since $\partial \lambda^*(x_1, M_e)/\partial M_e \leq 0$, $\partial c^*(x_1, M_e)/\partial M_e = 0$, and M_e rises, $\partial (c^*)'_x (x_1, M_e)/\partial M_e < 0$ (the right-hand side of the above equation decreases at $x_1 = 0$ with a rise in M_e). Note that $\partial c^*(x_1, M_e)/\partial M_e = 0$ and $\partial (c^*)'_x (x_1, M_e)/\partial M_e < 0$ imply that $\partial c^*(x, M_e)/\partial M_e < 0$ in some right neighborhood of $x_1 = 0$.

Case 2: Assume that $x_2 = \overline{x}^*(M_e)$. We have $\lambda^*(\overline{x}^*(M_e), M_e) = \lambda_{min}$. This implies that

$$\frac{\partial \lambda^*(\overline{x}^*(M_e), M_e)}{\partial x} \frac{d\overline{x}^*(M_e)}{dM_e} + \frac{\partial \lambda^*(\overline{x}^*(M_e), M_e)}{\partial M_e} = 0.$$

The second term in the left-hand side of the above equation is non-positive (as assumed). Recall that $\lambda^*(x, M_e)$ is strictly decreasing in x. As a result, $d\overline{x}^*(M_e)/dM_e \leq 0$. Combining this with the fact $(\overline{x}^*(M_e), \overline{c}^*(M_e)) \in C$, where C is the downward sloping cutoff curve, we get: $\mathrm{d}\overline{c}^*(M_e)/\mathrm{d}M_e \geq 0$. That is,

$$\frac{\partial c^*\left(\overline{x}^*(M_e), M_e\right)}{\partial x} \frac{\mathrm{d}\overline{x}^*(M_e)}{\mathrm{d}M_e} + \frac{\partial c^*\left(\overline{x}^*(M_e), M_e\right)}{\partial M_e} \geq 0,$$

where the first term is non-positive because, as shown above, $\mathrm{d}\overline{x}^*(M_e)/\mathrm{d}M_e \leq 0$, while $\partial c^*\left(\overline{x}^*(M_e), M_e\right)/\partial x$ 0. Hence, the second term, $\partial c^*\left(\overline{x}^*(M_e), M_e\right)/\partial M_e$, must be non-negative. If $\partial c^*\left(\overline{x}^*(M_e), M_e\right)/\partial M_e = 0$, then one can show that $\partial (c^*)'_x (\overline{x}^*(M_e), M_e)/\partial M_e < 0$. Here, we use again the fact that

$$\frac{\mathrm{d}c}{\mathrm{d}x} = \frac{1}{M_e} \frac{(V')^{-1}(1/\lambda)}{g(c)\, u(q_x)}.$$

This in turn implies that $\partial c^*\left(\overline{x}^*(M_e), M_e\right)/\partial M_e > 0$ in some left neighborhood of $x_2 = \overline{x}^*(M_e)$.

Case 3: Assume that $0 < x_1 < x_2 < \overline{x}^*(M_e)$. Because $[x_1, x_2]$ is non-extendable, there exists a small open left half-neighborhood $\mathcal{N}_1$ of x_1, and a small right half-neighborhood $\mathcal{N}_2$ of x_2, such that $\partial\lambda^*(x, M_e)/\partial M_e > 0$ for all $x \in \mathcal{N} \equiv \mathcal{N}_1 \cup \mathcal{N}_2$. Hence, for a c-type firm where $c = c^*(x, M_e)$ with $x \in [x_1, x_2]$, relocating marginally beyond $[x_1, x_2]$ in response to a marginal increase in M_e is not profit-maximizing behavior. Indeed, that $\partial\lambda^*(x, M_e)/\partial M_e \leq 0$ over $[x_1, x_2]$ means that the profit function increases uniformly over $[x_1, x_2]$, while $\partial\lambda^*(x, M_e)/\partial M_e > 0$ for all $x \in \mathcal{N}$ means that relocating from $[x_1, x_2]$ into $\mathcal{N}$ would lead to a reduction of maximum feasible profit.[2] This immediately imply that

$$\frac{\partial c^*(x_1, M_e)}{\partial M_e} \leq 0, \qquad \frac{\partial c^*(x_2, M_e)}{\partial M_e} \geq 0.$$

