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Abstract

We model the dynamic contest between two players as a game of tug-of-war with a Tullock contest success function (CSF). We show that (pure strategy) Markov perfect equilibrium of this game exists, and it is unique. In this equilibrium - in stark contrast to a model of tug-of-war with an all pay auction CSF - players exert positive efforts until the very last battle. Since the outcome of an individual battle is determined stochastically, even disadvantaged players who fell behind will occasionally win battles and hence the advantage likely change hands. We deliver a set of empirically appealing results on effort dynamics.

JEL-Codes: C720, D720, D740.

Keywords: contests, discouragement effect, perseverance, stochastic games, tug-of-war, Tullock contest success function.

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1 Introduction

Many competitive settings involve multiple, sequential contests where parties compete against each other. The final outcome of such dynamic contests depend on the outcomes of individual battles (or sub-contests). Examples abound: firms in an R&D race competing for a patent, committee decision-making in organizations, candidates in election campaigns, political parties in coalition talks, teams or singles in sports competitions, adversaries in public debates, countries in wars, etc. In a multi-battle (or dynamic) contest, efforts in an individual battle and the corresponding battle outcome may influence future efforts and battle outcomes. When such an influence exists, this may have implications for parties' behavior in earlier rounds (see Konrad, 2012). For instance, if losing the first battle makes the whole contest a write-off, then this may induce excessive efforts in the first battle. The presence of such dynamic linkages gives rise to some interesting questions such as "How do efforts vary across battles?", "How do efforts vary with intensity of rivalry?", and "Who exerts a greater effort: the leader or the follower?" (see Harris and Vickers, 1987).

The literature on contest games produced various models of dynamic contests, such as *race*, *tug-of-war*, *elimination contests*, *war of attrition*, and *repeated incumbency fights* (see Konrad, 2012 for a review). The current paper is concerned with the questions mentioned above, and it focuses on the model of *tug-of-war* between two players. A tug-of-war is a multi-battle contest game with a finite number of (ordered) states and potentially infinite number of battles. Players start at an initial state (neutral or non-neutral) and simultaneously exert effort in each battle to win the contest. Winning a battle moves the state towards the winning player's favorite terminal state. This game can be illustrated by a horizontal line with an interior point that represents the initial state and two end-points that represent respective players' favorite terminal states –resembling the sports competition after which the model is named. A player wins the prize/award if he has won sufficiently many battle victories to pull the state to his terminal state. The winner of each battle is determined by a *contest success function* (CSF for short), which takes players' efforts as an input. As the description of the game above reveals, what matters in tug-of-war is not the absolute number of battle victories, but the difference between the two players' numbers of victories.

Harris and Vickers (1987) argued that tug-of-war is possibly the simplest framework to address questions regarding the effort dynamics since the model has a single state variable –a measure of the distance between players. Another reason why tug-of-war is an appealing modeling device for dynamic contests in multiple disciplines is its empirical relevance. Harris and Vickers (1987) modeled an R&D competition between firms whereas Budd, Harris, and Vickers (1993) modeled dynamic evolution of a duopoly as tugs of war. In international relations, Yoo (2001) described the relations between the US and the North Korea, whereas Organski and Lust-Okar (1997) described the Israeli-Palestinian conflict on the status of Jerusalem as tugs of war. Schutten et al. (1996) described the ongoing debate in the manufacturing literature on efficient production vs delivery performance as tug-of-war.[1]

The following observation is worth mentioning here since it is one of the motivations for the current work: a common element in many dynamic conflicts is that the laggard does not immediately surrender, but instead exerts effort and occasionally wins some battles as the contest continues. This perseverance, and the suspense it produces are two trademarks of many real-life dynamic contests.

Our second observation concerns how the existing models of tug-of-war handle uncertainty. Exogenous noise is a ubiquitous aspect of many real-life contests (see Konrad, 2007 for examples and Jia, 2008 for foundations): battle outcomes are affected by factors such as weather conditions, contestants' or jury members' moods, technical breakdowns, most of which are stochastic by nature. The all-pay-auction CSF used in Konrad and Kovenock (2005), Agastya and McAfee (2006) and the more recent work on tug-of-war team contests rules out exogenous stochastic factors (i.e., whoever exerts a greater effort wins the battle for sure) that may affect the outcome of a battle. A natural alternative is the Tullock CSF (Tullock, 1980), which is possibly the most frequently employed CSF in the literature (see Konrad, 2007); and it incorporates a stochastic component. More precisely, in the Tullock CSF, the probability of a player winning a battle is given by the ratio of that player's effort to total effort exerted by all players. Therefore, even the player who

[1] There are examples from other disciplines as well. Larsson et al. (2004) described the interactions between viruses and the dendritic cells as a tug-of-war. Bradley et al. (2005) reported that their data suggests a tug-of-war scenario for the battle between certain male members of wild mountain gorilla groups for the control over reproduction. Recently, Tsend-Ayush et al. (2016) found that the hormone, known as glucagon-like peptide-1 has conflicting functions in the platypus and described the interactions between these functions as tug-of-war.

exerts a lower effort has a chance of winning. Besides its empirical appeal, the presence of exogenous noise allows for the existence of a pure strategy equilibrium.[2] A practical downside is that it is much more difficult to analytically solve for the equilibrium of tug-of-war with Tullock CSF. In fact, to the best of our knowledge, characterizing the equilibrium of a tug-of-war game between two players with Tullock CSF was an open problem up to date.

The research reported here is primarily motivated by these two observations. First, we observe more suspense and perseverance in real-life tugs of war (see Deck and Sheremeta, 2015 for a recent experimental evidence) –an observation not completely captured by the theoretical literature. Second, we observe that exogenous noise is an essential feature of many real-life contests (see Thorngate and Carroll, 1987). Consequently, we study a tug-of-war game between two players, where the battle outcomes are determined by a Tullock CSF. There are no intermediate prizes, the cost of effort is convex, and players do not discount future.

We completely characterize the Markov perfect equilibrium of this game, under a regularity assumption. The equilibrium strategies are deterministic (i.e., pure). Furthermore, we prove that the equilibrium is unique. We also offer a set of results on effort dynamics and some comparative statics. Our main results are as follows:

(i) equilibrium efforts in all interior (i.e., non-terminal) states are positive,

(ii) the player who is closer to winning (advantaged player) exerts a higher effort than the other player,

(iii) the ratio of the advantaged player's effort to the disadvantaged player's effort increases as the former approaches his favorite terminal state,

(iv) the sum of players' efforts decreases as either player gets closer to winning,

(v) players' equilibrium efforts follow monotonic paths across interior nodes, and

[2] See Vojnović (2015) for some natural instances where the ratio-form CSFs (generalized form of Tullock CSF) is a good fit (pages 164-166), and Fu and Lu (2012) and Lu and Wang (2015) for further justification for Tullock CSF.

(vi) equilibrium effort levels at all interior nodes increase with the difference between the values of winner and loser prizes and decrease with the threshold level of victory difference.

Our first result is in stark contrast with the equilibrium of tug-of-war with all-pay auction CSF (see Konrad and Kovenock, 2005). Also note that positive equilibrium efforts in all interior states and the stochastic nature of the Tullock CSF imply that there will be swings back-and-forth (i.e., the advantage may change hands). On the other hand, (ii), (iii), and (iv) show that a partial discouragement is still present. We will talk about the implications of our results in greater detail in Section 2.

This paper contributes to the theoretical literature on dynamic contests. Our model delivers empirically more appealing predictions on effort dynamics compared to the model of tug-of-war with all-pay auction CSF. Our results are also of interest from a design perspective: a contest-designer who values neck-to-neck competition or suspense (e.g., since it is a desirable feature from audience's perspective in sports competitions) should prefer Tullock CSF to all-pay auction CSF, in tug-of-war. Finally, we believe that our model will be of practical value for experimental economists studying dynamic contests, due to the existence of pure strategy equilibrium, which is easier to interpret/identify empirically, and a rich set of testable hypotheses.

