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8215
2020

April 2020

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CESifo

Impressum:

CESifo Working Papers
ISSN 2364-1428 (electronic version)
Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH
The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute
Poschingerstr. 5, 81679 Munich, Germany
Telephone +49 (0)89 2180-2740, Telefax +49 (0)89 2180-17845, email office@cesifo.de
Editor: Clemens Fuest
https://www.cesifo.org/en/wp
An electronic version of the paper may be downloaded
· from the SSRN website: www.SSRN.com
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CESifo *Working Paper No. 8215*

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Abstract

This paper develops a static model of endogenous task-based technical progress to study how factor scarcity induces technological progress and changes in factor prices. The equilibrium technology is multi-dimensional and not strongly factor-saving in the sense of Acemoglu (2010). Nevertheless, labor scarcity induces labor productivity growth. There is a weak but no strong absolute equilibrium bias. This model provides a plausible interpretation of the famous contention of Hicks (1932) about the role of factor prices and factor endowments for induced innovations. It may serve as a micro-foundation for canonical macro-economic models. Moreover, it accommodates features like endogenous factor supplies and a binding minimum wage.

JEL-Codes: O310, D920, O330, O410.

Keywords: economic growth, endogenous technical change, direction of technical change, biased technology.

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This Version: March 31, 2020
I gratefully acknowledge financial assistance under the Inter Mobility Program of the FNR Luxembourg ("Competitive Growth Theory - CGT"). For valuable comments I would like to thank two anonymous referees and the editor. This article also benefited from useful suggestions provided by Hippolyte d'Albis, Martin Hellwig, Ka-Kit Iong, Anastasia Litina, Amer Tabakovic, and Benteng Zou.

1 Introduction

In a competitive environment, a process innovation allows firms to increase profits through a reduction of costs. A firm adopts such an innovation or attempts to invent it if the cost advantage due to an improved productivity of the factors of production outweighs the cost of making the associated innovation investment. At the level of the individual firm, a process innovation allows to produce the same amount of output with fewer resources. For the economy as a whole, aggregate output will increase if the factors of production set free by the process innovation either produce more in the industry where the innovation occurs or produce other commodities elsewhere. The question about the economic phenomena that explain when and why firms adopt a new process innovation or engage in its invention is therefore crucial for our understanding of the observed productivity differences across firms, industries, and countries.

The present paper addresses this question in a static version of the dynamic competitive economy with endogenous task-based technical progress devised in Irmen (2017) and Irmen and Taboković (2017). This analytical framework formalizes a central idea of John Hicks' *The Theory of Wages* according to which (relative) factor endowments affect (relative) factor prices and induce firms to implement or invent new technologies that replace the more expensive factor (Hicks (1932)).[2]

The analysis starts from the premise that a firm is an economic unit where tasks are performed to produce output. Accordingly, a firm's production function relates performed tasks to final output. These tasks are executed by two factors of production, capital and labor. New technologies are process innovations with the potential to increase the productivity of capital and labor in performing tasks. The factor productivity of both factors is endogenous and hinges on the firm's willingness to make innovation investments. Since factor markets clear, the equilibrium number of performed tasks depends on the productivity of each factor and on the economy's factor endowments. Moreover, technical change is factor augmenting at the macroeconomic level.

Two main sets of results are derived. The first concerns *induced productivity growth* and underlines the role of factor scarcity as an important determinant of technology choice. The second set of results deals with *factor price biases* due to technological change and changing factor endowments. Two complementary analytical strategies are used to establish these findings. The first strategy is based on comparative statics of the competitive equilibrium. The second strategy relies on the notions of *net output* and *net marginal product* at given factor endowments. It establishes and exploits the fact that the

[2]Similar arguments have been put forward to shed light on various episodes in economic history. Examples include Habbakuk's explanation of the differential technological progress in the United States and Britain in the 19th century (Habakkuk (1962)), or Elvin's argument for why a spinning wheel was abandoned in fourteenth-century China (Elvin (1972)). More recently Allen (2009) and Broadberry and Gupta (2009) use it to explain why major inventions were adopted in eighteenth-century Britain and not elsewhere.

equilibrium technology maximizes net output at given factor endowments and that the equilibrium factor prices are equal to the respective net marginal product at given factor endowments.

As to induced productivity growth, the equilibrium technology is neither strongly factor saving nor strongly factor complementary in the sense of Acemoglu (2010), i. e., depending on the kind of process innovation the marginal product of capital and labor may increase or fall. However, even without these regularity conditions the equilibrium productivity of a factor of production will be higher if this factor becomes scarcer. For instance, less labor increases the equilibrium incentives to substitute labor with technology and leads to a higher equilibrium labor productivity. At the same time, it weakens the incentives to substitute capital with technology. Accordingly, the equilibrium productivity of capital will be lower. It is in this sense that factor endowments determine the direction of technical change.

As to factor price biases - in the taxonomy of Acemoglu (2007) - technologies are shown to be absolutely and relatively biased towards the complementary factor, i. e., at given factor endowments a higher productivity of labor increases the real rental rate of capital and reduces the real wage. Moreover, there is neither a strong absolute nor a strong relative bias.[3] Hence, labor scarcity leads to a higher equilibrium real wage and a higher relative price of capital. The latter finding is driven by a partial and a general equilibrium effect of opposite sign. The partial equilibrium effect captures the effect of changes in a factor endowment for a given technology and is negative. The general equilibrium effect captures the effect of a change in factor endowments on factor prices through induced technical change. This effect is positive, i. e., there is a weak absolute and a weak relative equilibrium bias. Hence, labor scarcity induces technical change that increases the real wage and reduces the relative price of capital. Since the partial equilibrium effect dominates the general equilibrium effect, the long-run demand schedule of a factor is declining in its price.[4]

Additional sets of new results are derived in the 'extensions' section. First, the link between the task-based model of this paper and some of the author's earlier work including Irmen (2011) and Hellwig and Irmen (2001a) is discussed. The former contribution studies a competitive three-sector economy. It is shown that the equilibrium of a static version of this multi-sector economy is isomorphic to the one derived in the present

[3]Absence of a strong absolute bias means that the equilibrium price of a factor cannot increases in response to an increase in its supply. Absence of a strong relative equilibrium bias means that the relative demand curve for the two factors cannot be upward-sloping (Acemoglu (2007), p. 1372).

[4]These findings are in line with those derived in Acemoglu (2007) for an economy where a factor's (gross) marginal product coincides with its net marginal product and both are equal to its equilibrium factor price. In the present setting, the equilibrium remuneration of the factors of production is equal to their respective net marginal product at given factor endowments which differs from the respective (gross) marginal product. This distinction reflects the marginal contribution to final output of a factor and the additional investment outlays related to the tasks performed by it.

task-based model. The key is that the first-order condition determining the aggregate number of tasks performed in the task-based model coincides with the free-entry, zero-profit condition of the intermediate-good sectors of the three-sector economy. As a consequence, the implications of factor scarcity for innovation incentives and factor prices derived in the present paper carry over to this multi-sector environment. The analysis of the link to a static version of the competitive growth model proposed by Hellwig and Irmen (2001a) reveals that this one-sector model has no weak absolute bias since the equilibrium technology maximizes the real wage.

Second, the analysis turns to the role of endogenous factor supplies. Intuition suggests that the link between the scarcity of a factor, a higher factor price, and induced innovation may be counteracted by an increase in the aggregate supply of this factor. The analysis confirms this intuition for a scenario where either individuals supply more hours in response to a higher real wage or where the supply of labor increases in the rental rate of capital. However, this tendency does not invalidate the key predictions derived in the basic version of the model with inelastic factor supplies. It does however weaken the link between factor endowments, innovation incentives, and factor prices via a general equilibrium effect.

The third extension allows for one factor price to be exogenous. This turns the economy either into one with a minimum wage or into a small open economy. Both setups yield similar results concerning the role of changing factor endowments for the equilibrium technology, the remaining endogenous factor price, and employment levels. The analysis focusses on the case of a minimum wage. Then, the economy under scrutiny is similar to a static version of the one analyzed in Hellwig and Irmen (2001b). A binding minimum wage is found to entirely determine the direction of technical change as well as the rental rate of capital. Compared to the equilibrium under laissez-faire it reinforces the incentive to save labor, reduces the incentive to raise the productivity of capital, and implies a lower rental rate of capital. Changing the economy's capital endowment leaves these variables unaffected but leads to adjustments of the level of employment.

The present paper builds on and contributes to at least two strands of the literature. First, it makes a contribution to the theory of endogenous capital- and labor-saving technical change that has its roots in the so-called 'induced innovations' literature of the 1960s (see Fellner (1961), von Weizsäcker (1962), Kennedy (1964), Samuelson (1965), Drandakis and Phelps (1966), and von Weizsäcker (1966)). A main focus of this literature is on the link between (relative) factor prices and induced technical change as envisaged by Hicks (1932). However, its lack of a sound micro-foundation has often been criticized (see, e. g., Salter (1966), Burmeister and Dobell (1970), Chapter 3, Nordhaus (1973), Funk (2002), or Acemoglu (2003)). It assumes competitive firms with access to a constant-returns-to-scale production function $F(bK, aL)$ where K is capital, L is labor, and b and a are capital- and labor-augmenting technology terms. Obviously, profit-maximization with respect to (b, a, K, L) is not well defined since F has increasing returns in all four variables. To circumvent this problem, firms maximize instead the current rate of cost reduction subject to some invention possibility frontier. While this ad-hoc heuristic leads to results in support of Hicks' argument, the question remains open as to whether these findings would still hold under a sound micro-foundation.

The model developed in this paper provides such a micro-foundation. It gives rise to an endogenous 'technology frontier' along which b and a cannot simultaneously in- or decrease (see, Section 3.1), a property that is key to the exogenous invention possibility frontier of the 'induced innovations' literature (Burmeister and Dobell (1970)). Moreover, Section 5.2 establishes the close link between the comparative statics of my model and Hicks' famous contention.[5]

Second, this paper complements the literature on a class of competitive models with endogenous technological change where the technology has a tendency to be *strongly capital saving* or *strongly labor saving*. As argued in Acemoglu (2010), models with this property include, e. g., Champernowne (1961) or Zeira (1998). However, the multi-dimensional technology in the model of the present paper does not comply with this regularity condition.[6] Nevertheless, labor (capital) scarcity induces a higher equilibrium productivity of labor (capital). From the results of the 'extensions' section, it is evident that this property is also shared by the competitive endogenous growth models proposed in Irmen (2011).

Another important dimension with respect to which the present setup differs from existing competitive models with endogenous technical change is that technical progress applies to tasks and requires an innovation investment for each of them. Therefore the gross marginal product of a factor exceeds its net marginal product which is equal to the respective equilibrium factor price. Nevertheless, most of the findings on absolute and relative factor price biases are consistent with those of Acemoglu (2007) where gross and net marginal products coincide.

This paper is organized as follows. Section 2 presents the model with endogenous task-based technical progress. Section 3 establishes the existence of a general equilibrium, discusses its welfare properties, and introduces key concepts such as net output, equilibrium technology, and net marginal product. In Section 4, the link between factor scarcity, the equilibrium technology, and net output is discussed. The link between factor prices, technical progress and factor scarcity is the focus of Section 5. Section 6 has the above mentioned extensions. Section 6.1 establishes the equivalence between the model of Section 2 and a three-sector model of (Irmen (2011). Moreover, it discusses the link to a static version of the one-sector model of Hellwig and Irmen (2001a). Section 6.2 deals

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[5]See Irmen (2018) for an analysis of the relationship between the model of this paper and the steady-state requirements of canonical growth models. Alternative approaches to characterize the technology frontier and its relationship to the aggregate production function include Jones (2005), Growiec (2013), Growiec (2018), and León-Ledesma and Satchi (2019).

[6]Therefore, the derived effects of factor scarcity on technological progress are not covered by Theorem 1 in Acemoglu (2010) when applied to a competitive environment.

with the role of an endogenous labor supply. Finally, Section 6.3 studies the effect of an exogenous factor price with a focus on a binding minimum wage. Section 7 concludes. Proofs are relegated to Section 8, the Appendix.

2 The Model

Consider a static economy endowed with capital and labor. The respective factor endowments are denoted by $K \in \mathbb{R}_{++}$ and $L \in \mathbb{R}_{++}$. These inputs are inelastically supplied.[7] There is a single manufactured final good that can be consumed or invested. If invested it may increase the productivity of capital and/or labor in the performance of factor-specific tasks. The economy is perfectly competitive. All agents' preferences are defined over the consumption of the final good which also serves as numéraire.

Throughout this paper subscripts are often used to denote partial derivatives. For functions of one variable, it is the argument that appears as a subscript, for instance, $f_\kappa(\kappa) \equiv df(\kappa)/d\kappa$. In the context of functions of several variables numbers are used as, for example, in $F_{12}(M, N) \equiv \partial^2 F(M, N)/\partial M \partial N$.