Moreover, for $j = 1, 2$ we have (the proof is the same as in the previous cases)

$$\frac{\partial c^*(x_j, M_e)}{\partial M_e} = 0 \Rightarrow \frac{\partial (c^*)'_x (x_j, M_e)}{\partial M_e} < 0.$$

The findings in the above cases allow us to formulate the following important result. *There exists a location x_4 in an arbitrary small right half-neighborhood of x_1, such that $\partial c^*(x_4, M_e)/\partial M_e < 0$. Similarly, there exists a location x_5 in an arbitrary small left half-neighborhood of x_2, such that $\partial c^*(x_5, M_e)/\partial M_e > 0$.*

By the intermediate value theorem, there must exist a location $x_3 \in (x_4, x_5) \subset [x_1, x_2]$ such that

$$\frac{\partial c^*(x_3, M_e)}{\partial M_e} = 0, \quad \frac{\partial (c^*)'_x (x_3, M_e)}{\partial M_e} \geq 0.$$

The non-negative sign of the derivative follows from the fact that $c^*(x, M_e)$ is increasing in x. This in turn implies that the derivative of

$$\frac{1}{M_e} \frac{(V')^{-1}(1/\lambda^*(x_3, M_e))}{g(c^*(x_3, M_e))\, u(q(\lambda^*(x_3, M_e)c^*(x_3, M_e)))}$$

with respect to M_e is non-negative. That is, the derivative of

$$\frac{(V')^{-1}(1/\lambda^*(x_3, M_e))}{g(c^*(x_3, M_e))u(q(\lambda^*(x_3, M_e)c^*(x_3, M_e)))}$$

with respect to M_e is strictly positive. This means that $\partial\lambda^*(x_3, M_e)/\partial M_e > 0$ (recall that $\partial c^*(x_3, M_e)/\partial M_e = 0$). However, since $x_3 \in [x_1, x_2]$, it must be that $\partial\lambda^*(x_3, M_e)/\partial M_e \leq 0$, which is a contradiction. This completes the proof of uniqueness of the equilibrium.

[2]One may wonder why no firm would relocate from $[x_1, x_2]$ to somewhere beyond $\mathcal{N}$ in response to a marginal increase of M_e. This would mean, for at least some firm type c, that the firm's profit-maximizing location choice $x^*(c, M_e)$ has a discontinuity in M_e. However, by the maximum theorem **(Sundaram, 1996),** $x^*(c, M_e)$ must be upper-hemicontinuous in M_e. Furthermore, by strict quasi-concavity of the profit function, $x^*(c, M_e)$ is single-valued. For single-valued mappings, upper-hemicontinuity implies continuity. Hence, $x^*(c, M_e)$ cannot exhibit discontinuities.

The proof of Proposition 5

To prove the proposition, we use the equilibrium conditions for $\lambda'(x)$ and $c'(x)$. Specifically, from (3.11) and (3.9),

$$\lambda'(x) = \frac{l'(x)\lambda(x)}{l(x)} \frac{p(x,c(x)) - c(x)}{p(x,c(x))},$$

$$M_e g\left(c(x)\right) c'(x) u\left(q(x,c(x))\right) = \left(V'\right)^{-1}\left(1/\lambda(x)\right) \iff c'(x) = \frac{\left(V'\right)^{-1}\left(1/\lambda(x)\right)}{M_e g\left(c(x)\right) u\left(q(x,c(x))\right)}.$$

Hence,

$$(\lambda(x)c(x))'_x = c(x)\lambda'(x) + \lambda(x)c'(x)$$

$$= \frac{\lambda(x)}{g\left(c(x)\right)}\left[c(x)g\left(c(x)\right)\frac{l'(x)}{l(x)}\frac{p(x,c(x)) - c(x)}{p(x,c(x))} + \frac{\left(V'\right)^{-1}\left(1/\lambda(x)\right)}{M_e u\left(q(x,c(x))\right)}\right].$$