2 Literature Review

We restrict our attention mostly to the literature on tug of war here. Theoretical work on tug-of-war mostly developed in the realm of economics. Harris and Vickers (1987) were the first to study it, formally. They modeled an R&D competition between firms as such. The outcome of a battle was determined probabilistically and was a function of firms' efforts. Harris and Vickers presented some qualitative results but did not characterize equilibrium with complete generality –mainly due to the probabilistic dependence of battle outcomes on firm efforts.

Konrad and Kovenock (2005) used an all-pay-auction (without noise) as a CSF to study tug-of-war. Accordingly, the player who exerts the highest effort wins the battle for sure (see Hillman and Riley, 1989; Baye, Kovenock, and de Vries, 1996) in their model. They analytically solved for equilibrium and provided conditions for uniqueness. Since their CSF is deterministic, the equilibrium is in mixed strategies. Perhaps, the most striking result of the paper is the extreme *discouragement effect* (see Konrad, 2012 for a review) that emerges in equilibrium: players exert considerable effort (using mixed strategies) in the first battle and zero efforts in all the remaining battles.[3] Consequently, the player who wins the first battle wins the contest without exerting any further effort. The deterministic nature of the CSF employed was the major reason behind this result (see Konrad, 2010).

Later –building on McAfee (2000)'s analysis– Agastya and McAfee (2006) also investigated a model of tug-of-war using an all-pay-auction CSF. Two major differences from Konrad and Kovenock (2005) were (i) the presence of a negative loser prize in Agastya and McAfee (2006) and (ii) the way their CSF broke ties. Now, the disadvantaged party may have a reason to continue exerting effort: escaping from the negative loser prize. These authors showed that there exists two types of stationary equilibria with very different characteristics. In one of them, effort tends to rise as either player gets close to winning, whereas in the other one, players remain in an interior state forever, not fighting against each other. The latter equilibrium resembles a discouragement effect different from the one in Konrad and Kovenock (2005). Moscarini and Smith (2007) extended Harris and Vickers (1987)'s model to a continuous-time and continuous state-space environment. In their model, a player continously exerts effort at a quadratic cost to produce a flow output, and when a predetermined output difference is reached, the game ends. A player's effort controls (linearly) the drift of the Brownian motion, which governs his cumulative output. The authors' main focus is on the optimal contest design (optimal prize and optimal scoring rule). They showed that the optimal prize (maximizing expected total output) is finite, and conjectured that the optimal scoring rule penalizes the leader so that the laggard does not give up, for which they provided numerical results.

Recently, there has been an increased interest in team contests using tug-of-war and other multi-battle contest designs (see Fu, Lu, and Pan, 2015; Häfner and Konrad, 2016; Häfner, 2017). On the other hand, to the best of our knowledge, Deck and Sherameta (2015) is the only experimental work on tug-of-war contests. Their experimental design builds on Konrad and Kovenock (2005). That said, their results are very different from the predic-

[3] Konrad and Kovenock (2005) assumed that the discount factor is strictly less than 1. Later, Vojnović (2015) proved that the same result holds for the no-discounting case, as well.

tions of Konrad and Kovenock (2005). In particular, they observed that participants invested fewer resources than what the model would predict in the first battle, and more resources than what the model would predict in the following battles. Furthermore, they observed that resources invested increase in the duration of the tug-of-war, which is also contrary to the theoretical prediction. Finally, very recently, Ewerhart and Teichgräber (2019) studied a class of dynamic contests, using finite automata techniques. They restricted their attention to dynamic contests that can be defined using a finite state machine and satisfy three assumptions (exchangability, monotonicity, and centeredness). Tug of war is of special interest. The alternative description of the problem and the methods they use allow them to prove that tug of war admits a unique, symmetric, interior Markov perfect equilibrium under a general form of Tullock contest success function.

3 The Results

We consider a tug-of-war contest, a multi-stage game with observed actions and potentially infinite horizon, characterized by the following elements.[4] The set of players is $\mathcal{N} = \{L, R\}$. The set of states of the contest (or war) is given by finite ordered nodes of a regular grid.[5] Probability of moving from one node to another is determined by the Tullock contest success function (Tullock, 1980). More precisely, if the players' efforts at a (non-terminal) node k are (l_k, r_k), the probability of a win by Player L is given by

$$p(l_k, r_k) = \begin{cases} \frac{l_k}{l_k + r_k} \text{ if } \max\{l_k, r_k\} > 0, \\ \frac{1}{2} \text{ otherwise.} \end{cases}$$

and the probability of a win by Player R is given by $1 - p(l_k, r_k)$. The cost of effort is determined by a quadratic cost function. In particular, the cost of spending effort e is $C(e) = \frac{e^2}{2}$. In this contest, (i) a win by Player L results in a move towards left; (ii) a win by Player R results in a move towards the right. The game ends when one of the players wins sufficient number of battles to drive the state of the game to his favored terminal state. Corresponding prizes are awarded to the winner/loser of the game.

[4] To make it clear, effort choices in each battle are simultaneous, but all past actions and battle outcomes are publicly observable.

[5] We use the terms, *state* and *node*, interchangeably.

There are no intermediate prizes, and the players do not discount future. We denote this game by Γ.

3.1 Equilibrium

In this section, we analyze the equilibrium of Γ. Under no intermediate prize and no discounting assumptions, tug-of-war turns out to be a multi-stage game with a simple Markov structure. In particular, players do not distinguish among different histories that may lead them to a given state when choosing their efforts at a given state (see Konrad 2012, page 9). In what follows, we restrict our attention to Markov strategies that end the game in finite time with probability 1.[6] We solve for the (pure strategy) equilibrium for odd and even number of nodes, separately. The analysis of even number of nodes is relegated to the Appendix since the differences from the odd number of nodes case are marginal.

Suppose that we have $2n+1$ nodes, and the set of states is given by $\mathcal{K}^n = \{-n, -(n-1), ..., 0, ..., (n-1), n\}$ for all $n \in \mathbb{Z}_{++}$. We denote the set of terminal states by $\mathcal{T}^{\ n} = \{-n, n\}$. A winning (losing) prize is awarded to player L (R) if the terminal state $-n$ is reached and, vice-a-versa if the terminal state n is reached (see Figure 1).

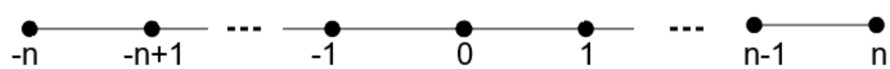

Figure 1. Tug-of-War with Odd Number of States

Consider now the node $-(n-1)$, which represents the state of the game in which Player L has $(n-1)$ victories advantage over Player R. At this node, Player L maximizes

$$p\left(l_{-(n-1)}, r_{-(n-1)}\right) V_{-n} + \left(1 - p\left(l_{-(n-1)}, r_{-(n-1)}\right)\right) V_{-(n-2)} - C\left(l_{-(n-1)}\right)$$

where V_{-n} is the value of winning the tug of war contest and $V_{-(n-2)}$ is the equilibrium expected value for Player L seen from node $-(n-2)$. In a

[6]Without this assumption, there exists Markov strategies that prolong the game indefinitely, which induces infinite costs. This is a well-known problem in stochastic games (see Kushner and Chamerlain, 1969); and the literature tackles this problem in the same way we do (see Bertsekas and Tsitsiklis, 1991; Patek and Bertsekas, 1999 among others).

similar vein, at node $-(n-1)$, Player R maximizes

$$p\left(l_{-(n-1)}, r_{-(n-1)}\right) V_n + \left(1 - p\left(l_{-(n-1)}, r_{-(n-1)}\right)\right) V_{(n-2)} - C\left(r_{-(n-1)}\right)$$

where V_n is the value of losing the contest and $V_{(n-2)}$ is the equilibrium expected value for Player R seen from node $-(n-2)$. Note that $V_{-k}^{Left} \equiv V_{-k}$ and $V_{-k}^{Right} \equiv V_k$. Hence, given that the game ends in finite time with probability 1, we have $V_{-n} > V_{-n+1} > \cdots > V_0 > ... > V_n$ in equilibrium, where

$$V_{-k} = \max_{l_{-k}} \frac{l_{-k}}{l_{-k} + r_{-k}} V_{-k-1} + \frac{r_{-k}}{l_{-k} + r_{-k}} V_{-k+1} - \frac{l_{-k}^2}{2}, \tag{1}$$

$$V_k = \max_{r_{-k}} \frac{l_{-k}}{l_{-k} + r_{-k}} V_{k+1} + \frac{r_{-k}}{l_{-k} + r_{-k}} V_{k-1} - \frac{r_{-k}^2}{2}. \tag{2}$$

Our first proposition shows that there exists a unique profile of Markov strategies such that no one-shot deviation is profitable.[7] Furthermore, we characterize the equilibrium value of each non-terminal node as a convex combination of the values of the terminal nodes.These weights (i.e., b_i and $1 - b_i$) will be derived in a constructive fashion in the proof of Proposition 1.