Technology

The production sector has a continuum $[0, 1]$ of competitive firms. Without loss of generality, their behavior may be analyzed through the lens of a competitive representative firm. Two types of tasks have to be performed to produce output. The first type needs capital, the second labor as the only input. Let $m \in \mathbb{R}_+$ denote a task performed by capital and $n \in \mathbb{R}_+$ a task performed by labor. Then, $m \in [0, M]$ and $n \in [0, N]$ where M and N denotes the total 'number' of tasks of each type performed by the representative firm.

Tasks of the same type are identical. Therefore, total output depends only on M and N. Let $F : \mathbb{R}^2_+ \to \mathbb{R}_+$ denote the production function of the representative firm. It assigns the maximum output, Y, to each pair $(M, N) \in \mathbb{R}^2_+$, i. e.,

$$Y = F(M, N). \tag{2.1}$$

The function F is $\mathcal{C}^2$ with $F_1 > 0 > F_{11}$ and $F_2 > 0 > F_{22}$ for all $(M, N) > 0$. While tasks of each type are identical, they differ with respect to their marginal product. Moreover,

[7] The labels capital and labor are used for convenience only. They provide the link to the neoclassical production function as introduced by Solow (1956) and Swan (1956), the extension being that technical change is endogenous here. However, from a purely mathematical point of view, K and L may represent any pair of distinguishable inputs that are inelastically supplied. I discuss the role of endogenous factor supplies in Section 6.2.

F exhibits constant returns to scale (CRS) with respect to both task types. For further reference, let κ denote the task intensity of the firm, i. e.,

$$\kappa \equiv \frac{M}{N}. \tag{2.2}$$

Then, the production function in intensive form is $F(\kappa, 1) \equiv f(\kappa)$, where $f : \mathbb{R}_+ \to \mathbb{R}_+$, with $f_\kappa(\kappa) > 0 > f_{\kappa\kappa}(\kappa)$ for all $\kappa > 0$.

A task m requires $k(m) = 1/b(m)$ units of capital, a task n needs $l(n) = 1/a(n)$ units of labor. Hence, $b(m)$ and $a(n)$ denote the productivity of capital and labor, respectively. They are equal to

$$b(m) = 1 + q^b(m) \quad \text{and} \quad a(n) = 1 + q^a(n), \tag{2.3}$$

where $q^b(m) \in \mathbb{R}_+$ and $q^a(n) \in \mathbb{R}_+$ are indicators of productivity growth associated with task m and n, respectively. These productivity levels require investments of $i\left(q^b(m)\right) \geq 0$ and $i\left(q^a(n)\right) \geq 0$ units of the final output. The investment cost function $i : \mathbb{R}_+ \to \mathbb{R}_+$ is the same for all tasks, $\mathcal{C}^2$, increasing and strictly convex. Hence, higher levels of productivity require ever larger investments. Moreover, it satisfies for all tasks and $j = a, b$

$$\lim_{q^j \to 0} i(q^j) = \lim_{q^j \to 0} i_q(q^j) = 0, \quad \text{and} \quad \lim_{q^j \to \infty} i\left(q^j\right) = \lim_{q^j \to \infty} i_q\left(q^j\right) = \infty. \tag{2.4}$$

One may think of an investment as a decision to adopt a new technology that is available in differing degrees of sophistication or as R&D outlays in the spirit of the lab-equipment model of Rivera-Batiz and Romer (1991). In both cases, q^j measures the productivity gain that results from an investment that costs $i(q^j)$ units of output.

Profit-Maximization

The representative firm takes the vector (R, w) of the real rental rate of capital and the real wage as given and chooses a plan comprising $\left(q^b(m), k(m)\right)$ for all $m \in [0, M]$ and $(q^a(n), l(n))$ for all $n \in [0, N]$ as well as the choice of how many tasks (M, N) to perform. This plan is to maximize profits

$$\Pi \equiv F(M, N) - C, \tag{2.5}$$

where C is the firm's cost reflecting factor and investment costs for each task, i. e.,

$$C \equiv \int_0^M \left[Rk(m) + i(q^b(m))\right] dm + \int_0^N \left[wl(n) + i(q^a(n))\right] dn. \tag{2.6}$$

With (2.3) one has $k(m) = 1/(1 + q^b(m))$ and $l(n) = 1/(1 + q^a(n))$. Accordingly, the firm's problem may be split up in two parts. First, for each $m \in [0, M]$ and each $n \in$

$[0, N]$, the values $\left(q^b(m), q^a(n)\right) \in \mathbb{R}_+^2$ are to minimize C. This leads to the first-order (sufficient) conditions

$$q^b(m) \quad : \quad \frac{-R}{(1+q^b(m))^2} + i_q\left(q^b(m)\right) = 0, \quad \forall m \in [0, M], \tag{2.7}$$

$$q^a(n) \quad : \quad \frac{-w}{(1+q^a(n))^2} + i_q\left(q^a(n)\right) = 0, \quad \forall n \in [0, N]. \tag{2.8}$$

For each task of the respective type, these conditions equate the marginal reduction of the firm's capital cost/wage bill to the marginal increase in its investment costs. Assuming $R > 0$ and $w > 0$, the convexity of the investment cost function and the fact that $\lim_{q^j \to 0} i_q(q^j) = 0$, $j = a, b$, imply that these conditions determine a unique $q^b(m) = q^b > 0$ and $q^a(n) = q^a > 0$ for either task type. Accordingly, $b(m) = b$, $a(n) = a$, $k = 1/b$, and $l = 1/a$.

Second, each performed task must be profitable, i. e.,

$$F_1(m, N) - \frac{R}{1+q^b(m)} - i\left(q^b(m)\right) \quad \geq \quad 0, \quad \forall m \in [0, M], \tag{2.9}$$

$$F_2(M, n) - \frac{w}{1+q^a(n)} - i\left(q^a(n)\right) \quad \geq \quad 0, \quad \forall n \in [0, N]. \tag{2.10}$$

Hence, for a task to be performed, its marginal value product must be at least as large as its cost. The former is equal to $F_1(m, N)$ and $F_2(M, n)$, respectively. The latter is the sum of the capital or wage cost and the investment outlays of the respective task. Since each task is associated with a strictly positive input requirement $k(m) = 1/\left(1 + q^b(m)\right) > 0$ and $a(n) = 1/\left(1 + q^a(n)\right) > 0$, M and N must be finite in equilibrium to exclude an excess demand for capital or labor. In other words, in equilibrium conditions (2.9) and (2.10) must hold as an equality. Since Π has CRS in (M, N) at $\left(q^b, q^a\right)$, this also implies that equilibrium profits are zero.

Finally, observe that conditions (2.9) and (2.10) will only pin down the task intensity $\kappa = M/N$ since F has CRS in (M, N). The number of tasks will be determined by market clearing conditions.

3 Equilibrium Analysis

3.1 Definition and Characterization

An equilibrium consists of a plan

$$\left(\left(q^b(m), k(m)\right)_{m \in [0,M]}, \left(q^a(n), l(n)\right)_{n \in [0,N]}, M, N\right)$$

and factor prices (R,w) that solve (2.7) - (2.10) and the factor market clearing conditions

$$\begin{aligned} R &\geq 0, \quad \int_0^M k(m)dm \leq K, \quad R\left(\int_0^M k(m)dm - K\right) = 0, \\ w &\geq 0, \quad \int_0^N l(n)dn \leq L, \quad w\left(\int_0^N l(n)dn - L\right) = 0. \end{aligned} \tag{3.1}$$

Hence, at positive factor prices there must not be an excess demand, and equilibrium factor prices can only be strictly positive if there is full-employment of the respective factor.

Let $\theta \equiv K/L$ denote the capital intensity. Then the following holds.

Theorem 1 *There is a unique competitive equilibrium for any $(K,L) \in \mathbb{R}^2_{++}$. The equilibrium technology satisfies*

$$\bar{q}^b = g^b(\bar{\kappa}) \quad \text{and} \quad \bar{q}^a = g^a(\bar{\kappa}), \tag{3.2}$$

where $g^j : \mathbb{R}_{++} \to \mathbb{R}_{++}$, $j = a,b$, with $g^a_\kappa(\kappa) > 0 > g^b_\kappa(\kappa)$. Equilibrium factor prices are

$$\bar{R} = R(\bar{\kappa}) > 0 \qquad \text{and} \quad \bar{w} = w(\bar{\kappa}) > 0, \tag{3.3}$$

where $R : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $R_\kappa(\kappa) < 0$ and $w : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $w_\kappa(\kappa) > 0$. Moreover, the equilibrium task intensity, $\bar{\kappa}$, satisfies

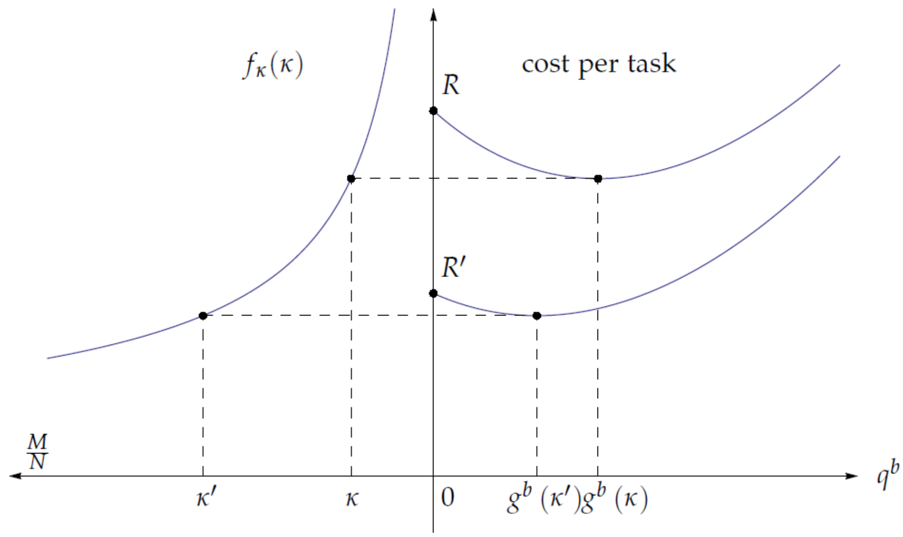
$$\bar{\kappa} = \kappa(\theta) > 0, \tag{3.4}$$

where $\kappa : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $\kappa_\theta(\theta) > 0$.

Theorem 1 suggests a simple way to characterize the equilibrium. Both, the equilibrium technology and equilibrium factor prices depend on the equilibrium task intensity, $\bar{\kappa}$, which, in turn, is pinned down by the capital intensity, θ. The intuition behind this comes in two steps.

The first step addresses the dependency of the equilibrium technology and the equilibrium factor prices on the task intensity κ as stated in (3.2) and (3.3). This property reflects the first-order conditions (2.7) - (2.10) and is illustrated in Figure 3.1 for tasks performed by capital. The left panel shows the marginal value product of the marginal task where use is made of the fact that $F_1(M,N) = f_\kappa(\kappa)$. According to (2.7) and (2.9), this value product must be equal to the minimized cost of tasks performed by capital. The cost-minimization is shown in the right panel. Hence, $(\kappa, R, g^b(\kappa))$ is an admissible solution to these two equations. As $f_{\kappa\kappa}(\kappa) < 0$, increasing the task intensity from κ to κ' means that the cost minimum must fall. This requires a lower real rental rate of capital $R' < R$, hence $R_\kappa(\kappa) < 0$. Since a lower capital cost reduces the marginal advantage of a productivity enhancing investment, the new cost-minimum is reached at a lower

Figure 3.1: The Link between κ, R, and $g^b(\kappa)$.



level of q^b, i. e., $g^b(\kappa') < g^b(\kappa)$ and $g^b_\kappa(\kappa) < 0$. The same line of reasoning shows why $w_\kappa(\kappa) > 0$ and $g^a_\kappa(\kappa) > 0$. The key difference here is that the marginal value product $F_2(M, N) = f(\kappa) - \kappa f_\kappa(\kappa)$ increases in κ.

The second step concerns the determination of the equilibrium task intensity as stated in (3.4). From the first step, the equilibrium technology depends on the task intensity. However, the market clearing conditions (3.1) reveal that in equilibrium $M = (1 + \bar{q}^b)K$ and $N = (1 + \bar{q}^a)L$, i. e., the task intensity depends on the equilibrium technology. Combining factor market clearing and (3.2) shows that the task intensity that performs both functions, $\bar{\kappa}$, must be a solution to

$$\bar{\kappa} = \frac{1 + g^b(\bar{\kappa})}{1 + g^a(\bar{\kappa})}\,\theta. \tag{3.5}$$

According to (3.4), there is a unique $\bar{\kappa} > 0$ that satisfies this equation. Moreover, $\bar{\kappa}$ increases in the capital intensity, θ, since $g^a_\kappa(\kappa) > 0 > g^b_\kappa(\kappa)$. For further reference, let me express this last result in terms of elasticities, i. e.,

$$\varepsilon^\kappa_\theta = \frac{1}{1 + \varepsilon^b_\kappa + \varepsilon^a_\kappa} \in (0, 1), \tag{3.6}$$

where

$$\varepsilon^\kappa_\theta \equiv \frac{\kappa_\theta(\theta)\,\theta}{\bar{\kappa}}, \quad \varepsilon^b_\kappa \equiv \frac{-g^b_\kappa(\bar{\kappa})\,\bar{\kappa}}{1 + g^b(\bar{\kappa})} > 0, \quad \text{and} \quad \varepsilon^a_\kappa \equiv \frac{g^a_\kappa(\bar{\kappa})\,\bar{\kappa}}{1 + g^a(\bar{\kappa})} > 0.$$

Hence, due to induced technical change the response of the equilibrium task intensity to changes in the capital-labor ratio is less than proportionate.