Consider,

$$(\lambda(x)c(x))'_{x=0} = \frac{\lambda(0)}{g\left(c_{min}\right)}\left(c_{min}\, g\left(c_{min}\right)\frac{l'(0)}{l(0)}\frac{p(0,c_{min}) - c_{min}}{p(0,c_{min})} + \frac{\left(V'\right)^{-1}\left(1/\lambda(0)\right)}{M_e u\left(q(0,c_{min})\right)}\right).$$

Since $g\left(c\right)$ is a density function, $lim_{c_{min}\to 0} c_{min}\, g\left(c_{min}\right) = 0$. Hence, if $\mid l'(0) \mid < \infty$, then for sufficiently low c_{min},

$$c_{min}\, g\left(c_{min}\right)\frac{l'(0)}{l(0)}\frac{p(0,c_{min}) - c_{min}}{p(0,c_{min})} + \frac{\left(V'\right)^{-1}\left(1/\lambda(0)\right)}{M_e u\left(q(0,c_{min})\right)} > 0.$$

Similarly,

$$(\lambda(x)c(x))'_{x=\bar{x}} = \frac{\lambda(\bar{x})}{g\left(\bar{c}\right)}\left(\bar{c}\, g\left(\bar{c}\right)\frac{l'(\bar{x})}{l(\bar{x})}\frac{p(\bar{x},\bar{c}) - \bar{c}}{p(\bar{x},\bar{c})} + \frac{\left(V'\right)^{-1}\left(1/\lambda(\bar{x})\right)}{M_e u\left(q(\bar{x},\bar{c})\right)}\right).$$

Note that, as there is the fixed cost of production f, $p(\bar{x},\bar{c}) > \bar{c}$. Moreover, $\lambda(\bar{x}) = 1/V'\left(0\right)$ in the equilibrium, implying that $\left(V'\right)^{-1}\left(1/\lambda(\bar{x})\right) = 0$ (this also means that $c'(\bar{x}) = 0$). As a result, since $l'(\bar{x}) < 0$,

$$\bar{c}\, g\left(\bar{c}\right)\frac{l'(\bar{x})}{l(\bar{x})}\frac{p(\bar{x},\bar{c}) - \bar{c}}{p(\bar{x},\bar{c})} + \frac{\left(V'\right)^{-1}\left(1/\lambda(\bar{x})\right)}{M_e u\left(q(\bar{x},\bar{c})\right)} < 0.$$

To prove the third statement of the proposition, we rewrite $(\lambda(x)c(x))'_x$ in the following way:

$$(\lambda(x)c(x))'_x = \frac{\lambda(x)}{g\left(c\left(x\right)\right)}\left(\frac{l'(x)}{l(x)}c\left(x\right)g\left(c\left(x\right)\right)\mathcal{M}(\lambda(x)c(x)) + \frac{\left(V'\right)^{-1}\left(1/\lambda(x)\right)}{M_e u\left(q(\lambda(x)c(x))\right)}\right),$$

where $\mathcal{M}(.)$ is the markup function. Let us denote $\tilde{x} \in (0, \bar{x})$ as an interior extremum of $\lambda(x)c(x)$: $(\lambda(\tilde{x})c(\tilde{x}))'_x = 0$. We know that $(\lambda(x)c(x))'_{x=0} > 0$ and $(\lambda(x)c(x))'_{x=\bar{x}} < 0$. Hence, $\lambda(x)c(x)$ has at least one interior local maximizer.

Next, we show that, for any $\tilde{x}$, $(\lambda(\tilde{x})c(\tilde{x}))''_{xx} < 0$. We have

$$(\lambda(\tilde{x})c(\tilde{x}))''_{xx} = \left(\tfrac{\lambda(\tilde{x})}{g(c(\tilde{x}))}\right)' \left(\tfrac{l'(\tilde{x})}{l(\tilde{x})} c\left(\tilde{x}\right) g\left(c\left(\tilde{x}\right)\right) \mathcal{M}(\lambda(\tilde{x})c(\tilde{x})) + \tfrac{(V')^{-1}(1/\lambda(\tilde{x}))}{M_e u(q(\lambda(\tilde{x})c(\tilde{x})))}\right)$$

$$+\tfrac{\lambda(\tilde{x})}{g(c(\tilde{x}))} \left(\tfrac{l'(\tilde{x})}{l(\tilde{x})} c\left(\tilde{x}\right) g\left(c\left(\tilde{x}\right)\right) \mathcal{M}(\lambda(\tilde{x})c(\tilde{x})) + \tfrac{(V')^{-1}(1/\lambda(\tilde{x}))}{M_e u(q(\lambda(\tilde{x})c(\tilde{x})))}\right)'_x.$$