Proposition 1. *There exists a unique Markov perfect Nash equilibrium of* Γ*. In this equilibrium,*

$$V_i = b_i V_{-n} + (1 - b_i) V_n, \ \forall i \in \mathcal{K}^n \backslash \mathcal{T}^n,$$

where $0 < b_{i+1} < b_i < 1$*. Furthermore, equilibrium strategies are given by*

$$l_i = \frac{t_{-i}^{1/2} (V_{i-1} - V_{i+1})^{1/2}}{1 + t_{-i}}$$

$$r_i = \frac{t_{-i}^{1/2} (V_{-i-1} - V_{-i+1})^{1/2}}{1 + t_{-i}}$$

where $t_{-i} = \left(\frac{V_{i-1} - V_{i+1}}{V_{-i-1} - V_{-i+1}}\right)^{1/2}$.

[7] Note that since we consider Markov strategies that end the game in finite time with probability 1, payoffs are continuous at infinity, which allows us to use *one-deviation property* in the proof of Proposition 1.

Proof. Consider the optimization problems in (1) and (2). By the first order conditions,

$$\frac{r_{-k}}{(l_{-k}+r_{-k})^2}(V_{-k-1}-V_{-k+1}) = l_{-k} \tag{3}$$

$$\frac{l_{-k}}{(l_{-k}+r_{-k})^2}(V_{k-1}-V_{k+1}) = r_{-k} \tag{4}$$

we obtain that

$$\frac{l_{-k}}{r_{-k}} = \left(\frac{V_{-k-1}-V_{-k+1}}{V_{k-1}-V_{k+1}}\right)^{1/2}, \tag{5}$$

$$l_{-k}+r_{-k} = (V_{-k-1}-V_{-k+1})^{1/4}(V_{k-1}-V_{k+1})^{1/4}. \tag{6}$$

Now, let $t_k \equiv \left(\frac{V_{-k-1}-V_{-k+1}}{V_{k-1}-V_{k+1}}\right)^{1/2}$ and $x_k \equiv \frac{V_{-k}-V_{-k+1}}{V_{k-1}-V_k}$. Note that $t_k = 1/t_{-k}$ represents the relative efforts of the players at node $-k$, and x_k represents the ratio of the changes in the valuations of the players when Player R wins at node $-k$. We have then

$$\frac{l_{-k}}{l_{-k}+r_{-k}} = \frac{t_k}{1+t_k}, \tag{7}$$

$$\frac{r_{-k}}{l_{-k}+r_{-k}} = \frac{1}{1+t_k}, \tag{8}$$

so that equilibrium efforts can be given as

$$l_{-k} = \frac{(V_{-k-1}-V_{-k+1})^{1/2}\,t_k^{1/2}}{1+t_k}, \tag{9}$$

$$r_{-k} = \frac{(V_{k-1}-V_{k+1})^{1/2}\,t_k^{1/2}}{1+t_k}. \tag{10}$$

By substituting the solutions of the first order conditions (5) and (6) in (1) and (2), we obtain

$$\frac{V_{-k}-V_{-k+1}}{V_{-k-1}-V_{-k+1}} = \frac{(2t_k+1)t_k}{2(1+t_k)^2}. \tag{11}$$

Thus,

$$\frac{V_{k-1}-V_k}{V_{k-1}-V_{k+1}} = \frac{(2t_k+3)t_k}{2(1+t_k)^2}. \tag{12}$$

By (11) and (12), we can deduce

$$\frac{t_k^2}{x_k} = \frac{2t_k+3}{2t_k+1},$$

and hence

$$\frac{t_{k+1}^2}{x_{k+1}} = \frac{2t_{k+1}+3}{2t_{k+1}+1}. \tag{13}$$

Similarly, since $\frac{V_{-k}-V_{-k+1}}{V_{-k-1}-V_{-k+1}} + \frac{V_{-k-1}-V_{-k}}{V_{-k-1}-V_{-k+1}} = 1$, we can write

$$\frac{V_{-k-1}-V_{-k}}{V_{-k-1}-V_{-k+1}} = \frac{3t_k+2}{2(1+t_k)^2} \text{ and } \frac{V_k-V_{k+1}}{V_{k-1}-V_{k+1}} = \frac{t_k+2}{2(1+t_k)^2}.$$

Thus,

$$\frac{t_k^2}{x_{k+1}} = \frac{t_k+2}{3t_k+2}. \tag{14}$$

By combining (13) and (14), we can get

$$\frac{(3t_k+2)t_k^2}{t_k+2} = x_{k+1} = \frac{(2t_{k+1}+1)t_{k+1}^2}{2t_{k+1}+3}. \tag{15}$$

Now, let's define $f(x) \equiv \frac{(3x+2)x^2}{x+2}$ and $g(x) \equiv \frac{(2x+1)x^2}{2x+3}$. Clearly $f(0) = g(0) = 0$ and $f(\infty) = g(\infty) = \infty$. Moreover,

$$f'(x) = \frac{6x^3+20x^2+8x}{(x+2)^2}.$$

So, for all $x > 0$, $f'(x) > 0$. Similarly, $g'(x) > 0$ can be shown. Now, note that we have $f(t_k) = g(t_{k+1})$. Any $t_k > 0$ value implies $f(t_k) > 0$. Moreover, since $g(x)$ is continuous and increasing at $(0,\infty)$, it is one-to-one; thus $t_{k+1} = g^{-1}(f(t_k))$ is a uniquely determined positive value. Similarly, $t_k = f^{-1}(g(t_{k+1}))$.

We know $t_0 = 1$. Thus, by the equations above, we can find all values of t_n sequence for all $n \in \mathbb{Z}$. Now, let's prove that $t_k \geq 1$ implies $t_{k+1} \geq t_k$. If $t_k > 1$ then $g(t_k) < t_k^2 < f(t_k)$. Thus $g(t_k) < g(t_{k+1})$ and $t_k < t_{k+1}$.