Finally, observe that Theorem 1 implicitly defines a 'technology frontier' and a 'factor-price frontier' (Samuelson (1960)). The technology frontier links any pair $\left(q^b, q^a\right) > 0$ that satisfies (3.2). It may be stated as

$$q^b = g^b\left[(g^a)^{-1}(q^a)\right] \equiv g(q^a), \tag{3.7}$$

where $(g^a)^{-1}(q^a)$ is the inverse of $g^a(\kappa)$ and, accordingly, $g : \mathbb{R}_{++} \to \mathbb{R}_{++}$. Since $g^a_\kappa(\kappa) > 0$, the slope of the inverse is also strictly positive. Hence, $dq^b/dq^a < 0$, i. e., q^b and q^a cannot increase simultaneously. Notice that unlike the exogenous invention possibility frontier stipulated by the 'induced innovations' literature of the 1960s, the technology frontier of the present model is the result of profit-maximizing behavior. The factor-price frontier is defined for any pair of factor prices $(R, w) > 0$ that satisfies (3.3) with $dR/dw < 0$, i. e., R and w cannot increase simultaneously. I shall explore these properties in Sections 4 and 5.

3.2 Net Output, Equilibrium Technology, and Equilibrium Factor Prices

This section introduces the notions of net output and net marginal product at given factor endowments. These concepts are later used to establish key properties of the equilibrium technology and factor prices.

Net Output

Net output is aggregate output minus aggregate investment outlays. Let $\left(q^b, q^a\right) \in \mathbb{R}^2_+$ denote the vector of *symmetric technology choices*. It presumes a firm behavior where the same amount of investment is allocated to all tasks of the same type (though, not necessarily the profit-maximizing amount of investment). Then, *net output at symmetric technology choices* is defined as

$$V\left(q^b, q^a, M, N\right) \equiv F(M, N) - Mi\left(q^b\right) - Ni(q^a). \tag{3.8}$$

The argument $\left(q^b, q^a, K, L\right)$ is used to study the effect of technical change at given factor endowments. Besides symmetric technology choices, it reflects the additional use of the market clearing conditions (3.1), i. e., $M = \left(1 + q^b\right) K$ and $N = (1 + q^a) L$. Then, *final output at given factor endowments* is defined as

$$Y\left(q^b, q^a, K, L\right) \equiv F\left(\left(1 + q^b\right) K, (1 + q^a) L\right). \tag{3.9}$$

This reveals that i) technical change is factor augmenting, ii) a better technology means more output of the final good, i. e., $Y_1 = KF_1 > 0$ and $Y_2 = LF_2 > 0$, and iii) $Y\left(q^b, q^a, K, L\right)$ is (strictly) super-modular in $\left(q^b, q^a\right)$, i. e., $Y_{12} = KLF_{12} > 0$. Here, super-modularity follows since F has positive, yet diminishing, marginal products, and CRS to scale in

(M, N). Using (3.9) and the market clearing conditions (3.1) in (3.8) gives rise to the definition of *net output at given factor endowments*, i. e.,

$$V\left(q^b, q^a, K, L\right) \equiv Y\left(q^b, q^a, K, L\right) - \left(1+q^b\right) Ki\left(q^b\right) - \left(1+q^a\right) Li\left(q^a\right), \qquad (3.10)$$

where $\left(1+q^b\right) Ki\left(q^b\right) + \left(1+q^a\right) Li\left(q^a\right)$ are aggregate investment outlays given full employment of both factors of production.

Equilibrium Technology

The following proposition derives an important property of the equilibrium technology.

Proposition 1 *If $\left(\bar{q}^b, \bar{q}^a\right)$ is the equilibrium technology then*

$$\left(\bar{q}^b, \bar{q}^a\right) = argmax_{\left(q^b, q^a\right) \in \mathbb{R}_+^2} \quad V\left(q^b, q^a, K, L\right). \qquad (3.11)$$

Moreover, any $\left(q^b, q^a\right) \in \mathbb{R}_+^2$ that solves (3.11) is an equilibrium technology.

Hence, both the first and the second welfare theorem hold in this economy.[8] This finding confirms the claim that the static technology choice in competitive environments tends to be welfare maximizing (see, e. g., Acemoglu (2007) or Zeira (1998)). However, in the present model there is a novel perspective on the equilibrium technology which will prove useful later. Indeed, the presence of the technology frontier (3.7) and the fact that the equilibrium technology is a global maximizer of $V\left(q^b, q^a, K, L\right)$ leads immediately to the following corollary to Proposition 1.

Corollary 1 *The equilibrium technology $\left(\bar{q}^b, \bar{q}^a\right)$ is the solution to*

$$\bar{q}^a = argmax_{q^a \in \mathbb{R}_+} \quad V\left(g\left(q^a\right), q^a, K, L\right) \quad and \quad \bar{q}^b = g\left(\bar{q}^a\right). \qquad (3.12)$$

For further reference, the maximum of net output at given factor endowments is henceforth referred to as *equilibrium net output* and denoted by $V(K, L)$, i. e.,

$$\begin{aligned} V(K, L) &\equiv V\left(\bar{q}^b, \bar{q}^a, K, L\right) \\ &= Y\left(\bar{q}^b, \bar{q}^a, K, L\right) - \left(1+\bar{q}^b\right) Ki\left(\bar{q}^b\right) - \left(1+\bar{q}^a\right) Li\left(\bar{q}^a\right). \end{aligned} \qquad (3.13)$$

[8] At first sight, Proposition 1 may seem restrictive because it presumes a symmetric technology choice. However, this turns out to be a valid short cut since a planner who chooses $\left(q^b(m)\right)_{m\in[0,M]}$, $\left(q^a(n)\right)_{n\in[0,N]}$, and (M, N) to maximize net output, $F(M, N) - \int_0^M i\left(q^b(m)\right) dm - \int_0^N i\left(q^a(n)\right) dn$, subject to the resource constraints $\int_0^M \left(1+q^b(m)\right)^{-1} dm \leq K$ and $\int_0^N \left(1+q^a(n)\right)^{-1} dn \leq L$ will pick the technology $\left(\bar{q}^b, \bar{q}^a\right)$ that also solves (3.11).

Equilibrium Factor Prices and Net Marginal Products

The *net marginal product of capital at given factor endowments* is the additional net output at $\left(q^b, q^a, K, L\right)$ that results from a small increase in K. Analogously, for a small increase in L, one has the *net marginal product of labor at given factor endowments*. To develop an intuition for these concepts consider capital.

If the economy's capital stock is fully employed then $M = \left(1+q^b\right)K$, and a small increase in capital means $dM = \left(1+q^b\right)dK$ additional tasks. On the one hand, this implies an increase in the output of the final good equal to $dY = F_1\left(\left(1+q^b\right)K, \left(1+q^a\right)L\right)dM = F_1\left(\left(1+q^b\right)K, \left(1+q^a\right)L\right)\left(1+q^b\right)dK$. On the other hand, aggregate investments increase by $i(q^b)dM = i(q^b)\left(1+q^b\right)dK$. The net marginal product of capital at given factor endowments is then the difference between these two effects and equal to

$$\frac{\partial V\left(q^b, q^a, K, L\right)}{\partial K} \equiv \left(1+q^b\right)\left[F_1\left(\left(1+q^b\right)K, \left(1+q^a\right)L\right) - i\left(q^b\right)\right]. \tag{3.14}$$

Analogously, the net marginal product of labor at given factor endowments is

$$\frac{\partial V\left(q^b, q^a, K, L\right)}{\partial L} = \left(1+q^a\right)\left[F_2\left(\left(1+q^b\right)K, \left(1+q^a\right)L\right) - i\left(q^a\right)\right]. \tag{3.15}$$

This leads to the following result.

Proposition 2 *The equilibrium factor prices satisfy*

$$\bar{R} = \frac{\partial V\left(\bar{q}^b, \bar{q}^a, K, L\right)}{\partial K} \quad \textit{and} \quad \bar{w} = \frac{\partial V\left(\bar{q}^b, \bar{q}^a, K, L\right)}{\partial L}. \tag{3.16}$$

Hence, the equilibrium factor prices are equal to the respective net marginal products at given factor endowments evaluated at $\left(\bar{q}^b, \bar{q}^a\right)$. Intuitively, equilibrium factor prices adjust so that (2.9) and (2.10) hold as equality. This requires $\bar{R}$ and $\bar{w}$ to be equal to their respective net marginal products.

4 Factor Scarcity, Equilibrium Technology, and Net Output

This section explores the role of factor scarcity for the equilibrium technology and for equilibrium net output.

Factor Scarcity and Equilibrium Technology

The main result of this section is given in the following proposition.

Proposition 3 *At* $\left(\bar{q}^b, \bar{q}^a, K, L\right)$ *it holds that*

$$\frac{d\bar{q}^b}{dK} < 0, \quad \frac{d\bar{q}^a}{dK} > 0, \quad \frac{d\bar{q}^a}{dL} < 0, \quad \frac{d\bar{q}^b}{dL} > 0. \tag{4.1}$$

Hence, a larger capital stock induces a lower $\bar{q}^b$ and a higher $\bar{q}^a$, and, mutatis mutandis, for a larger labor force. In other words, the equilibrium incentive to equip a factor with a better factor-augmenting technology declines if the factor becomes more abundant. At the same time, the equilibrium incentives for a better technology that augments the other factor increases. It is in this sense that a factor and 'its' technology are substitutes, whereas a factor and the 'other' technology are complements.

The intuition for these findings is closely linked to the one underlying Theorem 1 since changes in factor endowments increase or decrease the capital intensity, θ. For instance, a higher K increases θ so that the equilibrium task intensity, $\bar{\kappa}$, shifts upwards and induces a lower $\bar{q}^b$ and a higher $\bar{q}^a$ in accordance with the technology frontier defined above.[9]

To place Proposition 3 in a broader context recall from Proposition 1 that the equilibrium technology satisfies $V_1\left(\bar{q}^b, \bar{q}^a, K, L\right) = V_2\left(\bar{q}^b, \bar{q}^a, K, L\right) = 0$. Restricting attention to labor, total differentiation of these two conditions at $\left(\bar{q}^b, \bar{q}^a, K, L\right)$ delivers

$$\frac{d\bar{q}^b}{dL} = \frac{V_{12}V_{24} - V_{22}V_{14}}{V_{11}V_{22} - V_{12}^2}, \tag{4.2}$$

$$\frac{d\bar{q}^a}{dL} = \frac{V_{21}V_{14} - V_{11}V_{24}}{V_{11}V_{22} - V_{12}^2}. \tag{4.3}$$

To sign these derivatives note the following. First, since $\left(\bar{q}^b, \bar{q}^a\right)$ maximizes $V\left(q^b, q^a, K, L\right)$, it holds that $V_{11} < 0, V_{22} < 0$ and $V_{11}V_{22} - V_{12}^2 > 0$. Second, $V_{12} = KLF_{12} > 0$ since $Y\left(q^b, q^a, K, L\right)$ is (strictly) super-modular in $\left(q^b, q^a\right)$. Finally,

$$V_{14} = K\left(1 + \bar{q}^a\right)F_{12} > 0, \tag{4.4}$$

$$V_{24} = L\left(1 + \bar{q}^a\right)F_{22} < 0. \tag{4.5}$$

Hence, a higher q^b increases the net marginal product of labor at given factor endowments whereas a higher q^a reduces it. In other words, V neither exhibits strictly decreasing nor strictly increasing differences in $\left(q^b, q^a, L\right)$.[10] Therefore, in the taxonomy of Acemoglu (2010), the technology $\left(q^b, q^a\right)$ is neither *strongly labor saving* nor *strongly labor complementary*.[11] As a consequence, the products in the numerators of (4.2) and (4.3) are

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[9]In fact, Proposition 3 may also be expressed in terms of the relative scarcity of factors of production measured by θ. Then, it would state that $d\bar{q}^b/d\theta = g_{\kappa}^b\left(\kappa(\theta)\right)\kappa_\theta(\theta) < 0$ and $d\bar{q}^a/d\theta = g_{\kappa}^a\left(\kappa(\theta)\right)\kappa_\theta(\theta) > 0$ where the signs follow from Theorem 1 according to which $\bar{q}^b = g^b\left(\kappa\left(\theta\right)\right)$ and $\bar{q}^a = g^a\left(\kappa\left(\theta\right)\right)$. To the extent that changes in θ may result from simultaneous variations in capital *and* labor, rephrasing Proposition 3 in this way is slightly more general.