Note that the first term in the right hand side of the above formula is equal to zero. Thus, we have (recall that $(\lambda(\tilde{x})c(\tilde{x}))'_x = 0$)

$$(\lambda(\tilde{x})c(\tilde{x}))''_{xx} = \tfrac{\lambda(\tilde{x})}{g(c(\tilde{x}))} \left(\tfrac{l'(\tilde{x})}{l(\tilde{x})} c\left(\tilde{x}\right) g\left(c\left(\tilde{x}\right)\right) \mathcal{M}(\lambda(\tilde{x})c(\tilde{x})) + \tfrac{(V')^{-1}(1/\lambda(\tilde{x}))}{M_e u(q(\lambda(\tilde{x})c(\tilde{x})))}\right)'_x$$

$$= \tfrac{\lambda(\tilde{x})}{g(c(\tilde{x}))} \left(\left(\tfrac{l'(\tilde{x})}{l(\tilde{x})} c\left(\tilde{x}\right) g\left(c\left(\tilde{x}\right)\right)\right)'_x \mathcal{M}(\lambda(\tilde{x})c(\tilde{x})) + \tfrac{\left((V')^{-1}(1/\lambda(\tilde{x}))\right)'_x}{M_e u(q(\lambda(\tilde{x})c(\tilde{x})))}\right).$$

We have

$$\left(\frac{l'(x)}{l(x)} c(x) g\left(c(x)\right)\right)'_x = \frac{l'(x)}{l(x)} \left(c(x) g\left(c(x)\right)\right)'_x + c(x) g\left(c(x)\right) \left(\frac{l'(x)}{l(x)}\right)'_x < 0,$$

since $c'(x) > 0$, $g'(c) \geq 0$, and $(l'(x)/l(x))'_x \leq 0$. At the same time, $(V')^{-1}(1/\lambda(x))$ is decreasing in x as $V''(\cdot) < 0$ and $\lambda'(x) < 0$. Hence, $(\lambda(\tilde{x})c(\tilde{x}))''_{xx} < 0$.

We now finish the proof of part (iii) of Proposition 3. As derived above, $\lambda(x)c(x)$ has no interior local minimum over $(0, \overline{x})$ and at least one interior local maximizer. Assume that $\lambda(x)c(x)$ has at least two distinct local maximizers. Then, there must be a local minimizer in between, which contradicts our above finding. We conclude that $\lambda(x)c(x)$ is bell-shaped in x, while the markup function $\mathcal{M}(\lambda(x)c(x))$ is U-shaped in x. This completes the proof.

The proof of Lemma 2

Note that in this proof it is important that $\partial\lambda(x, M_e, \delta)/\partial\delta$ and $\partial c(x, M_e, \delta)/\partial\delta$ are analytic in x over $(0, \overline{x})$, meaning that they can be represented by convergent power series (this is the case, when, for instance, the primitives in the model are analytic):

$$\frac{\partial\lambda(x, M_e, \delta)}{\partial\delta} = \sum_{k=0}^{\infty} a_k(M_e, \delta) x^k, \qquad \frac{\partial c(x, M_e, \delta)}{\partial\delta} = \sum_{k=0}^{\infty} b_k(M_e, \delta) x^k.$$

This makes the case when $\partial\lambda(x, M_e, \delta)/\partial\delta = 0$ and $\partial(\lambda)'_x(x, M_e, \delta)/\partial\delta = 0$ at some x impossible. Why? If this is the case, then $\partial c(x, M_e, \delta)/\partial\delta = 0$ and $\partial(c)'_x(x, M_e, \delta)/\partial\delta = 0$ as well implying that the derivatives of all orders of $\partial\lambda(x, M_e, \delta)/\partial\delta$ w.r.t. x at this point equal to zero. An analytic function with this property must be identically zero (Courant and John 2012, p. 545). This in turn means that $\lambda(x)$ does not change on the whole interval $[0, \overline{x}]$ when δ changes, which is impossible. For the same reason, it is not possible that $\partial c(x, M_e, \delta)/\partial\delta = 0$ and $\partial(c)'_x(x, M_e, \delta)/\partial\delta = 0$ at some x.