We can determine x_k sequence as well. By definition, we know

$$\frac{(V_{-k-1} - V_{-k}) + (V_{-k} - V_{-k+1})}{(V_k - V_{k+1}) + (V_{k-1} - V_k)} = t_k^2.$$

Since

$$x_{k+1} = \frac{V_{-k-1} - V_{-k}}{V_k - V_{k+1}} \text{ and } x_k = \frac{V_{-k} - V_{-k+1}}{V_{k-1} - V_k},$$

we can get

$$(V_{-k-1} - V_{-k}) + (V_{-k} - V_{-k+1}) = \frac{t_k^2}{x_{k+1}}(V_{-k-1} - V_{-k}) + \frac{t_k^2}{x_k}(V_{-k} - V_{-k+1}).$$

Thus,

$$\frac{V_{-k-1} - V_{-k}}{V_{-k} - V_{-k+1}} = \frac{\frac{t_k^2}{x_k} - 1}{1 - \frac{t_k^2}{x_{k+1}}} = \frac{\frac{2}{2t_k+1}}{\frac{2t_k}{3t_k+2}} = \frac{3t_k + 2}{(2t_k + 1)t_k} \equiv h(t_k). \tag{16}$$

Similarly,

$$\frac{V_k - V_{k+1}}{V_{k-1} - V_k} = \frac{t_{-k} + 2}{(2t_{-k} + 3)t_{-k}}.$$

Now, let us define a_j, $j = 1, 2, ..., 2n$ according to

$$\begin{aligned} a_1 &= V_{-n} - V_{-n+1}, \\ a_2 &= V_{-n+1} - V_{-n+2}, \\ &\vdots \\ a_j &= V_{-n+j-1} - V_{-n+j}, \\ a_{j+1} &= V_{-n+j} - V_{-n+j+1}, \\ &\vdots \\ a_{2n} &= V_{n-1} - V_n. \end{aligned} \tag{17}$$

It is clear that

$$\sum_{j=1}^{2n} a_j = V_{-n} - V_n. \tag{18}$$

By (16), one can easily find a_j, $j = 1, 2, ..., 2n$, and hence, the valuation of the players at each node as a convex combination of V_{-n} (winning) and V_n

(losing) prizes.

First, note from (16) that $\frac{a_j}{a_{j+1}} = h\left(t_{n-j}\right)$, $j = 1, 2, ..., 2n-1$, which implies

$$a_j = \frac{a_1}{\prod_{i=1}^{j-1} h\left(t_{n-i}\right)}, \forall j.$$

From (18), we have

$$a_1 = \frac{V_{-n} - V_{n\cdot}}{\sum_{j=1}^{2n} \left(\frac{1}{\prod_{i=1}^{j-1} h(t_{n-i})}\right)}.$$

From the definition of a_j, $j = 1, 2, ..., 2n$, we deduce that

$$\forall i \in \mathcal{K}^n \backslash \mathcal{T}^n,\ V_i = V_{-n} - \sum_{j=1}^{n+i} a_j.$$

Equivalently,

$$\begin{aligned} V_i &= V_{-n} - \sum_{j=1}^{n+i} \left(\frac{a_1}{\prod_{s=1}^{j-1} h\left(t_{n-s}\right)}\right) \\ &= V_{-n} - \frac{V_{-n} - V_{n\cdot}}{\sum_{j=1}^{2n} \left(\frac{1}{\prod_{s=1}^{j-1} h(t_{n-s})}\right)} \sum_{j=1}^{n+i} \left(\frac{1}{\prod_{s=1}^{j-1} h\left(t_{n-s}\right)}\right), \forall i \in \mathcal{K}^n \backslash \mathcal{T}^n. \end{aligned}$$

Hence,

$$V_i = \underbrace{\left(1 - \frac{\sum_{j=1}^{n+i}\left(\frac{1}{\prod_{s=1}^{j-1} h(t_{n-s})}\right)}{\sum_{j=1}^{2n}\left(\frac{1}{\prod_{s=1}^{j-1} h(t_{n-s})}\right)}\right)}_{b_i} V_{-n} + \underbrace{\left(\frac{\sum_{j=1}^{n+i}\left(\frac{1}{\prod_{s=1}^{j-1} h(t_{n-s})}\right)}{\sum_{j=1}^{2n}\left(\frac{1}{\prod_{s=1}^{j-1} h(t_{n-s})}\right)}\right)}_{(1-b_i)} V_n, \qquad (19)$$

where $0 < b_{i+1} < b_i < 1$, $\forall i \in \mathcal{K}^n \backslash \mathcal{T}^n$. ■

The fact that any intermediate node has an equilibrium value, which can be expressed as a convex combination of the values of terminal nodes has important implications for equilibrium effort levels. In particular, it implies that any non-terminal node has a positive value, which leads to positive equilibrium efforts by both players in any such node (given in the proposition). This shows that the extreme discouragement (e.g., losing the first battle makes the whole contest a write-off) is not present in our model, and every node in the game is reached with a positive probability.

3.2 Effort Dynamics

Now, we investigate the changes in individual efforts and the sum of individual efforts across different states. Note that by (9) and (10), we can compute the equilibrium effort levels of both players. In the following Proposition, we first show that the player who is closer to winning (i.e., the advantaged player) exerts a higher effort than the other (disadvantaged) player, and the ratio of the advantaged player's effort to the disadvantaged player's effort increases as the advantaged player approaches his favorite terminal node. Second, we show that the sum of players' efforts decreases as the advantaged player approaches to his favorite terminal node.

Proposition 2. *Given V_{-n} and V_n, the equilibrium effort choices of the two players in Γ satisfy*
i) $\forall k \in \{1, 2, ..., n-1\}$, $l_{-k} > r_{-k}$ and, moreover, $\frac{l_{-k}}{r_{-k}}$ increases in k.
ii) $\forall k \in \{1, 2, ..., n-1\}$, $l_{-k+1} + r_{-k+1} > l_{-k} + r_{-k}$.

Proof. i) Recall from (7) and (8) that $\frac{l_{-k}}{r_{-k}} = t_k$, $\forall k$ and $t_0 = 1$. As $t_{k+1} > t_k$, $\forall k$ (see the proof of Proposition 1), the result immediately follows.
ii) Using (6) and (17), we have

$$l_{-k} + r_{-k} = l_k + r_k = (a_{n-k} + a_{n-k+1})^{1/4} (a_{n+k} + a_{n+k+1})^{1/4}$$

$\forall k \in \{1, 2, ..., n-1\}$.

Also recall from (16) that $\frac{a_j}{a_{j+1}} = h(t_{n-j})$, $j = 1, 2, ..., 2n$. These allow us to write

$$\frac{l_{-k+1} + r_{-k+1}}{l_{-k} + r_{-k}} = \left(\frac{a_{n-k+1} + a_{n-k+2}}{a_{n-k} + a_{n-k+1}}\right)^{1/4} \left(\frac{a_{n+k-1} + a_{n+k}}{a_{n+k} + a_{n+k+1}}\right)^{1/4}$$
$$= \left(\frac{a_{n-k+1} + a_{n-k+2}}{h(t_k)\, a_{n-k+1} + h(t_{k-1})\, a_{n-k+2}}\right)^{1/4} \left(\frac{h(t_{-k+1})\, a_{n+k} + h(t_{-k})\, a_{n+k+1}}{a_{n+k} + a_{n+k+1}}\right)^{1/4}$$
$$> \left(\frac{h(t_{-k+1})}{h(t_{k-1})}\right)^{1/4}, \forall k \in \{1, 2, ..., n-1\},$$

as $t_{k+1} > t_k$, $\forall k$ and $h(.)$ is a strictly decreasing function. Noting that $\frac{h(t_{-k+1})}{h(t_{k-1})} = 1$ for $k = 1$ and, moreover, $\frac{h(t_{-k})}{h(t_k)}$ is increasing in k ends the proof.
■

Proposition 2 shows that a *partial* discouragement is still present in our game: the disadvantaged player exerts a lower effort (than the advantaged player), and the gap between their efforts (measured as a ratio) widens as the advantaged player approaches victory. That said, the advantage can still change hands since the disadvantaged player keeps exerting a positive effort (till the very end) and as long as he does so, he has a chance to win.

Proposition 2 makes a point about the relative values and the sums of individual efforts, but does not pin down the dynamics of individual efforts, separately. Proposition 3 below completes the picture by analytically showing that –except around the central node– players' individual efforts follow monotonic paths (across nodes) in equilibrium.[8]

[8]Since in the case of even number of nodes, there is no (single) central node, the statement of the corresponding Proposition for the even number of nodes case (i.e., Proposition 7) is slightly different (see Appendix).