[10]If a function $f\left(\mathbf{x}, t\right)$ defined on $\mathbb{R}^n \times \mathbb{R}$ is twice differentiable on some open set, then for each $i = 1, ..., n$ increasing (decreasing) differences means $\partial f^2(\mathbf{x}, t)/\partial x_i \partial t \geq 0$ $\left(\partial f^2(\mathbf{x}, t)/\partial x_i \partial t \leq 0\right)$.

[11]According to Acemoglu (2010), Definition 1, p. 1050, a technology is said to be *strongly labor saving*

of the same sign.[12] Nevertheless, the overall sign of these numerators is unequivocal. The positive sign of $d\bar{q}^b/dL$ follows since

$$V_{12}V_{24} - V_{22}V_{14} = Lc_q\left(\bar{q}^a\right)V_{14} > 0.$$

The negative sign of $d\bar{q}^a/dL$ results since

$$V_{21}V_{14} - V_{11}V_{24} = Kc_q\left(\bar{q}^b\right)V_{24} < 0,$$

where $c_q\left(q^j\right) \equiv 2i_q\left(q^j\right) + \left(1+q^j\right)i_{qq}\left(q^j\right) > 0$, $j = a, b$, is the slope of the minimized cost per task.

An alternative and insightful interpretation of the comparative statics stated in (4.2) and (4.3) can be gained from Corollary 1. Recall that the technologies q^b and q^a are linked via the technology frontier $q^b = g\left(q^a\right)$ introduced in (3.7). Along this frontier, the technology becomes effectively single-dimensional and net output at given factor endowments can be stated as $V\left(g\left(q^a\right), q^a, K, L\right)$. Then, it is readily verified that

$$\frac{\partial^2 V}{\partial L \partial q^a} = g_q\left(\bar{q}^a\right)V_{14} + V_{24} < 0,$$

where V is evaluated at $\left(\bar{q}^b, \bar{q}^a, K, L\right)$ and $g_q\left(\bar{q}^a\right) < 0$ is the slope of the technology frontier. In light of (4.4) and (4.5), both summands are negative. Hence, the technology q^a is 'strongly labor saving along the technology frontier'. Moreover, total differentiation of the first-order condition associated with (3.12) and evaluation at $\left(\bar{q}^b, \bar{q}^a, K, L\right)$ delivers

$$\frac{d\bar{q}^a}{dL} = -\frac{g_q\left(\bar{q}^a\right)V_{14} + V_{24}}{g_q\left(\bar{q}^a\right)\left(V_{11}g_q\left(\bar{q}^a\right) + 2V_{12}\right) + V_{22}} < 0 \quad and \quad \frac{d\bar{q}^b}{dL} = \frac{d\bar{q}^a}{dL}g_q\left(\bar{q}^a\right) > 0 \qquad (4.6)$$

which coincides with (4.2) and (4.3) but unequivocally reveals the sign of the comparative statics.

Finally, observe that the qualitative results of Proposition 3 carry over to a world where firms have access to only one of the two technologies. For instance, without means to raise the productivity of capital, $q^b = 0$, $k = 1$, and the equilibrium technology

(*strongly labor complementary*) if improvements in the technology reduce (increase) the net marginal product of labor at $\left(\bar{q}^b, \bar{q}^a, K, L\right)$. Analogously, it is *strongly capital saving* (*strongly capital complementary*) if improvements in the technology reduce (increase) the net marginal product of capital at $\left(\bar{q}^b, \bar{q}^a, K, L\right)$. Here, 'improvements in the technology' refer to higher levels of both elements of the technology vector $\left(q^b, q^a\right)$.

[12] Mutatis mutandis, the qualitative results of (4.2) - (4.5) and the ensuing interpretation are analogous for changes in the capital endowment. Hence, V has neither strictly decreasing nor strictly increasing differences in $\left(q^b, q^a, K\right)$. Accordingly, the technology $\left(q^b, q^a\right)$ is neither *strongly capital saving* nor *strongly capital complementary*. Moreover, the logic behind the unequivocal signs of changes in K on the equilibrium technology is analogous to the one for changes in L.

$\bar{q}^a$ maximizes net output at given factor endowments given by $V(0, q^a, K, L)$. Implicit differentiation of $V_2(0, \bar{q}^a, K, L) = 0$ delivers

$$\frac{d\bar{q}^a}{dK} = \frac{-V_{23}}{V_{22}} > 0 \quad \text{and} \quad \frac{d\bar{q}^a}{dL} = \frac{-V_{24}}{V_{22}} < 0, \tag{4.7}$$

again confirming the signs obtained in Proposition 3. More capital fosters innovation investments that increase the productivity of the complementary factor since $V_{23} = LF_{12} > 0$. Moreover, labor scarcity increases innovation incentives since, as in (4.5), $V_{24} = (1 + \bar{q}^a) LF_{22} < 0$.[13] Analogous results obtain if only the productivity of capital can be increased by means of innovation investments.

Factor Scarcity and Equilibrium Net Output

How does the equilibrium net output of (3.13) respond to changes in factor endowments? The answer is given by the *equilibrium net marginal product of capital*. Using Proposition 1 - 3, the latter is

$$\begin{aligned} \frac{dV(K,L)}{dK} &= \left[V_1\left(\bar{q}^b, \bar{q}^a, K, L\right) \frac{d\bar{q}^b}{dK} + V_2\left(\bar{q}^b, \bar{q}^a, K, L\right) \frac{d\bar{q}^a}{dK} \right] \\ &+ \frac{\partial V\left(\bar{q}^b, \bar{q}^a, K, L\right)}{\partial K} = \bar{R}. \end{aligned} \tag{4.8}$$

The first line captures the effect of induced technical change on net equilibrium output. In light of Proposition 3 it holds that $d\bar{q}^a/dK > 0 > d\bar{q}^b/dK$, i.e., the productivity of labor increases whereas the one of capital falls. However, the effect of these incremental adjustments on equilibrium net output is negligible since, according to Proposition 1, the equilibrium technology has already been chosen to maximize net output at given factor endowments. Accordingly, the partial derivatives $V_1\left(\bar{q}^b, \bar{q}^a, K, L\right)$ and $V_2\left(\bar{q}^b, \bar{q}^a, K, L\right)$ are zero in equilibrium, and the first line of (4.8) vanishes. As a consequence, a small increase in capital augments equilibrium net output only to the extent that more tasks can be performed using the given technology $\left(\bar{q}^b, \bar{q}^a\right)$. According to Proposition 2, this effect is equal to the equilibrium real rental rate of capital. An analogous argument shows that the effect of changing labor on equilibrium net output is equal to $\bar{w}$.

5 Factor Prices, Factor Scarcity, and Equilibrium Technology

This section studies the role of factor scarcity and technical progress for the levels of *absolute* and *relative* factor prices.

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[13] This is in line with the key finding of Acemoglu (2010). Since there is only one technology $V_{24} < 0$ means that the equilibrium technology is strongly labor-saving. Hence, labor scarcity increases $\bar{q}^a$.

5.1 Absolute Factor Prices

Denote the equilibrium factor prices of (3.16) by $\bar{R}\left(\bar{q}^b, \bar{q}^a, K, L\right)$ and $\bar{w}\left(\bar{q}^b, \bar{q}^a, K, L\right)$, respectively. The following proposition states the main result of this section.

Proposition 4 *At $\left(\bar{q}^b, \bar{q}^a, K, L\right)$ it holds that*

$$\frac{d\bar{R}}{dK} < 0, \quad \frac{d\bar{w}}{dK} > 0, \quad \frac{d\bar{w}}{dL} < 0, \quad \frac{d\bar{R}}{dL} > 0. \tag{5.1}$$

Hence, a larger capital stock lowers the equilibrium rental rate of capital and increases the real wage, and, mutatis mutandis, for a larger labor force. To get the intuition consider $d\bar{R}/dK < 0$. From Theorem 1 the equilibrium rental rate of capital declines in the task intensity which, in turn, increases in the capital-labor ratio. In other words, since a higher K increases θ, $\bar{R}$ must fall. Moreover, in accordance with the factor-price frontier, $\bar{w}$ increases.[14]

To place Proposition 4 in a broader context note that the effects stated in (5.1) may be split up into a partial and a general equilibrium effect. Indeed, with Proposition 2 one finds

$$\frac{d\bar{R}}{dK} = \frac{\partial\bar{R}}{\partial K} + \left[\frac{\partial\bar{R}}{\partial q^a}\frac{d\bar{q}^a}{dK} + \frac{\partial\bar{R}}{\partial q^b}\frac{d\bar{q}^b}{dK}\right], \quad \frac{d\bar{w}}{dL} = \frac{\partial\bar{w}}{\partial L} + \left[\frac{\partial\bar{w}}{\partial q^a}\frac{d\bar{q}^a}{dL} + \frac{\partial\bar{w}}{\partial q^b}\frac{d\bar{q}^b}{dL}\right], \tag{5.2}$$

$$\frac{d\bar{R}}{dL} = \frac{\partial\bar{R}}{\partial L} + \left[\frac{\partial\bar{R}}{\partial q^a}\frac{d\bar{q}^a}{dL} + \frac{\partial\bar{R}}{\partial q^b}\frac{d\bar{q}^b}{dL}\right], \quad \frac{d\bar{w}}{dK} = \frac{\partial\bar{w}}{\partial K} + \left[\frac{\partial\bar{w}}{\partial q^a}\frac{d\bar{q}^a}{dK} + \frac{\partial\bar{w}}{\partial q^b}\frac{d\bar{q}^b}{dK}\right]. \tag{5.3}$$

Here, the first term of each expression captures the partial equilibrium effect of changing factor endowments for a given technology. The terms in brackets represent the general equilibrium effects due to induced technical progress.

Partial Equilibrium Effects

To understand the link between Proposition 4, (5.2), and (5.3) consider the real wage. From (3.15), the sign of the partial equilibrium effects are determined by diminishing returns to labor and the super-modularity of F as $\partial\bar{w}/\partial L = \left(1+\bar{q}^a\right)^2 F_{22}\left(\bar{q}^b, \bar{q}^a, K, L\right) < 0$ and $\partial\bar{w}/\partial K = \left(1+\bar{q}^b\right)\left(1+\bar{q}^a\right) F_{21}\left(\bar{q}^b, \bar{q}^a, K, L\right) > 0$. An analogous argument applies to the equilibrium rental rate of capital.

General Equilibrium Effects

To provide an understanding of the general equilibrium effects, one needs to study first the partial effect of technical change on factor prices.

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[14] In fact, Proposition 4 may also be expressed in terms of the relative scarcity of factors of production measured by θ. Then, it would state that $d\bar{R}/d\theta = R_\kappa\left(\kappa(\theta)\right)\kappa_\theta(\theta) < 0$ and $d\bar{w}/d\theta = w_\kappa\left(\kappa(\theta)\right)\kappa_\theta(\theta) > 0$ where the signs follow from Theorem 1 according to which $\bar{R} = R\left(\kappa\left(\theta\right)\right)$ and $\bar{w} = w\left(\kappa\left(\theta\right)\right)$. To the extent that changes in θ may result from simultaneous variations in capital *and* labor, rephrasing Proposition 4 in this way is slightly more general.

Proposition 5 *At* $\left(\bar{q}^b, \bar{q}^a, K, L\right)$*, it holds that*

$$\frac{\partial \bar{R}}{\partial q^b} < 0, \quad \frac{\partial \bar{R}}{\partial q^a} > 0, \quad \frac{\partial \bar{w}}{\partial q^b} > 0, \quad \frac{\partial \bar{w}}{\partial q^a} < 0. \tag{5.4}$$

Since equilibrium factor prices are equal to the equilibrium net marginal products of the corresponding factor, the findings of Proposition 5 follow immediately from (4.4) and (4.5) and the corresponding expressions for capital $V_{23} > 0 > V_{13}$. In other words, they reflect the fact that the technology $\left(q^b, q^a\right)$ is neither strongly factor saving nor strongly factor complementary.[15]

To highlight the importance of Proposition 5 it is worth contrasting the effects of technical change on factor prices with those of exogenous factor-augmenting technical change that arise in the neoclassical growth model. In this model and the present notation, final output equals $F\left(\left(1+q^b\right)K, \left(1+q^a\right)L\right)$. Marginal cost pricing leads to an equilibrium real wage equal to $\hat{w} = \left(1+q^a\right)F_2$ so that

$$\frac{\partial \hat{w}}{\partial q^a} = F_2 + \left(1+q^a\right)LF_{22} \gtreqless 0, \tag{5.5}$$

where F is evaluated at $\left(\left(1+q^b\right)K, \left(1+q^a\right)L\right)$. Hence, technology q^a may increase or decrease the price of labor. This reflects the tension between a positive productivity effect and a negative effect due to diminishing returns (see Irmen (2014) for details). With endogenous technical change, the effect of q^a on $\bar{w}\left(\bar{q}^b, \bar{q}^a, K, L\right)$ is derived from Proposition 2 as

$$\begin{aligned} \frac{\partial \bar{w}}{\partial q^a} &= \left[F_2 - \left(1+\bar{q}^a\right)i_q(\bar{q}^a) - i(\bar{q}^a)\right] + \left(1+q^a\right)LF_{22} \\ &= \left(1+q^a\right)LF_{22} < 0, \end{aligned} \tag{5.6}$$

where F is evaluated at $\left(\left(1+\bar{q}^b\right)K, \left(1+\bar{q}^a\right)L\right)$. The sign is unequivocally negative since $V_2\left(\bar{q}^b, \bar{q}^a, K, L\right) = L\left[F_2 - \left(1+\bar{q}^a\right)i_q(\bar{q}^a) - i(\bar{q}^a)\right] = 0$. This suggests that the ambiguity of (5.5) is due to an asymmetry in the analytical setup rather than to properties of the production function: if technical change is exogenous, then competitive firms compete in factor markets but not for their technology. If technical progress is endogenous, then firms compete for the resources that make technical progress happen. As a consequence, the positive productivity effect that appears in (5.5) is competed away. In other words, a higher q^a cannot have a positive effect on the equilibrium real wage since the competitive equilibrium technology maximizes net output. Mutatis mutandis, the same reasoning applies to the effect of q^b on the equilibrium rental rate of capital.