To simplify the exposition of the proof, we divide it into several parts.

Part 1

In this part, we prove that $\partial\overline{x}(M_e, \delta)/\partial\delta > 0$. Assume, on the contrary, that $\partial\overline{x}(M_e, \delta)/\partial\delta \leq 0$. Then, because an increase in δ leads to an upward shift of the cutoff curve C, it must be that $\partial\overline{c}(M_e, \delta)/\partial\delta > 0$. Note also that if $\partial\overline{x}(M_e, \delta)/\partial\delta < 0$, then (by continuity) $\lambda(x, M_e, \delta)$ must decrease w.r.t. δ in some neighborhood of $\overline{x}$ (as $\lambda(x, M_e, \delta)$ is decreasing in x). If $\overline{x}$ does not change with the change in δ, one can derive from (3.14) that $\partial\left(-(\lambda)'_x(\overline{x}, M_e, \delta)\right)\partial\delta < 0$. This is because $\partial\overline{c}(M_e, \delta)/\partial\delta > 0$ and $\lambda(\overline{x}, M_e, \delta) = \lambda_{\min}$. This in turn also means that $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$ in some neighborhood of $\overline{x}$. That is, if $\partial\overline{x}(M_e, \delta)/\partial\delta \leq 0$, $\lambda(x, M_e, \delta)$ must decrease w.r.t. δ over some interval $(x_1, \overline{x})$. Two cases may arise.

Case 1: $x_1 = 0$. In this case, $\partial\lambda(0, M_e, \delta)/\partial\delta < 0$. Then, taking into account the boundary condition $c(0, M_e, \delta) = c_{\min}$, it is straightforward to see from the equilibrium condition in (3.15) that $\partial(c)'_x(0, M_e, \delta)/\partial\delta < 0$. This in turn implies that $\partial c(x, M_e, \delta)/\partial\delta < 0$ in the vicinity of $x = 0$ (since $c(0, M_e, \delta) = c_{\min}$ is not affected by δ). As a result, we have the following situation: given the rise in δ, $c(x)$ falls in the neighborhood of zero and rises in the neighborhood of $\overline{x}$ as $\partial\overline{c}(M_e, \delta)/\partial\delta > 0$. This implies that there exists $x_2 \in (0, \overline{x})$ such that $\partial c(x_2, M_e, \delta)/\partial\delta = 0$ - the value of $c(x)$ at x_2 is not affected by the rise in δ. Moreover, $\partial(c)'_x(x_2, M_e, \delta)/\partial\delta > 0$ (as $c(x)$ falls around zero). This in turn means (here we use the equilibrium condition in (3.15)) that $\partial\lambda(x_2, M_e, \delta)/\partial\delta > 0$ which contradicts the assumption that $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$ for all $x > 0$. *Note that we will use this particular way of deriving the contradiction throughout the whole proof of the lemma.*

Case 2 $x_1 > 0$. In this case, it must be true that $\partial\lambda(x_1, M_e, \delta)/\partial\delta = 0$. Moreover, the absolute value of the slope of $\lambda(x)$ at this point increases: $\partial\left(-(\lambda)'_x(x_1, M_e, \delta)\right)/\partial\delta > 0$, as $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$ on $(x_1, \overline{x})$. In this case, from the equilibrium condition in (3.14) we derive that $\partial c(x_1, M_e, \delta)/\partial\delta < 0$. Now, we use the same argument as in the previous case. There exists $x_3 \in (x_1, \overline{x})$ such that $\partial c(x_3, M_e, \delta)/\partial\delta = 0$ and $\partial(c)'_x(x_3, M_e, \delta)/\partial\delta > 0$. This in turn implies that $\partial\lambda(x_3, M_e, \delta)/\partial\delta > 0$ which contradicts the assumption that $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$ for all $x > x_1$.

Thus, we show that $\partial\overline{x}(M_e, \delta)/\partial\delta > 0$.