Proposition 3. *The equilibrium effort choices of an advantaged player in* Γ *follow a monotonic path, i.e.,*

$$l_{-1} > l_{-2} > ... > l_{-n+1} \text{ or } r_1 > r_2 > ... > r_{n-1}.$$

The equilibrium effort choices of a disadvantaged player also follow a monotonic path, i.e.,

$$l_1 > l_2 > ... > l_{n-1} \text{ or } r_{-1} > r_{-2} > ... > r_{-n+1}.$$

However, $l_{-1} > l_0$ *and* $r_0 < r_1$.

Proof. By (9), we have

$$\frac{l_{-k}}{l_{-k-1}} = \left(\frac{V_{-k-1} - V_{-k+1}}{V_{-k-2} - V_{-k}}\right)^{1/2} \frac{\phi(t_k)}{\phi(t_{k+1})},$$

where $\phi(t_k) = \frac{t_k^{1/2}}{1+t_k}$, $\forall k \in \{0, 1, 2, ..., n-1\}$. It follows from (17) and (16) that

$$\begin{aligned}\frac{V_{-k-1} - V_{-k+1}}{V_{-k-2} - V_{-k}} &= \frac{a_{n-k} + a_{n-k+1}}{a_{n-k-1} + a_{n-k}} = \frac{a_{n-k+1}(1 + h(t_k))}{a_{n-k}(1 + h(t_{k+1}))} \\ &= \frac{1}{h(t_k)} \left(\frac{1 + h(t_k)}{1 + h(t_{k+1})}\right).\end{aligned}$$

Accordingly, we can write $\frac{l_{-k}}{l_{-k-1}} = \Phi_k$, $\forall k \in \{0, 1, 2, ..., n-1\}$, where

$$\Phi_k \equiv \left[\frac{1}{h(t_k)} \left(\frac{1 + h(t_k)}{1 + h(t_{k+1})}\right)\right]^{1/2} \frac{\phi(t_k)}{\phi(t_{k+1})}.$$

Recall $t_0 = 1$ and $t_{k+1} = g^{-1}(f(t_k))$. Then, from the proof of Proposition 1, we know that $t_{k+1} \geq t_k$ for all k. Hence, $t_k \geq 1$ for all k. Now, note that $t_k > 1$ implies $\phi'(t_k) < 0$, which further implies $\frac{\phi(t_k)}{\phi(t_{k+1})} > 1$. Moreover, recall that $h(.)$ is a strictly decreasing function. This implies that $h(t_k)\, h(t_{k+1}) < 1$. Accordingly, $\left[\frac{1}{h(t_k)}\left(\frac{1+h(t_k)}{1+h(t_{k+1})}\right)\right]^{1/2} > 1$. Thus, $\Phi_k > 1$ for all $k \geq 1$. So, $\frac{l_{-k}}{l_{-k-1}} > 1$, $\forall k \in \{1, 2, ..., n-1\}$, and $l_{-1} > l_0$. As $r_k = l_{-k}$, $\forall k$, we can also conclude that $\frac{r_k}{r_{k+1}} > 1$, $\forall k \in \{1, 2, ..., n-1\}$, and $r_0 < r_1$. The fact that

$\frac{r_{-k}}{r_{-k-1}} > 1$ or $\frac{l_k}{l_{k+1}} > 1$, $\forall k \in \{1, 2, ..., n-1\}$ can also be shown in a similar way. ■

3.3 Comparative Statics

In this subsection we investigate how the players' equilibrium efforts respond to changes in (i) the difference between winning and losing prizes and (ii) the required victory threshold. More precisely, in the next Proposition, we show that an increase in the difference between the winning prize and the consolation (or losing) prize encourages players to exert more effort in every interior node. We also show that an increase in the required winning threshold (the difference in the number of victories a player has to reach to win the contest) decreases players' efforts in every interior node.

Proposition 4. *Players' equilibrium efforts at any interior node (i) increase with* $(V_{-n} - V_n)$ *and (ii) decrease with the number of nodes.*

Proof. (i) Note from (15) that t_k and x_k sequences are both independent from V_{-n} and V_n. Then, equation (19) shows that b_i's are also independent from V_{-n} and V_n. Recalling the equation (9) and applying simple algebra obtains

$$l_{-k} = \left(\left(b_{-k-1} - b_{-k+1}\right)\left(V_{-n} - V_n\right)\right)^{1/2} \frac{t_k^{1/2}}{(1 + t_k)},$$

which clearly shows that l_{-k} is strictly increasing in $(V_{-n} - V_n)$. The fact that r_{-k} is strictly increasing in $(V_{-n} - V_n)$ can be shown in a similar way.

(ii) We will now prove that the equilibrium effort choices of the players decrease with the number of nodes. To do so, consider the tug-of-war contest under two different sets of states: $\mathcal{K}^n = \{-n, -n+1, ..., 0, ..., (n-1), n\}$ and $\mathcal{K}^{\tilde{n}} = \{-\tilde{n}, -\tilde{n}+1, ..., 0, ..., (\tilde{n}-1), \tilde{n}\}$, where $n, \tilde{n} \in \mathbb{Z}_{++}$. Without loss of generality, assume that $\tilde{n} > n$. Let the corresponding equilibrium values of the nodes be denoted by V_i, $i \in \mathcal{K}^n$ and $\widetilde{V}_i$,$i \in \mathcal{K}^{\tilde{n}}$. Since the winning and losing prizes are not altered, we have

$$\widetilde{V}_{-n} < \widetilde{V}_{-\tilde{n}} = V_{-n} \text{ and } V_n = \widetilde{V}_{\tilde{n}} < \widetilde{V}_n.$$

This implies

$$\widetilde{V}_{-n} - \widetilde{V}_n < V_{-n} - V_n.$$

As we have already shown in (i) that the equilibrium effort choices of the players increase with $(V_{-n} - V_n)$, which ends the proof. ■

We state an immediate consequence of Proposition 4 on the sum of efforts below, as a corollary.

Corollary 1. *The sum of equilibrium individual efforts at any interior node (i) increase with* $(V_{-n} - V_n)$ *and (ii) decrease with the number of nodes.*

Proof. Follows directly from Proposition 4. ■

Finally, we finish this subsection by stating another consequence of Proposition 4.

Corollary 2. *Consider two different sets of states,* $\mathcal{K}^n = \{-n, -n+1, ..., 0, ..., (n-1), n\}$ *and* $\mathcal{K}^{\tilde{n}} = \{-\tilde{n}, -\tilde{n}+1, ..., 0, ..., (\tilde{n}-1), \tilde{n}\}$*, where* $n, \tilde{n} \in \mathbb{Z}_{++}$*. For any node* $k < \min(n, \widetilde{n})$*,* $\frac{l_{-k}}{r_{-k}}$ *is the same in both games.*

Proof. Follows from (5) and (19), and Proposition 4. ■

This last corollary shows that in two games with different numbers of nodes, $\frac{l_{-k}}{r_{-k}}$ ratio is identical in each interior node common to both games. This has an interesting implication: the ratio of players' efforts –the density of competition, in a sense– depends on how far the state in question is from the initial state but not on how far it is from the terminal states. In other words, the density of competition has a *backward-focus* rather than a *forward-focus*, which is an interesting theoretical result that calls for an empirical test.

4 Concluding Remarks

Many real-life dynamic contests are described as tugs of war. In most of these contests, (i) exogenous stochastic factors (e.g., chance) play a non-negligible role in determining battle outcomes, (ii) parties do not give up *easily*, and (iii) advantage occasionally changes hands. We model a multi-battle contest between two players as a tug-of-war where the battle outcome is determined by a Tullock CSF. The equilibrium of such a contest game was an open question. We provided a characterization of pure strategy Markov perfect equilibrium under a regularity assumption. As we expected, in the equilibrium, both players exert positive efforts until the very last battle. The player closer to winning exerts more effort and the asymmetry between the two players' efforts increase as the advantageous player approaches winning the contest. Furthermore, the total effort decreases as either player gets closer to winning. Finally, equilibrium effort levels at all interior nodes increase with the difference between the values of winning and losing prizes, and decrease with the threshold level of victory difference.