[15] In the taxonomy of Acemoglu (2007), technology q^b is *absolutely biased* towards labor whereas technology q^a is *absolutely biased* towards capital.

Finally, observe that the models with and without endogenous technical change predict the same factor price movements for the cross-effects. Here, only the properties of F matter. More precisely, its super-modularity means that $\partial\hat{R}/\partial q^a = \partial\bar{R}/\partial q^a = V_{32} > 0$ and $\partial\hat{w}/\partial q^b = \partial\bar{w}/\partial q^b = V_{14} > 0$.

In light of Proposition 3 and Proposition 5, it is now straightforward to sign the general equilibrium effects. For brevity, I denote those of (5.2) by $\Delta\bar{R}_K$ and $\Delta\bar{w}_L$, and the ones of (5.3) by $\Delta\bar{R}_L$ and $\Delta\bar{w}_K$.

Proposition 6 *It holds that*

$$\Delta\bar{R}_K > 0, \quad \Delta\bar{w}_L > 0, \quad \Delta\bar{R}_L < 0, \quad \Delta\bar{w}_K < 0. \tag{5.7}$$

Hence, a larger capital stock leads to induced technical change that increases the price of capital and decreases the price of labor, and, mutatis mutandis, for labor. These results follow immediately from Propositions 3 and 5. The latter implies that all products that appear in (5.2) are strictly positive and those of (5.3) are strictly negative. As a result, induced technical change increases the price of the factor that has become more abundant and reduces the one of the factor that becomes scarcer.

To grasp the intuition consider $\Delta\bar{R}_K > 0$. If K becomes more abundant, then there are two effects on $\bar{R}$. On the one hand, the incentive to substitute capital with the capital-augmenting technology falls and $\bar{q}^b$ declines. This diminishes the efficient amount of capital and increases $\bar{R}$ due to diminishing returns. On the other hand, a higher productivity of the complementary factor becomes more valuable. Therefore, $\bar{q}^a$ increases which increases $\bar{R}$, due to the super-modularity of F.[16] Mutatis mutandis, the intuition is analogous for the remaining three Δs.

Total Effects

Summing up, the total effects shown in Proposition 4 reflect the tension between partial and general equilibrium effects of opposite sign. However, in all cases the partial effect dominates the general equilibrium effect.[17] In particular, (equilibrium) inverse factor demand functions are declining in the respective factor endowment. However, due to

[16] In the taxonomy of Acemoglu (2007), there is *weak absolute equilibrium bias* with respect to K and L since $\Delta\bar{R}_K > 0$ and $\Delta\bar{w}_L > 0$. Therefore, Proposition 6 is in line with Theorem 2 in Acemoglu (2007), p. 1394 saying that under fairly mild conditions there is weak absolute equilibrium bias. However, here the intuition for this result is quite different from Acemoglu's given on page 1373. Moreover, it should be noted that the signs in (5.2) and (5.3) can only be different from zero if the equilibrium technology does not maximize equilibrium factor prices. See Section 6.1.2 for further discussion.

[17] In the taxonomy of Acemoglu (2007), there is no *strong absolute equilibrium bias* since $d\bar{R}/dK < 0$ and $d\bar{w}/dL < 0$. This result may also be traced back to the fact that the Hessian of $V\left(\bar{q}^b, \bar{q}^a, K, L\right)$ in $\left(q^b, q^a, K\right)$ or $\left(q^b, q^a, L\right)$ is negative definite. In fact, V is jointly strictly concave in both $\left(q^b, q^a, K\right)$ and $\left(q^b, q^a, L\right)$ as $V_{14} > 0$ and $V_{23} > 0$.

induced technical progress, the response of a factor price to a change in 'its' factor endowment is attenuated.

Finally, observe that Proposition 4 encompasses the cases where only one or none of the technologies are available.[18] If only one technology is available, then one product representing a general equilibrium effect in (5.2) and (5.3) vanishes. If no technology is available, only the partial effect matters and we are back in the neoclassical growth model without technical change.

5.2 Relative Factor Prices

The symmetry of the results on absolute factor prices leads to clear-cut predictions for the effect of technical progress and factor scarcity on the relative factor price. Throughout, the findings of this section are expressed in terms of the relative price of capital $\bar{R}\left(\bar{q}^b, \bar{q}^a, K, L\right) / \bar{w}\left(\bar{q}^b, \bar{q}^a, K, L\right)$. The main result is the following.

Proposition 7 *At $\left(\bar{q}^b, \bar{q}^a, K, L\right)$ it holds that*

$$\begin{aligned}
\frac{d(\bar{R}/\bar{w})}{dK} &= \frac{\partial(\bar{R}/\bar{w})}{\partial K} + \left[\frac{\partial(\bar{R}/\bar{w})}{\partial q^a}\frac{d\bar{q}^a}{dK} + \frac{\partial(\bar{R}/\bar{w})}{\partial q^b}\frac{d\bar{q}^b}{dK}\right] < 0, \\
\frac{d(\bar{R}/\bar{w})}{dL} &= \frac{\partial(\bar{R}/\bar{w})}{\partial L} + \left[\frac{\partial(\bar{R}/\bar{w})}{\partial q^a}\frac{d\bar{q}^a}{dL} + \frac{\partial(\bar{R}/\bar{w})}{\partial q^b}\frac{d\bar{q}^b}{dL}\right] > 0.
\end{aligned} \tag{5.8}$$

Proposition 7 establishes two distinct outcomes. First, it shows that the total effect of a larger capital stock on the relative price of capital is negative whereas the effect of a larger labor endowment is positive. This finding can be directly deduced from Proposition 4. For instance, since a larger capital stock reduces $\bar{R}$ and increases $\bar{w}$, the relative price of capital must also fall.

Second, the derivatives on the left hand side of (5.8) represent total effects that may be decomposed into a partial and a general equilibrium effect. From the discussion of the partial effects on absolute factor prices, it is immediate that $\partial\left(\bar{R}/\bar{w}\right)/\partial K < 0$ and $\partial\left(\bar{R}/\bar{w}\right)/\partial L > 0$. To sign the general equilibrium effects, observe that Proposition 5 implies that

$$\frac{\partial\left(\bar{R}\left(\bar{q}^b, \bar{q}^a, K, L\right) / \bar{w}\left(\bar{q}^b, \bar{q}^a, K, L\right)\right)}{\partial q^b} < 0 \quad \text{and} \quad \frac{\partial\left(\bar{R}\left(\bar{q}^b, \bar{q}^a, K, L\right) / \bar{w}\left(\bar{q}^b, \bar{q}^a, K, L\right)\right)}{\partial q^a} > 0.$$

[18] This is immediate from the proof of Proposition 4. Indeed, if only one technology is available, then one of the elasticities in the denominator of (3.6) is zero. If none of the technologies are available, then $\bar{\kappa} = \theta$ and $\varepsilon_\theta^\kappa = 1$.

It is in this sense that technology q^b is biased towards labor whereas technology q^a is biased towards capital.

In the taxonomy of Hicks (1932), p. 121-122, technical change associated with an increase in q^b is called *capital-saving* since it decreases the ratio of the (net) marginal product of capital to that of labor. Technical change associated with an increase in q^a is called *labor-saving* since it increases this ratio. Hence, 'labor-saving inventions' in the terminology of Hicks are those that exhibit a relative bias towards capital in the sense that $\partial\left(\bar{R}\left(\bar{q}^b,\bar{q}^a,K,L\right)/\bar{w}\left(\bar{q}^b,\bar{q}^a,K,L\right)\right)/\partial q^a>0$. Here, $\bar{q}^a$ captures this type of invention.

Using these results in conjunction with Proposition 3 reveals that the general equilibrium effects satisfy[19]

$$\frac{\partial(\bar{R}/\bar{w})}{\partial q^a}\frac{d\bar{q}^a}{dK}+\frac{\partial(\bar{R}/\bar{w})}{\partial q^b}\frac{d\bar{q}^b}{dK}>0, \quad \text{and} \quad \frac{\partial(\bar{R}/\bar{w})}{\partial q^a}\frac{d\bar{q}^a}{dL}+\frac{\partial(\bar{R}/\bar{w})}{\partial q^b}\frac{d\bar{q}^b}{dL}<0. \tag{5.9}$$

In light of Proposition 3 and Proposition 5 all products associated with dK are positive whereas those associated with dL are negative. Hence, the total effects shown in Proposition 7 reveal a tension between partial and general equilibrium effects. While their sign is determined by the partial effect, the general equilibrium effects weakens the response of the relative price of capital to changes in the respective factor endowment.

At this stage, the link between the results derived so far and the famous contention of John Hicks becomes apparent (Hicks (1932)). On page 124 he asserts that

> A change in the relative prices of the factors of production is itself a spur to invention, and to invention of a particular kind - directed to economising the use of a factor which has become relatively expensive.

These incentives are implied by the first-order conditions of cost-minimization (2.7) and (2.8). At $\left(q^b,q^a\right)$, they may be rearranged to

$$\frac{R}{w}=\frac{\left(1+q^b\right)^2 i_q\left(q^b\right)}{\left(1+q^a\right)^2 i_q\left(q^a\right)}. \tag{5.10}$$

Since the numerator of the right-hand side increases in q^b and the denominator increases in q^a, a hike in R/w induced either by an increase in R or by a decrease in w implies a greater ratio q^b/q^a. In other words, the higher firms expect the relative price of capital to be, the more attractive is it for them to substitute capital with technology rather than labor with technology.

Of course, factor prices and the technology are endogenous. Hicks suggests changing factor endowments as the driving force behind factor prices and technical change (Hicks (1932), 124-125):

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[19] According to the taxonomy of Acemoglu (2007) the signs of the expressions in (5.9) imply a *weak relative equilibrium bias* with respect to K and L.

> The general tendency to a more rapid increase of capital than labour which has marked European history during the last few centuries has naturally provided a stimulus to labour-saving inventions.

Following the discussion of Propositions 3 and 4, it is indeed the case that a higher capital intensity, θ, means a lower $\bar{q}^b$, a larger $\bar{q}^a$, a lower $\bar{R}$, and a higher $\bar{w}$. Hence, when firms expect a larger capital intensity, they rightly anticipate the price of labor to increase and the price of capital to fall. The induced innovation investments attenuate these price movements. This is the role of the general equilibrium effects. However, in spite of induced innovations, the relatively scarcer factor becomes more expensive.

6 Extensions

6.1 Alternative Environments

This section studies two alternative economic environments and establishes the relationship to the model of endogenous task-based technical progress discussed so far. Section 6.1.1 establishes the equivalence between the model introduced in Section 2 and the three-sector economy studied in Irmen (2011) where tasks correspond to intermediate goods. Section 6.1.2 shows the link between the model of Section 2 and a static variant of the competitive one-sector growth model proposed in Hellwig and Irmen (2001a). Here, the key difference is the absence of diminishing returns of tasks in the production of the final good.

6.1.1 Competitive Three-Sector Economy

Consider a competitive economy with a final-good sector and two intermediate-good sectors, one producing the first intermediate with capital, the other producing the second intermediate with labor. Refer to these intermediates as the capital-intensive and the labor-intensive one. The final good serves as numéraire.

Technology and Profit-Maximization

The representative final-good firm manufactures the final-good out of two intermediate inputs according to the production function (2.1). Now, M and N denote the respective aggregate amounts of the capital-intensive and the labor-intensive intermediate inputs. The firm maximizes profits equal to $Y - p_K M - p_L N$ where p_K and p_L denote the real price of the respective intermediate. The respective first-order conditions are $p_K = F_1(M, N)$ and $p_L = F_2(M, N)$.