Part 2

Next, we show that $\partial\lambda(x, M_e, \delta)/\partial\delta > 0$ for all x. Assume that, on the contrary, there exists a non-extendable interval $(x_4, x_5) \subset [0, \overline{x}]$ such that $\partial\lambda(x, M_e, \delta)/\partial\delta \leq 0$ on this interval. Note that since $\overline{x}$ rises (implying that $\partial\lambda(x, M_e, \delta)/\partial\delta > 0$ in some neighborhood of $\overline{x}$), $x_5 < \overline{x}$. Consider again two cases.

Case 1: $x_4 > 0$. In this case, because (x_4, x_5) is a non-extendable interval where $\partial\lambda(x, M_e, \delta)/\partial\delta < 0$, it must be that:

$$\frac{\partial\lambda(x_4, M_e, \delta)}{\partial\delta} = 0 = \frac{\partial\lambda(x_5, M_e, \delta)}{\partial\delta}.$$

Moreover,

$$\frac{\partial\left(-\left(\lambda\right)'_x(x_4, M_e, \delta)\right)}{\partial\delta} > 0 > \frac{\partial\left(-\left(\lambda\right)'_x(x_5, M_e, \delta)\right)}{\partial\delta}.$$

In this case, (3.14) implies that

$$\frac{\partial c(x_4, M_e, \delta)}{\partial\delta} < 0 < \frac{\partial c(x_5, M_e, \delta)}{\partial\delta}.$$

Hence, there exists $x_6 \in (x_4, x_5)$, such that

$$\frac{\partial c(x_6, M_e, \delta)}{\partial\delta} = 0, \qquad \frac{\partial\left(c\right)'_x(x_6, M_e, \delta)}{\partial\delta} > 0.$$

This means that $\partial\lambda(x_6, M_e, \delta)/\partial\delta > 0$, which contradicts the assumption that $\partial\lambda(x, M_e, \delta)/\partial\delta \leq 0$ for all $x \in (x_4, x_5)$.

Case 2: $x_4 = 0$. In this case, it can potentially be that $\partial\lambda(0, M_e, \delta)/\partial\delta = 0$ or $\partial\lambda(0, M_e, \delta)/\partial\delta < 0$. Note that if $\partial\lambda(0, M_e, \delta)/\partial\delta = 0$, then $\partial(\lambda)'_x(x, M_e, \delta)/\partial\delta = 0$ (as $\partial c(0, M_e, \delta)/\partial\delta = 0$). As discussed at the beginning of the proof, this case is impossible. If $\partial\lambda(0, M_e, \delta)/\partial\delta < 0$, then from (3.15), $\partial(c)'_x(0, M_e, \delta)/\partial\delta < 0$, meaning that in some neighborhood of zero $c(x)$ falls with the rise in δ. Then, we use again the logic from the previous case and, thereby, derive the contradiction.

Part 3

The next step is to show that $\partial c(x, M_e, \delta)/\partial\delta > 0$ for all $x \in (0, \overline{x}]$. Assume that, on the contrary, that there exists a non-extendable interval $(x_7, x_8) \subset [0, \overline{x}]$, such that $\partial c(x, M_e, \delta)/\partial\delta \leq 0$ on this interval. If $x_7 = 0$, then $\partial(c)'_x(0, M_e, \delta)/\partial\delta \leq 0$ and $\partial c(0, M_e, \delta)/\partial\delta = 0$. In this case, $\partial\lambda(0, M_e, \delta)/\partial\delta \leq 0$ which contradicts our previous results. If $x_7 > 0$, then again $\partial c(x_7, M_e, \delta)/\partial\delta = 0$ and $\partial\left(c\right)'_x(x_7, M_e, \delta)/\partial\delta < 0$ (recall that $\partial\left(c\right)'_x(x_7, M_e, \delta)/\partial\delta$ cannot be equal to zero). That is, we derive the contradiction: $\partial\lambda(x_7, M_e, \delta)/\partial\delta < 0$.

Finally, since $\partial c(x, M_e, \delta)/\partial\delta > 0$, $\partial\overline{x}(M_e, \delta)/\partial\delta > 0$, and $\left(c\right)'_x > 0$, $\partial\overline{c}(M_e, \delta)/\partial\delta > 0$.