It is worthwhile mentioning that McAfee (2000), Agastya and McAfee (2006), Häfner and Konrad (2016) and Häfner (2017) also report pervasiveness (i.e., all states are reached with positive probabilities) in a game of tug of war, despite using an all pay CSF. However, the first two papers added a negative losing prize and changed the tie-breaking assumption, whereas the last two papers studied team contests. In our model, we do not need negative losing prizes, alternative tie-breaking assumptions, or team structure: Tullock CSF provides sufficient incentives to players.

Recently, Deck and Sheremata (2015) conducted the first experiment on tug-of-war contests. Their experimental design was built on Konrad and Kovenock (2005). To bridge the gap between subjects' behavior and the predictions of a tug of war model with all-pay auction CSF, they offered behavioral explanations. Our model relies on standard preferences, yet delivers empirically more appealing predictions compared to tug of war with all-pay auction CSF. We believe that our model will be of practical value to experimental economists who study tug-of-war contests in the lab.

Our results could also be of interest from a design perspective: a contest-designer who values neck-to-neck competition (e.g., since it is a desirable feature from audience's perspective in sports competitions) should prefer Tullock CSF to all-pay auction CSF in a tug-of-war (see Moscarini and Smith, 2007 for a work in this spirit).

We focused on a standart Tullock CSF with an impact function of the form, $f(x) = x$. Whether our results generalize to an impact function of the form, $f(x) = x^r$, where $r < 1$ or $1 < r \leq 2$ is far from trivial, and hence left as an open question.

Future work may investigate tug-of-war team contests (see Häfner and Konrad, 2016; Häfner, 2017 for recent examples) or multi-player tugs of war (see Doğan et al., 2018 for a recent work on multi-player race), with Tullock CSF. Comparison of results with Tullock CSF and all-pay CSF would be of interest to researchers in the field.

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5 Appendix

5.1 Even number of Nodes

Suppose that we have $2n$ nodes and the set of states is given by $\mathcal{K}^n = \{-2n+1, -2n+3, ..., -1, 1, ..., 2n-3, 2n-1\}$ for all $n \in \mathbb{Z}_{++}$. We denote the set of terminal states by $\mathcal{T}^{\ n} = \{-2n+1, 2n-1\}$. The winning prize

V_{-2n+1} is awarded to player L if the terminal state $-2n+1$ is reached and, alternatively, it is awarded to player R if the terminal state $2n-1$ is reached.

We have $V_{-2n+1} > V_{-2n+3} > \cdots > V_{-1} > V_1 > ... > V_{2n-1}$ in equilibrium, where

$$V_{-k} = \max_{l_{-k}} \frac{l_{-k}}{l_{-k}+r_{-k}} V_{-k-2} + \frac{r_{-k}}{l_{-k}+r_{-k}} V_{-k+2} - \frac{l_{-k}^2}{2}, \tag{20}$$

$$V_k = \max_{r_{-k}} \frac{l_{-k}}{l_{-k}+r_{-k}} V_{k+2} + \frac{r_{-k}}{l_{-k}+r_{-k}} V_{k-2} - \frac{r_{-k}^2}{2}. \tag{21}$$

Proposition 5. *There exists a unique Markov perfect Nash equilibrium of* Γ*. In this equilibrium,*

$$V_i = b_i V_{-n} + (1-b_i) V_n, \ \forall i \in \mathcal{K}^n \backslash \mathcal{T}^n,$$

where $0 < b_{i+1} < b_i < 1$*. Furthermore, equilibrium strategies are given by*

$$l_i = \frac{t_{-i}^{1/2} (V_{i-2} - V_{i+2})^{1/2}}{1+t_{-i}}$$

$$r_i = \frac{t_{-i}^{1/2} (V_{-i-2} - V_{-i+2})^{1/2}}{1+t_{-i}}$$

where $t_{-i} = \left(\frac{V_{i-2}-V_{i+2}}{V_{-i-2}-V_{-i+2}}\right)^{1/2}$.

Proof. Consider (20) and (21). By the first order conditions of optimality,

$$\frac{r_{-k}}{(l_{-k}+r_{-k})^2} (V_{-k-2} - V_{-k+2}) = l_{-k} \tag{22}$$

$$\frac{l_{-k}}{(l_{-k}+r_{-k})^2} (V_{k-2} - V_{k+2}) = r_{-k} \tag{23}$$

we obtain that

$$\frac{l_{-k}}{r_{-k}} = \left(\frac{V_{-k-2}-V_{-k+2}}{V_{k-2}-V_{k+2}}\right)^{1/2}, \tag{24}$$

$$l_{-k}+r_{-k} = (V_{-k-2}-V_{-k+2})^{1/4} (V_{k-2}-V_{k+2})^{1/4}. \tag{25}$$

Let us define $t_k \equiv \left(\frac{V_{-k-2} - V_{-k+2}}{V_{k-2} - V_{k+2}}\right)^{1/2}$ and $x_k \equiv \frac{V_{-k} - V_{-k+2}}{V_{k-2} - V_k}$. Note that $t_k = 1/t_{-k}$ represents the relative efforts of the players at node $-k$ and x_k represents the ratio of the changes in the valuations of the players when Right wins at node $-k$. Accordingly, we continue to have

$$\begin{aligned} \frac{l_{-k}}{l_{-k} + r_{-k}} &= \frac{t_k}{1 + t_k}, \\ \frac{r_{-k}}{l_{-k} + r_{-k}} &= \frac{1}{1 + t_k}, \end{aligned}$$

so that

$$l_{-k} = \frac{(V_{-k-2} - V_{-k+2})^{1/2}\, t_k^{1/2}}{1 + t_k}, \tag{26}$$

$$r_{-k} = \frac{(V_{k-2} - V_{k+2})^{1/2}\, t_k^{1/2}}{1 + t_k}. \tag{27}$$

By substituting the solutions of the first order conditions in (20) and (21), we get

$$\frac{V_{-k} - V_{-k+2}}{V_{-k-2} - V_{-k+2}} = \frac{(2t_k + 1)t_k}{2(1 + t_k)^2}, \tag{28}$$

and hence

$$\frac{V_{k-2} - V_k}{V_{k-2} - V_{k+2}} = \frac{(2t_k + 3)t_k}{2(1 + t_k)^2}. \tag{29}$$

By (28) and (29), we can deduce

$$\frac{t_k^2}{x_k} = \frac{2t_k + 3}{2t_k + 1},$$

thus

$$\frac{t_{k+2}^2}{x_{k+2}} = \frac{2t_{k+2} + 3}{2t_{k+2} + 1}. \tag{30}$$

Similarly, since $\frac{V_{-k} - V_{-k+2}}{V_{-k-2} - V_{-k+2}} + \frac{V_{-k-2} - V_{-k}}{V_{-k-2} - V_{-k+2}} = 1$ we can write that

$$\frac{V_{-k-2} - V_{-k}}{V_{-k-2} - V_{-k+2}} = \frac{3t_k + 2}{2(1 + t_k)^2} \text{ and } \frac{V_k - V_{k+2}}{V_{k-2} - V_{k+2}} = \frac{t_k + 2}{2(1 + t_k)^2}.$$

Thus,

$$\frac{t_k^2}{x_{k+2}} = \frac{t_k+2}{3t_k+2}. \tag{31}$$

By combining these we can get

$$\frac{(3t_k+2)t_k^2}{t_k+2} = x_{k+2} = \frac{(2t_{k+2}+1)t_{k+2}^2}{2t_{k+2}+3}. \tag{32}$$

Now let's define $f(x) \equiv \frac{(3x+2)x^2}{x+2}$ and $g(x) \equiv \frac{(2x+1)x^2}{2x+3}$. Clearly $f(0) = g(0) = 0$ and $f(\infty) = g(\infty) = \infty$. Moreover,

$$f'(x) = \frac{(9x^2+4x)(x+2)-(3x+2)x^2}{(x+2)^2} = \frac{6x^3+20x^2+8x}{(x+2)^2}$$

so, when $x > 0$, $f'(x) > 0$. Similarly $g'(x) > 0$ can be shown as well. Now note that we have $f(t_k) = g(t_{k+2})$. Now for any $t_k > 0$ value, $f(t_k) > 0$ as well. Moreover, since $g(x)$ is continuous and increasing at $(0, \infty)$, it is one-to-one thus $t_{k+2} = g^{-1}(f(t_k))$ is a uniquely determined positive value. Similarly $t_k = f^{-1}(g(t_{k+2}))$.