Intermediate-good firms either belong to the capital- or to the labor-intensive intermediate-good sector. Each sector is represented by the set $\mathbb{R}_+$ with Lebesgue measure. All firms of a sector have access to the same sector-specific production function

$$y_k(m) = \min\{1, b(m)k(m)\} \quad \text{and} \quad y_l(n) = \min\{1, a(n)l(n)\}, \tag{6.1}$$

where $y_k(m)$ and $y_l(n)$ is the output of firm $m \in \mathbb{R}_+$ or $n \in \mathbb{R}_+$, respectively. There is a capacity limit equal to 1,[20] $b(m)$ and $a(n)$ denote a firm's capital and labor productivity, and $k(m)$ and $l(n)$ is the capital and labor input. The firms' respective capital and labor productivity is given by (2.3). Firms may increase their factor productivity by investing $i(q^b)$ and $i(q^a)$ units of the final good, where i has the same properties as stated in and before (2.4).

Intermediate-good firms maximize profits, i. e., $p_K y_k(m) - Rk(m) - i(q^b(m))$ or $p_L y_l(n) - wl(n) - i(q^a(n))$ where $p_K y_k(m)$, $p_L y_l(n)$, is the respective firm's revenue from output sales, $Rk(m)$, $wl(n)$, its capital cost or wage bill, and $i\left(q^b(m)\right)$, $i\left(q^a(n)\right)$, its outlays for the innovation investment. Each firm chooses a production plan $\left(y_k(m), k(m), q^b(m)\right)$ and $\left(y_l(n), l(n), q^a(n)\right)$ taking prices, (p_K, p_L, R, w) as given.

To derive the optimal production plan a firm reasons as follows. If it innovates, there will be an investment cost $i\left(q^j\right) > 0$. Such an innovation investment is only profit-maximizing if the firm's profit margin is strictly positive, i. e., if $p_K > R/b(m)$ or $p_L > w/a(n)$. If this is the case then there is a positive scale effect, i. e., an innovating firm wants to apply the innovation to as large an output as possible and produces at the capacity limit, i. e., $y_k(m) = 1$ or $y_l(n) = 1$. The choice of $\left(k(m), q^b(m)\right)$ and $\left(l(n), q^a(n)\right)$ must then minimize the costs of producing the capacity output, i. e., assuming $w > 0$ and $R > 0$ these input combinations must satisfy $k(m) = 1/\left(1 + q^b(m)\right)$, $l(n) = 1/\left(1 + q^a(n)\right)$ and solve

$$\min_{q^b(m)\geq 0} \quad \frac{R}{1+q^b(m)} + i\left(q^b(m)\right) \quad \text{and} \quad \min_{q^a(n)\geq 0} \quad \frac{w}{1+q^a(n)} + i\left(q^a(n)\right).$$

The solution to this problem gives rise to first-order (sufficient) conditions that coincide with (2.7) and (2.8).

The aggregate capital demand is equal to $\int_0^M k(m)dm$ where M is the 'number' of firms producing each one unit of the capital-intensive intermediate. Mutatis mutandis, the aggregate labor demand is $\int_0^N l(n)dn$. Accordingly, the factor market clearing conditions are given by (3.1). To prevent excess factor demands in both factor markets, M and N must be finite, i. e., in equilibrium some intermediate-good firms must not enter. Therefore, the maximum profit of any intermediate-good firm producing the capital- or the

[20] The analysis is easily generalized to allow for an endogenous capacity choice requiring additional capacity investments, with investment outlays being a strictly convex function of capacity (see, Hellwig and Irmen (2001a) for details).

labor-intensive intermediate must be zero in equilibrium.[21] Using $p_K = F_1(M,N)$ and $p_L = F_2(M,N)$, these zero-profit conditions coincide with (2.9) and (2.10) as equalities.

Equilibrium

An equilibrium of the three-sector economy consists of production-cum-entry decisions $\{y_k(m), k(m), q^b(m)\}$ of all firms $m \in [0, M]$ and $\{y_l(n), l(n), q^a(n)\}$ of all firms $n \in [0, N]$, measures (M, N) of entering firms in both sectors producing one unit each, and prices (p_K, p_L, R, w). These variables solve the firms' first-order conditions for cost-minimization, the zero-profit (free-entry) condition as well as the factor market clearing conditions (3.1) given $(K, L) \in \mathbb{R}^2_{++}$.

Proposition 8 *Given $(K, L) \in \mathbb{R}_{++}$, the competitive three-sector economy has a unique equilibrium. The equilibrium values for $(\bar{q}^b, \bar{q}^a)$, $\bar{R}$, $\bar{w}$, $\bar{M}$, and $\bar{N}$ coincide with those of Theorem 1. In addition, the equilibrium determines $\bar{y}_k(m) = \bar{y}_l(n) = 1$, $\bar{p}_K = F_1(\bar{M}, \bar{N})$, and $\bar{p}_L = F_2(\bar{M}, \bar{N})$.*

Proposition 8 holds since the conditions for profit-maximization and zero-profits of the three-sector economy coincide with (2.7) - (2.10). Moreover, in both economies the factor market clearing conditions are given by (3.1). As a consequence, all concepts derived in Section 3 and the results that appear in Section 4 and Section 5 carry over to the three-sector economy.

6.1.2 Competitive One-Sector Economy

In the competitive one-sector economy studied in Hellwig and Irmen (2001a) firms produce a final good with the production function $y_l(n)$ of (6.1). Firms are represented by the set $\mathbb{R}_+$ with Lebesgue measure, hence $n \in \mathbb{R}_+$ indexes firms. The final good is the numéraire. A new element is that innovation investments have to be undertaken and financed one period before they are used in production. Let $R = 1 + r > 0$ denote the exogenous real interest factor.[22] Firms maximize profits equal to $y_l(n) - wl(n) - Ri(q^a(n))$ where $y_l(n)$ is the respective firm's revenue from output sales, $wl(n)$, its wage bill, and $Ri(q^a(n))$, its outlays for the innovation investment.

Firms choose a production plan $(y_l(n), l(n), q^a(n))$ taking factor prices (R, w) as given. Let N denote the measure of firms that enter and produce output. Following the reasoning set out for the intermediate-good firms in the three-sector economy, this leads to

[21] Firms that do not enter choose the plan $(0, 0, 0)$ that delivers zero-profits just as for entering firms. This also applies to the competitive one-sector economy of Section 6.1.2.

[22] In Hellwig and Irmen (2001a), R is endogenous and coordinates the inter-temporal decisions of firms and households. Since the latter sector is missing here, R is taken to be exogenous as would be the case, e. g., in a small open economy.

symmetric profit-maximizing choices for all $n \in [0, N]$ firms satisfying $l = 1/(1+q^a)$ and $-w/(1+q^a)^2 + Ri_q(q^a) = 0$, which is the counterpart of (2.8). To exclude an excess demand for labor, N must be finite, i. e., in equilibrium some firms must stay out of the market. Therefore, entering and producing firms must earn zero-profits in equilibrium, i. e., $1 - w/(1+q^a) - Ri(q^a) = 0$. Finally, full employment of labor pins down the 'number' of entering firms as $N = (1+q^a)L$. These conditions determine unique equilibrium values $\left(\bar{q}^a, \bar{l}, \bar{w}, \bar{N}\right) \in \mathbb{R}^4_{++}$ as functions of R, and $\bar{y}_l(n) = 1$.

Interpreting $y_l(n)$ as the 'number' of tasks performed by firm n, the link to the model of Section 2 becomes obvious. In equilibrium, each firm performs one task at minimum costs. Moreover, aggregate output is equal to the 'number' of entering firms $\bar{N}$, i. e., there are no diminishing returns associated with the number of performed tasks. Moreover, net output at given factor endowments is the difference between aggregate output, $N = (1+q^a)L$, and total investment outlays, $(1+q^a)LRi\left(q^a\right)$, i. e., $V\left(q^a, L; R\right) = \left(1+q^a\right)\left[1 - Ri\left(q^a\right)\right]L$. This leads to the following results corresponding to Theorem 1, Proposition 1 and Proposition 2.

Proposition 9 *(One-Sector Economy)*

1. *Equilibrium Technology*

   (a) *The equilibrium technology satisfies*

$$\bar{q}^a = g\left(R\right), \text{ where } g : \mathbb{R}_{++} \to \mathbb{R}_{++} \text{ and } g_R\left(R\right) < 0. \tag{6.2}$$

   (b) *If $\bar{q}^a$ is the equilibrium technology then*

$$\bar{q}^a = \text{argmax}_{q^a \in \mathbb{R}_+} \quad V\left(q^a, L; R\right). \tag{6.3}$$

   Moreover, any $q^a \in \mathbb{R}^2_+$ that solves (6.3) is an equilibrium technology.

2. *Equilibrium Wage*

   (a) *The equilibrium wage satisfies*

$$\bar{w} = \max_{q^a \in \mathbb{R}_+} \frac{\partial V\left(\bar{q}^a, L; R\right)}{\partial L}. \tag{6.4}$$

   (b) *The equilibrium wage is independent of L. Moreover, there is a function $w : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that*

$$\bar{w} = w\left(R, g\left(R\right)\right) \quad \text{with} \quad \frac{d\bar{w}}{dR} = \frac{\partial \bar{w}}{\partial R} < 0. \tag{6.5}$$

According to Claim 1.(a), the equilibrium technology is independent of the economy's labor endowment. Hence, there is no direct analogy to Proposition 3. This is due to the absence of diminishing returns, i. e., each performed task adds the same amount to final output. Therefore, the equilibrium innovation incentives are the same for all firms. The equilibrium technology declines in the real interest factor reflecting higher marginal and total investment costs as R increases. Claim 1.(b) is the counterpart to Proposition 1 and confirms the validity of the two welfare theorems for this economy.

Claim 2.(a) states two findings. First, it confirms Proposition 2, that is, the equilibrium wage is equal to the net marginal product of labor at $\bar{q}^a$. Second, unlike the model of Section 2, the equilibrium technology also maximizes the real wage. The reason is again the absence of diminishing returns. Therefore, $V(q^a, L; R)$ is linear in L, the equilibrium wage is equal to net output per worker, and $\bar{q}^a$ maximizes net output and the real wage. Claim 2.(b) reveals that the equilibrium wage necessarily falls in response to an increase in the price of the innovation investment. This reflects the direct price effect on investment outlays. By the envelope theorem, the indirect effect via induced innovation investments, $(\partial \bar{w}/\partial q^a)(\partial \bar{q}^a/\partial R)$, is mute since the equilibrium technology maximizes the real wage.

To establish a closer link between Claim 1.(a) and Claim 2.(b) and a potential role of factor endowments, one may want to think of R as being determined by (the world's) capital stock K with $R_K(K) < 0$. Then, these claims imply that both $\bar{q}^a$ and $\bar{w}$ increase with K. This confirms $d\bar{q}^a/dK > 0$ of Proposition 3 as well as $d\bar{w}/dK > 0$ of Proposition 4. Moreover, the latter comparison leads to the interesting conclusion that there is no weak absolute bias (or weak relative) bias in models where the equilibrium technology maximizes equilibrium factor prices.

6.2 Endogenous Factor Supplies

Factor supplies may respond to changing factor prices. Accordingly, the relative scarcity of employed factors of production becomes endogenous. This section allows for the labor supply to depend on the real wage and for the capital supply to depend on the real rental rate. The question is then how changing factor endowments affect the equilibrium technology and equilibrium factor prices. Conceptually, these two cases differ insofar as the real wage affects the intensive margin of the supply of labor whereas the real rental rate determines the extensive margin of the supply of capital.

6.2.1 Endogenous Labor Supply

In the short run, individuals may want to increase their labor supply in anticipation of a higher wage. Under full employment this behavior reduces, ceteris paribus, the ratio of capital to employed labor. In this sense, labor becomes more abundant. Then, Proposition 3 and Proposition 4 suggest that the productivity of labor and the wage decline in equilibrium.

To address this tension assume that the individual labor supply is a function of the real wage. To be precise, denote $\tau \in [0,1]$ the fraction of an individual's time endowment that she supplies to the labor market and normalize this endowment to unity. Assume further that $\tau = \tau(w)$ where $\tau : \mathbb{R}_{++} \to (0,1)$ with $\tau_w(w) > 0$. From Theorem 1, $w = w(\kappa)$ which results from firms' first-order conditions. Accordingly, $\tau = \tau(w(\kappa))$, and

$$\varepsilon_{\kappa}^{\tau}(\kappa) \equiv \frac{d \ln \tau}{d \ln \kappa} > 0$$

is the elasticity of $\tau(w(\kappa))$ with respect to κ. To find the equilibrium task intensity note that the aggregate labor supply is equal to $L\tau(w(\kappa))$ and, for any strictly positive real wage, the labor market clearing condition (3.1) delivers $N = ((1+g^a(\kappa)) L\tau(w(\kappa))$.