We know $x_1 = \frac{V_{-1}-V_1}{V_{-1}-V_1} = 1$. By (32),

$$\frac{t_1^2}{x_1} = \frac{2t_1+3}{2t_1+1} \Longrightarrow 2t_1^3 + t_1^2 - 2t_1 - 3 = 0,$$

that induces a unique solution $t_1 = 1.25316$. We can then find all values of t_k sequence for all $k \in \mathcal{K}^n \backslash \mathcal{T}^{\,n}$.

Now let's prove $t_k \geq 1$ implies $t_{k+2} \geq t_k$. If $t_k > 1$ then $g(t_k) < t_k^2 < f(t_k)$. Thus $g(t_k) < g(t_{k+2})$ and $t_k < t_{k+2}$.

Also we can determine x_k sequence as well. By definition, we know

$$\frac{(V_{-k-2}-V_{-k})+(V_{-k}-V_{-k+2})}{(V_k-V_{k+2})+(V_{k-2}-V_k)} = t_k^2.$$

Since

$$x_{k+2} = \frac{V_{-k-2}-V_{-k}}{V_k-V_{k+2}} \text{ and } x_k = \frac{V_{-k}-V_{-k+2}}{V_{k-2}-V_k}$$

we can obtain

$$(V_{-k-2}-V_{-k})+(V_{-k}-V_{-k+2})=\frac{t_k^2}{x_{k+2}}(V_{-k-2}-V_{-k})+\frac{t_k^2}{x_k}(V_{-k}-V_{-k+2}).$$

Thus,

$$\frac{V_{-k-2}-V_{-k}}{V_{-k}-V_{-k+2}}=\frac{\frac{t_k^2}{x_k}-1}{1-\frac{t_k^2}{x_{k+2}}}=\frac{\frac{2}{2t_k+1}}{\frac{2t_k}{3t_k+2}}=\frac{3t_k+2}{(2t_k+1)t_k}\equiv h\left(t_k\right) \tag{33}$$

Similarly,

$$\frac{V_k-V_{k+2}}{V_{k-2}-V_k}=\frac{t_{-k}+2}{(2t_{-k}+3)t_k}.$$

Now let us define a_j, $j=1,2,...,2n-1$ according to

$$\begin{aligned}
a_1 &= V_{-2n+1}-V_{-2n+3},\\
a_2 &= V_{-2n+3}-V_{-2n+5},\\
&\vdots\\
a_j &= V_{-2n+2j-1}-V_{-2n+2j+1},\\
a_{j+1} &= V_{-2n+2j+1}-V_{-2n+2j+3},\\
&\vdots\\
a_{2n-1} &= V_{2n-3}-V_{2n-1}.
\end{aligned} \tag{34}$$

It is clear that

$$\sum_{j=1}^{2n-1}a_j=V_{-2n+1}-V_{2n-1}. \tag{35}$$

By (33), one can then easily compute a_j, $j=1,2,...,2n-1$, and hence, the valuation of the players at each node as a convex combination of V_{-2n+1} (winning) and V_{2n-1} (losing).

First, note from (33) that $\frac{a_j}{a_{j+1}}=h\left(t_{2n-2j-1}\right)$, $j=1,2,...,2n-1$, which implies

$$a_j=\frac{a_1}{\prod_{i=1}^{j-1}h\left(t_{2n-1-2i}\right)},\ \forall j.$$

From (35), we find that

$$a_1 = \frac{V_{-2n+1} - V_{2n-1}.}{\sum\limits_{j=1}^{2n-1} \left(\frac{1}{\prod\limits_{i=1}^{j-1} h(t_{2n-1-2i})} \right)}.$$

From the definition of a_j, $j = 1, 2, ..., 2n-1$, we deduce that

$$\forall i \in \mathcal{K}^n \backslash \mathcal{T}^n, \; V_i = V_{-2n+1} - \sum_{j=1}^{n+\frac{i-1}{2}} a_j.$$

Then,

$$\begin{aligned} V_i &= V_{-2n+1} - \sum_{j=1}^{n+\frac{i-1}{2}} \left(\frac{a_1}{\prod\limits_{s=1}^{j-1} h\left(t_{2n-1-2s}\right)} \right) \\ &= V_{-2n+1} - \frac{V_{-2n+1} - V_{2n-1}.}{\sum\limits_{j=1}^{2n-1} \left(\frac{1}{\prod\limits_{s=1}^{j-1} h(t_{2n-1-2s})} \right)} \sum_{j=1}^{n+\frac{i-1}{2}} \left(\frac{1}{\prod\limits_{s=1}^{j-1} h\left(t_{2n-1-2s}\right)} \right), \forall i \in \mathcal{K}^n \backslash \mathcal{T}^n. \end{aligned}$$

Hence,

$$V_i = \underbrace{\left(1 - \frac{\sum\limits_{j=1}^{n+\frac{i-1}{2}} \left(\frac{1}{\prod\limits_{s=1}^{j-1} h(t_{2n-1-2s})} \right)}{\sum\limits_{j=1}^{2n-1} \left(\frac{1}{\prod\limits_{s=1}^{j-1} h(t_{2n-1-2s})} \right)} \right)}_{b_i} V_{-2n+1} + \underbrace{\left(\frac{\sum\limits_{j=1}^{n+\frac{i-1}{2}} \left(\frac{1}{\prod\limits_{s=1}^{j-1} h(t_{2n-1-2s})} \right)}{\sum\limits_{j=1}^{2n-1} \left(\frac{1}{\prod\limits_{s=1}^{j-1} h(t_{2n-1-2s})} \right)} \right)}_{(1-b_i)} V_{2n-1}, \tag{36}$$

where $0 < b_{i+1} < b_i < 1$, $\forall i \in \mathcal{K}^n \backslash \mathcal{T}^n$. ■

Proposition 6. *Given V_{-2n+1} and V_{2n-1}, the equilibrium effort choices of the players in Γ satisfy*
i) $\forall k \in \{1, 3, 5..., 2n-3\}$, $l_{-k} > r_{-k}$ and, moreover, $\frac{l_{-k}}{r_{-k}}$ increases in k.
ii) $\forall k \in \{1, 3, 5, .., 2n-3\}$, $l_{-k+2} + r_{-k+2} > l_{-k} + r_{-k}$.