Proposition 10 *There is a unique equilibrium task intensity, $\bar{\kappa} > 0$, that solves*

$$\bar{\kappa} = \frac{1+g^b(\bar{\kappa})}{1+g^a(\bar{\kappa})} \frac{\theta}{\tau(w(\bar{\kappa}))}. \tag{6.6}$$

In addition,

$$\varepsilon_{\theta}^{\kappa} = \frac{1}{1+\varepsilon_{\kappa}^{b}+\varepsilon_{\kappa}^{a}+\varepsilon_{\kappa}^{\tau}} \in (0,1), \tag{6.7}$$

where all terms are evaluated at $\bar{\kappa}$.

Proposition 10 extends equations (3.5) and (3.6) to the case of an endogenous labor supply. It is readily verified that $\tau_w(w) > 0$ is sufficient for the existence of a unique $\bar{\kappa} > 0$ that satisfies (6.6). Equation (6.7) states the elasticity of $\bar{\kappa}$ to changes in θ which is positive. Hence, $\bar{\kappa}$ increases in θ. However, compared to (3.6), the responsiveness is weaker since $\varepsilon_{\kappa}^{\tau} > 0$. This is due to an adjustment in the individual supply of labor and leads to the main result of this section: if individuals increase their labor supply at the intensive margin in response to a higher wage, then the effect of changing θ of the equilibrium task intensity weakens but does not change its direction. Therefore, Proposition 3 and Proposition 4 remain valid.[23]

[23]Proposition 3 and Proposition 4 may also remain valid if one allows for $\tau_w(w) < 0$ for some $w > 0$. This means that $\varepsilon_{\kappa}^{\tau}(\kappa)$ is negative for some κ. As long as a unique $\bar{\kappa}$ exists and $\varepsilon_{\tau}^{\kappa}(\bar{\kappa})$ is not too negative, $\varepsilon_{\theta}^{\kappa}$ of (3.6) exists and remains positive. However, in this case a higher wage reduces the supply of labor at the intensive margin which strengthens the responsiveness of $\bar{\kappa}$ to changes in θ relative to the case of exogenous labor supply.

6.2.2 Endogenous Supply of Capital

Denote the supply of capital by $K = K(R)$ where $K : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $K_R(R) > 0$. Theorem 1 implies $R = R(\kappa)$ with $R_\kappa(\kappa) < 0$. Hence, the equilibrium capital supply satisfies $K = K(R(\kappa))$ and

$$\varepsilon_\kappa^K(\kappa) \equiv \frac{d \ln K}{d \ln \kappa} < 0.$$

The capital market clearing condition (3.1) delivers $M = \left(1 + g^b(\kappa)\right) K(R(\kappa))$. Hence, the equilibrium task intensity solves

$$\kappa = \frac{1 + g^b(\kappa)}{1 + g^a(\kappa)} \frac{K(R(\kappa))}{L}. \tag{6.8}$$

The same line of reasoning as used in Proposition 10 leads to the conclusion that (6.8) pins down a unique $\bar{\kappa} > 0$. Moreover, with $\varepsilon_L^\kappa = d \ln \kappa / d \ln L$ one readily verifies that

$$\varepsilon_L^\kappa = \frac{-1}{1 + \varepsilon_\kappa^b + \varepsilon_\kappa^a - \varepsilon_\kappa^K} < 0. \tag{6.9}$$

As expected, increasing the supply of labor reduces the equilibrium task intensity. Therefore, the qualitative predictions made in Proposition 3 and Proposition 4 concerning changes is the labor endowment remain valid. However, the elastic supply of capital weakens this link. The intuition for this is straightforward. Given K an increase in L reduces the task intensity. This shifts the rental rate of capital upwards, leads to an increased supply of capital, and, therefore, to a greater task intensity. This general equilibrium effect weakens but does not dominate the effect of L on the equilibrium task intensity.[24]

6.3 Exogenous Factor Prices: The Case of a Minimum Wage

What is the role of factor scarcity for technology and factor prices if one factor price is fixed above its equilibrium value determined in Theorem 1? To answer this question I introduce an exogenous minimum wage into the model of Section 2.[25] Accordingly, the

[24] Following similar arguments as in Footnote 23 one readily verifies that Proposition 3 and Proposition 4 may also remain valid if one allows for $K_R(R) < 0$ for some $R > 0$.

[25] Alternatively, one could address this topic in a small open economy (SOE) facing an exogenous rental rate of capital under perfect international capital mobility. If the rental rate paid in the worldwide capital market exceeds $\bar{R}$ of Theorem 1, then there will be strictly positive net capital exports in equilibrium. This is the counterpart to the equilibrium level of unemployment of labor under a binding minimum wage in the closed economy under scrutiny here. Moreover, in the SOE the effect of changing the labor endowment on the equilibrium values of technology, factor prices, and net capital exports mimics the effect of changing the capital endowment as stated in Proposition 11 below.

equilibrium factor market clearing condition for labor stated in (3.1) must be extended. With $w_{min} > 0$ denoting the real minimum wage this condition becomes

$$w \geq w_{min}, \quad \int_0^N l(n)dn \leq L, \quad (w - w_{min}) \left(\int_0^N l(n)dn - L \right) = 0. \qquad (6.10)$$

Hence, the actual wage must not be lower than the minimum wage, and, if it is equal to the minimum wage an excess supply of labor may occur in equilibrium. The equilibrium of the economy is then defined as in Section 3 with (6.10) replacing the respective condition in (3.1).[26]

Clearly, what matters is whether the minimum wage is binding or not. If it is not binding then $w > w_{min}$ and there must be full employment. In other words, the equilibrium is as described in Theorem 1. However, if it is binding then $w = w_{min}$ and the character of the equilibrium changes drastically. Firms adjust their investment behavior, the rental rate of capital falls, and there is unemployment of labor. To see why the remainder of this section assumes that the minimum wage exceeds its equilibrium level under laissez-faire, i. e., $w_{min} > \bar{w}$. Moreover, the subscript *min* is used to denote the equilibrium values associated with w_{min}.

The analysis starts with the cost-minimizing choice of $q^a(n)$ given by (2.8). Now, this equation directly determines q^a_{min}. To make this more precise, let $q^a(w)$ denote the functional relationship between q^a and w defined by (2.8). Then, $q^a : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and, with $c_q(q^a) \equiv 2i_q(q^a) + (1 + q^a)\, i_{qq}(q^a) > 0$, one has $q^a_w(w) = \left[(1 + q^a)\, c_q(q^a)\right]^{-1} > 0$. Hence,

$$q^a_{min} = q^a(w_{min}) > \bar{q}^a = q^a(\bar{w}). \qquad (6.11)$$

Intuitively, competitive firms must raise the productivity of labor to meet the challenge of an excessive real wage.

Upon combining (2.8) and (2.10) for a given wage one finds

$$f(\kappa) - \kappa f_\kappa(\kappa) - (1 + q^a(w))\, i_q(q^a(w)) - i(q^a(w)) = 0 \qquad (6.12)$$

as the equilibrium condition that equates the value product of the marginal task N to the minimized cost of tasks performed by labor. Since w pins down q^a, (6.12) implicitly defines a functional relationship $\kappa(w)$ where $\kappa : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $\kappa_w(w) = -c_q(q^a)\, q^a_w(w) / (\kappa f_{\kappa\kappa}(\kappa)) > 0$. Hence, the equilibrium task intensity under a binding minimum wage satisfies

$$\kappa_{min} = \kappa(w_{min}) > \bar{\kappa} = \kappa(\bar{w}). \qquad (6.13)$$

[26] To simplify the exposition, here $F(M, N)$ satisfies the usual Inada-conditions, i. e., $\lim_{\kappa \to 0} f_\kappa(\kappa) = \infty$ and $\lim_{\kappa \to \infty} f_\kappa(\kappa) = 0$.

Intuitively, the cost minimum of tasks performed by labor is higher under a binding minimum wage. Therefore, the value product of the marginal task N must also increase. Accordingly, the equilibrium task intensity increases.

A higher task intensity reduces the value product of the marginal task M. Therefore, the equilibrium incentive to invest in the capital-augmenting technology will fall. To confirm this intuition formally, combine (2.7) with (2.9), and use $\kappa(w)$ as defined above. This gives the equilibrium condition that equates the value product of the marginal task M to the minimized cost of tasks performed by capital as

$$f_\kappa(\kappa(w)) - \left(1+q^b\right) i_q\left(q^b\right) - i\left(q^b\right) = 0. \tag{6.14}$$

The latter defines a functional relationship $q^b(w)$, where $q : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $q^b_w(w) = f_{\kappa\kappa}(\kappa(w))\,\kappa_w(w)/c_q\left(q^b\right) < 0$ and $c_q\left(q^b\right) \equiv 2i_q\left(q^b\right) + \left(1+q^b\right) i_{qq}\left(q^b\right) > 0$. Hence,

$$q^b_{min} = q^b(w_{min}) < \bar{q}^b = q^b(\bar{w}). \tag{6.15}$$

As to the rental rate of capital, let $R\left(q^b\right)$ denote the functional relationship between R and q^b implied by (2.7). Then, $R : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $R_q\left(q^b\right) = \left[\left(1+q^b\right) c_q\left(q^b\right)\right]^{-1} > 0$. Now, the function $q^b(w)$ defined above provides the link between the rental rate of capital and the wage. Indeed, one has $R\left(q^b(w)\right)$ with $R_w = R_q\left(q^b(w)\right) q^b_w(w) < 0$. Accordingly, the equilibrium rental rate of capital under a binding minimum wage satisfies

$$R_{min} = R\left(q^b_{min}\right) < \bar{R} = R\left(\bar{q}^b\right). \tag{6.16}$$

Intuitively, the rental rate of capital falls below its laissez-faire level so that the cost-minimum is attained at $q^b_{min} < \bar{q}^b$.

Hence, under a binding minimum wage the equilibrium technology as well as the equilibrium rental rate of capital is fully determined by w_{min}. In addition, these variables also depend on the functional forms chosen for f and i but, unlike under laissez-faire, not on factor endowments. However, it is important to see that the capital stock is a determinant of the level of equilibrium employment.

Under a binding minimum wage, the level of employment is equal to the demand for labor, $L^d = N/(1+q^a(w)) < L$. At the same time full employment of capital requires $M/\left(1+q^b(w)\right) = K$. Then, the equilibrium level of employment satisfies $\kappa(w) = M/N$, or

$$\kappa(w) = \frac{(1+q^a(w))\,K}{(1+q^b(w))\,L^d}. \tag{6.17}$$

Since $q^a_w(w) > 0 > q^b_w(w)$, and $\kappa_w(w) > 0$, the latter implicitly defines a function $L^d(w,K)$ where $L^d : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$. A higher (minimum) wage reduces the level of employment, i. e., $\partial L^d(w, K)/\partial w < 0$. Intuitively, this reflects two reinforcing channels. First, the number of tasks performed by labor declines since $N_{min} = M_{min}/\kappa_{min} = \left(1 + q^b_{min}\right) K/\kappa_{min} < \bar{N}$. Second, each of the N_{min} tasks requires less labor as $q^a_{min} > \bar{q}^a$. Hence, induced labor-saving technical change reinforces the employment reducing effect of a binding minimum wage. As a consequence,

$$L_{min} = L(w_{min}, K) < L. \tag{6.18}$$

A higher capital endowment unequivocally increases the level of employment, i. e., $\partial L^d(w, K)/\partial K > 0$. Intuitively, for a given equilibrium technology, a higher K implies a proportionate increase in M. To keep $\kappa(w_{min})$ constant, this requires a proportionate increase in N, hence also in L^d. The following proposition summarizes the results derived in this section.

Proposition 11 *Consider a binding minimum wage in the economy of Section 2, i. e., $w_{min} > \bar{w}$. Then, the equilibrium technology satisfies*

$$q^a_{min} > \bar{q}^a \quad with \quad \frac{dq^a_{min}}{dw_{min}} > 0 \quad and \quad q^b_{min} < \bar{q}^b \quad with \quad \frac{dq^b_{min}}{dw_{min}} < 0.$$

Moreover, the equilibrium real rental rate of capital satisfies

$$R_{min} < \bar{R} \quad with \quad \frac{dR_{min}}{dw_{min}} < 0.$$

Finally, the level of employment satisfies

$$L^d_{min} < L \quad with \quad \frac{\partial L^d_{min}}{\partial w_{min}} < 0, \quad and \quad \frac{\partial L^d_{min}}{\partial K} > 0.$$

Finally, observe that net output under a binding minimum wage is strictly smaller than under laissez-faire. To see this, consider net output of (3.10) evaluated at $q^b(w)$, $q^a(w)$, and $L^d(w, K)$ as defined above. This gives

$$V(w) \equiv V\left(q^b(w), q^a(w), K, L^d(w, K)\right).$$

In light of (6.12) and (6.14), one finds

$$\frac{dV(w)}{dw} = F_2\left(\left(1 + q^b(w)\right) K, \left(1 + q^a(w)\right) L^d(w, K)\right) \left(1 + q^a(w)\right) \frac{\partial L^d(w, K)}{\partial w} < 0,$$

i. e., the adjustments of the equilibrium technology induced by a higher real wage have no first-order effect on net output. Intuitively, one may think of the competitive technology choice as maximizing net output in an economy where factor endowments are given by K and L^d_{min}. Accordingly, a higher minimum wage affects net output only because it reduces employment.