Proof. i) Recall that $\frac{l_{-k}}{r_{-k}} = t_k$, $\forall k$ and $t_1 = 1,25316$. As $t_{k+2} > t_k$, $\forall k$ (see the proof of Proposition 5), the result immediately follows. *ii)* Using (25) and (34), we have

$$\forall k \in \{1, 3, ..., 2n-3\}, \; l_{-k}+r_{-k} = l_k+r_k = \left(a_{\frac{2n-1-k}{2}} + a_{\frac{2n+1-k}{2}}\right)^{1/4} \left(a_{\frac{2n-1+k}{2}} + a_{\frac{2n+1+k}{2}}\right)^{1/4}.$$

Also recall from (33) that $\frac{a_j}{a_{j+1}} = h\left(t_{2n-2j-1}\right)$, $j = 1, 2, ..., 2n-1$. These allow us to write

$$\begin{aligned}
\frac{l_{-k+2} + r_{-k+2}}{l_{-k} + r_{-k}} &= \left(\frac{a_{\frac{2n+1-k}{2}} + a_{\frac{2n+3-k}{2}}}{a_{\frac{2n-1-k}{2}} + a_{\frac{2n+1-k}{2}}}\right)^{1/4} \left(\frac{a_{\frac{2n-3+k}{2}} + a_{\frac{2n-1+k}{2}}}{a_{\frac{2n-1+k}{2}} + a_{\frac{2n+1+k}{2}}}\right)^{1/4} \\
&= \left(\frac{a_{\frac{2n+1-k}{2}} + a_{\frac{2n+3-k}{2}}}{h\left(t_k\right) a_{\frac{2n+1-k}{2}} + h\left(t_{k-2}\right) a_{\frac{2n+3-k}{2}}}\right)^{1/4} \left(\frac{h\left(t_{-k+2}\right) \; a_{\frac{2n-1+k}{2}} + h\left(t_{-k}\right) a_{\frac{2n+1+k}{2}}}{a_{\frac{2n-1+k}{2}} + a_{\frac{2n+1+k}{2}}}\right)^{1/4} \\
&> \left(\frac{h\left(t_{-k+2}\right)}{h\left(t_{k-2}\right)}\right)^{1/4}, \; \forall k \in \{1, 3, ..., 2n-3\},
\end{aligned}$$

as $t_{k+2} > t_k$, $\forall k$ and $h(.)$ is a strictly decreasing function. Noting that $\frac{h(t_{-k+2})}{h(t_{k-2})} = 1$ for $k = 1$ and, moreover, $\frac{h(t_{-k})}{h(t_k)}$ is increasing in k ends the proof.
■

Proposition 7. *The equilibrium effort choices of an advantaged player follow a monotonic path in Γ, i.e.,*

$$l_{-1} > l_{-3} > ... > l_{-2n+1} \text{ or } r_1 > r_3 > ... > r_{2n-1}.$$

The equilibrium effort choices of a disadvantaged player also follow a monotonic path, i.e.,

$$l_1 > l_3 > ... > l_{2n-1} \text{ or } r_{-1} > r_{-3} > ... > r_{-2n+1}.$$

Proof. By (26), we have

$$\frac{l_{-k}}{l_{-k-2}} = \left(\frac{V_{-k-2} - V_{-k+2}}{V_{-k-4} - V_{-k}}\right)^{1/2} \frac{\phi\left(t_k\right)}{\phi\left(t_{k+2}\right)},$$

where $\phi\left(t_k\right) = \frac{t_k^{1/2}}{1+t_k}$. It follows from (34) and (33) that

$$\frac{V_{-k-2} - V_{-k+2}}{V_{-k-4} - V_{-k}} = \frac{a_{\frac{2n-k-1}{2}} + a_{\frac{2n-k+1}{2}}}{a_{\frac{2n-k-3}{2}} + a_{\frac{2n-k-1}{2}}}.$$

Accordingly, we can write $\frac{l_{-k}}{l_{-k-2}} = \Phi_k$, $\forall k \in \{1, 3, ..., 2n-3\}$ where

$$\Phi_k \equiv \left[\frac{1}{h\left(t_k\right)} \left(\frac{1 + h\left(t_k\right)}{1 + h\left(t_{k+2}\right)}\right)\right]^{1/2} \frac{\phi\left(t_k\right)}{\phi\left(t_{k+2}\right)}.$$

Recall $t_1 = 1.25316$ and $t_{k+2} = g^{-1}\left(f\left(t_k\right)\right)$. Then, from the proof of Proposition 5, we know that $t_{k+2} \geq t_k$ for all k. Hence, $t_k > 1$ for all k. Now, note that $t_k > 1$ implies $\phi'\left(t_k\right) < 0$, which further implies $\frac{\phi(t_k)}{\phi(t_{k+2})} > 1$. Moreover, recall that $h(.)$ is a strictly decreasing function. This implies that $h\left(t_k\right) h(t_{k+2}) < 1$. Accordingly, $\left[\frac{1}{h(t_k)}\left(\frac{1+h(t_k)}{1+h(t_{k+2})}\right)\right]^{1/2} > 1$. Thus, $\Phi_k > 1$ for all $k \geq 1$, which concludes that $\frac{l_{-k}}{l_{-k-2}} > 1$, $\forall k \in \{1, 3, ..., 2n-3\}$. As $r_k = l_{-k}$, $\forall k$, we can also conclude that $\frac{r_k}{r_{k+2}} > 1, \forall k \in \{1, 3, ..., 2n-3\}$. The fact that $\frac{r_{-k}}{r_{-k-2}} > 1$ or $\frac{l_k}{l_{k+2}} > 1$, $\forall k \in \{1, 3, ..., 2n-3\}$ can also be shown in a similar way. ■

Proposition 8. *The equilibrium effort choices of the players at any interior node (i) increase with* $(V_{-2n+1} - V_{2n-1})$ *and (ii) decrease with the number of nodes.*

Proof. (i) Note from (32) that t_k and x_k sequences are both independent of V_{-n} and V_n. The equation (36) then depicts that b_i's are also independent from V_{-n} and V_n. Recalling the equation (9), simple algebra is applied to arrive at

$$l_{-k} = \left(\left(b_{-k-2} - b_{-k+2}\right)\left(V_{-2n+1} - V_{2n-1}\right)\right)^{1/2} \frac{t_k^{1/2}}{\left(1 + t_k\right)}$$

which clearly depicts that l_{-k} is strictly increasing in $(V_{-2n+1} - V_{2n-1})$. The fact that r_{-k} increases with $(V_{-2n+1} - V_{2n-1})$ can also be shown in a similar way.

(ii) We will now prove that the equilibrium effort choices of the players decrease with the number of nodes. To do so, consider the tug of war contest under two different sets of states: $\mathcal{K}^n = \{-2n+1, -2n+3, ..., -1, 1, ..., 2n-3, 2n-1\}$ and $\mathcal{K}^{\tilde{n}} = \{-2\tilde{n}+1, -2\tilde{n}+3, ..., -1, 1..., 2\tilde{n}-3, 2\tilde{n}-1\}$ where $n, \tilde{n} \in \mathbb{Z}_{++}$. Without loss of generality, assume that $\tilde{n} > n$. Let the equilibrium values of the nodes be denoted by V_i, $i \in \mathcal{K}^n$ and $V_i, i \in \mathcal{K}^{\tilde{n}}$. Since the winning and the consolation prizes are not altered, we have

$$\widetilde{V}_{-2n+1} < \widetilde{V}_{-2\tilde{n}+1} = V_{-2n+1} \text{ and } V_{2n+1} = \widetilde{V}_{2\tilde{n}+1} < \widetilde{V}_{2n+1}.$$

This implies

$$\widetilde{V}_{-2n+1} - \widetilde{V}_{2n-1} < V_{-2n+1} - V_{2n-1},$$

which ends the proof as we have already shown that the equilibrium effort choices of the players increase with $(V_{-2n+1} - V_{2n-1})$. ■

Corollary 3. *The sum of equilibrium individual efforts at any interior node (i) increase with* $(V_{-2n+1} - V_{2n-1})$ *and (ii) decrease with the number of nodes.*

Proof. Follows directly from Proposition 8. ■

Finally, we finish this subsection by stating another consequence of Proposition 4.

Corollary 4. *Consider two different sets of states,* $\mathcal{K}^n = \{-n, -n+1, ..., 0, ..., (n-1), n\}$ *and* $\mathcal{K}^{\tilde{n}} = \{-\tilde{n}, -\tilde{n}+1, ..., 0, ..., (\tilde{n}-1), \tilde{n}\}$*, where* $n, \tilde{n} \in \mathbb{Z}_{++}$*. For any node* $k < \min(n, \widetilde{n})$*,* $\frac{l_{-k}}{r_{-k}}$ *is the same in both games.*

Proof. Follows from (24) and (36), and Proposition 8. ■