7 Concluding Remarks

Many macroeconomic studies stipulate an aggregate production function with an exogenous factor-augmenting technology of the form $Y = F(bK, aL)$. The present paper shows that this feature obtains as an equilibrium phenomenon in a competitive model of endogenous task-based technical progress. Firms choose the technology to minimize costs per task. Since tasks are factor-specific, this choice hinges on factor prices. Moreover, since the total number of tasks performed by each factor is determined through a full-employment condition, it hinges on factor endowments.

This setup captures some fundamental ideas enunciated by Hicks and later by the literature on "induced innovations", namely, that factor scarcity and factor prices play an essential role for our understanding of the direction of technical change. Here, it is shown that the scarcity of a factor increases its equilibrium productivity. This occurs even though the equilibrium technology is not strongly factor saving in the sense of Acemoglu (2010). Moreover, changing either factor endowment has an intuitive effect on absolute and relative factor prices. For instance, the long-run factor demand schedules remain declining in the respective factor price.

Clearly, there are open and new questions left for future research. They include the robustness of the results for heterogeneous tasks or for more than two factors of production. Moreover, to establish the link between the present setup and the modern literature on Schumpeterian growth initiated by Romer (1990), Segerstrom, Anant, and Dinopoulos (1990), Grossman and Helpman (1991), and Aghion and Howitt (1992), one may want to allow for technology monopolists developing and selling the factor saving technologies to the competitive firm sector considered here. The approach proposed in Acemoglu (2007) and Acemoglu (2010) suggests a way to accomplish this.

8 Appendix

8.1 Proof of Theorem 1

The choice of (M, N) is only consistent with finite factor supplies if (2.9) and (2.10) hold as an equality. Then, combining (2.7), (2.9) with $F_1(M, N) = f_\kappa(\kappa)$, and (2.8), (2.10), with $F_2(M, N) = f(\kappa) - \kappa f_\kappa(\kappa)$ delivers

$$f_\kappa(\kappa) = c\left(q^b\right) \quad \text{and} \quad f(\kappa) - \kappa f_\kappa(\kappa) = c\left(q^a\right), \tag{8.1}$$

respectively. Here, $c\left(q^b\right) \equiv i(q^b) + (1+q^b)i_q(q^b)$ and $c\left(q^a\right) \equiv i(q^a) + (1+q^a)i_q(q^a)$ are the minimized costs per task of the respective types. Using the properties of f and i it is straightforward to show that there are maps $g^b : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $g^a : \mathbb{R}_{++} \to \mathbb{R}_{++}$ such that $q^b = g^b(\kappa) > 0$ with $g^b_\kappa(\kappa) < 0$, and $q^a = g^a(\kappa) > 0$ with $g^a_\kappa(\kappa) > 0$. Using these findings in (2.7) and (2.8) reveals that the factor prices satisfy

$$\begin{aligned} R &= \left(1+g^b(\kappa)\right)^2 i_q\left(g^b(\kappa)\right) \equiv R(\kappa) > 0 \quad \text{with} \quad R_\kappa(\kappa) < 0, \\ w &= \left(1+g^a(\kappa)\right)^2 i_q\left(g^a(\kappa)\right) \equiv w(\kappa) > 0 \quad \text{with} \quad w_\kappa(\kappa) > 0, \end{aligned} \tag{8.2}$$

where $R : \mathbb{R}_{++} \to \mathbb{R}_{++}$ and $w : \mathbb{R}_{++} \to \mathbb{R}_{++}$. Moreover, with $k = 1/b$ and $l = 1/a$ in (3.1), one has $M = bK = \left(1+g^b(\kappa)\right)K$ and $N = aL = \left(1+g^a(\kappa)\right)L$. Hence, the equilibrium task intensity is indeed determined by (3.5). There is a unique $\bar{\kappa} > 0$ that solves the latter equation. To see this, denote its right-hand side by $RHS(\kappa)$, which is a continuous function $RHS : \mathbb{R}_{++} \to \mathbb{R}_{++}$ with $RHS_\kappa(\kappa) < 0$ since $g^a_\kappa(\kappa) > 0 > g^b_\kappa(\kappa)$. Moreover, it satisfies $\lim_{\kappa\to 0} RHS(\kappa) > 0$. Hence, there is a unique $\bar{\kappa} > 0$ that satisfies $\bar{\kappa} = RHS\left(\bar{\kappa}\right)$. Implicit differentiation reveals that $\bar{\kappa} = \kappa(\theta)$ with $\kappa_\theta(\theta) > 0$ and $\theta \equiv K/L$. ■

8.2 Proof of Proposition 1

($\Rightarrow$) By construction, the equilibrium technology $\left(\bar{q}^b, \bar{q}^a\right)$ satisfies the first-order conditions (2.7) - (2.10) as equalities and the full employment conditions (3.1). I show that the solution to $\max_{(q^b,q^a)\in\mathbb{R}_+} V\left(q^b, q^a, K, L\right)$ coincides with (3.2). This establishes the first part of the proposition.

Consider $V\left(q^b, q^a, K, L\right)$ of (3.10) and recall that, by definition, net output at given factor endowments includes the full employment conditions (3.1) for symmetric technology choices, i. e., $M = \left(1+q^b\right)K$ and $N = \left(1+q^a\right)L$. Then, with $\left(\left(1+q^b\right)K, \left(1+q^a\right)L\right)$ being the argument of F, the first-order conditions are

$$V_1\left(q^b, q^a, K, L\right) = K\left[F_1 - c\left(q^b\right)\right] = 0, \quad \text{and} \quad V_2\left(q^b, q^a, K, L\right) = L\left[F_2 - c\left(q^a\right)\right] = 0. \tag{8.3}$$

These conditions deliver a global maximum since

$$\begin{aligned} & V_{11} = K\left[KF_{11} - c_q\left(q^b\right)\right] < 0, \quad V_{22} = L\left[LF_{22} - c_q\left(q^a\right)\right] < 0, \\ \text{and} \quad & V_{11}V_{22} - \left[V_{12}\right]^2 > 0 \quad \text{as} \quad c_q\left(q^b\right)c_q\left(q^a\right) - Kc_q\left(q^a\right)F_{11} - Lc_q\left(q^b\right)F_{22} > 0. \end{aligned} \tag{8.4}$$

As F has constant returns to scale, (8.3) may be written as (8.1). Hence, (8.3) gives rise to the same functions as stated in (3.2).

($\Leftarrow$) Suppose that $\left(q^b, q^a\right) \in \mathbb{R}^2_+$ solves (3.11). Then, it satisfies (8.3), hence $q^b = g^b\left(\kappa\right)$ and $q^a = g^a\left(\kappa\right)$. Since $V\left(q^b, q^a, K, L\right)$ implies the factor market conditions (3.1), the equilibrium task intensity $\bar{\kappa}$ must be given by (3.5). Hence, $q^b = g^b\left(\bar{\kappa}\right) = \bar{q}^b$ and $q^a = g^a\left(\bar{\kappa}\right) = \bar{q}^a$. ■

8.3 Proof of Corollary 1

From Theorem 1 any technology $\left(q^b, q^a\right)$ that qualifies as an equilibrium technology satisfies $q^b = g\left(q^a\right)$ of (3.7). From Proposition 1, the equilibrium technology is a global maximum of $V\left(q^b, q^a, K, L\right)$ on $\mathbb{R}^2_+$. Hence, the equilibrium technology also solves (3.12). ■

8.4 Proof of Proposition 2

The first-order conditions (2.7) and (2.8) deliver $q^b(m) = q^b$ and $q^a(n) = q^a$. Market clearing (3.1) means that $M = (1 + q^b)K$ and $N = (1 + q^a)L$. Using this information in (2.9) and (2.10) reveals that equilibrium factor prices are equal to the net marginal products of (3.14) and (3.15), respectively, where $\left(q^b, q^a\right) = \left(\bar{q}^b, \bar{q}^a\right)$. ■

8.5 Proof of Proposition 3

The equilibrium technology is given by $\bar{q}^b = g^b\left(\kappa(\theta)\right)$ and $\bar{q}^a = g^a\left(\kappa(\theta)\right)$. Therefore, $d\bar{q}^b/dK = g^b_\kappa(\bar{\kappa})\kappa_\theta(\theta)/L < 0$, $d\bar{q}^a/dL = g^a_\kappa(\bar{\kappa})\kappa_\theta(\theta) \times (-\theta/L) < 0$, $d\bar{q}^a/dK = g^a_\kappa(\bar{\kappa})\kappa_\theta(\theta)/L > 0$, and $d\bar{q}^b/dL = g^b_\kappa(\bar{\kappa})\kappa_\theta(\theta) \times (-\theta/L) > 0$. ■

8.6 Proof of Proposition 4

From Theorem 1, $\bar{R} = R(\bar{\kappa})$, $R_\kappa(\kappa) < 0$, and $d\bar{\kappa}/dK = \kappa_\theta(\theta)/L > 0$. Hence, $d\bar{R}/dK = \bar{R}_\kappa(\bar{\kappa})d\bar{\kappa}/dK < 0$. Similarly, using $\bar{w} = w(\bar{\kappa})$, $w_\kappa(\kappa) > 0$, and $d\bar{\kappa}/dL = \kappa_\theta(\theta)\left(-\theta/L\right) < 0$, I find $d\bar{w}/dL = w_\kappa(\bar{\kappa})d\bar{\kappa}/dL < 0$. In the same vein, $d\bar{R}/dL = \bar{R}_\kappa(\bar{\kappa})\kappa_\theta(\theta)\left(-\theta/L\right) > 0$ and $d\bar{w}/dK = w_\kappa(\bar{\kappa})\kappa_\theta(\theta)/L > 0$. ■

8.7 Proof of Proposition 5

One readily verifies that $\partial\bar{R}/\partial q^b = V_{13} = bKF_{11} < 0$, $\partial\bar{R}/\partial q^a = V_{23} = bLF_{12} > 0$, $\partial\bar{w}/\partial q^b = V_{14} > 0$, and $\partial\bar{w}/\partial q^a = V_{24} < 0$, where all derivatives are evaluated at $\left(\bar{q}^b, \bar{q}^a, K, L\right)$. ■

8.8 Proof of Proposition 6

Recall the partial effects derived in the proof of Proposition 5. Then, with (3.2) and (3.4) of Theorem 1 the proposition follows immediately. ■

8.9 Proof of Proposition 7

Use (3.3) to study the total effect of changing K and L on $\bar{R}/\bar{w}$, i. e.,

$$\frac{d\bar{R}/\bar{w}}{dK} = \frac{d(R(\bar{\kappa})/w(\bar{\kappa}))}{\kappa}\kappa_\theta(\theta)/L < 0 \quad \text{and} \quad \frac{d\bar{R}/\bar{w}}{dL} = \frac{d(R(\bar{\kappa})/w(\bar{\kappa}))}{d\kappa}\kappa_\theta(\theta) \times (-\theta/L) > 0. \tag{8.5}$$

The first term is strictly negative since $w_\kappa(\kappa) > 0 > R_\kappa(\kappa)$. Moreover, $\kappa_\theta(\theta) > 0$. ■

8.10 Proof of Proposition 8

To be found in the main text.

8.11 Proof of Proposition 9

Claim 1.(a): The equilibrium technology $\bar{q}^a$ minimizes costs and satisfies the zero-profit condition mentioned in the text. Upon combining these two conditions, i. e., $-w/(1+q^a)^2 + Ri_q(q^a) = 0$ and $1 - w/(1+q^a) - Ri(q^a) = 0$, one finds that the equilibrium technology is determined by $1/R = c(q^a)$, i. e., total minimized unit costs are equal to the present value of the firms' revenue. It is readily verified that the latter defines the function $g(R)$ mentioned in (6.2).

Claim 1.(b): One readily verifies that $V(q^a, L; R)$ is strictly concave in q^a. Then, the desired argument of the maximum is given by $\partial V(q^a, L; R)/\partial q^a = 1 - Rc(q^a) = 0$. From the proof of Claim 1.(a), this is the defining condition for $g(R)$.

Claim 2.(a): Consider the zero-profit condition at $q^a(n) = q^a$. Solving for the real wage gives $w = (1+q^a)[1 - Ri(q^a)] = \partial V(\bar{q}^a, L; R)/\partial L$. Since $V(\bar{q}^a, L; R)$ is linear in L, $\bar{q}^a$ also maximizes w.

Claim 2.(b): The equilibrium wage satisfies $\bar{w} = (1 + g(R))[1 - Ri(g(R))]$. This immediately delivers $d\bar{w}/dR = \partial\bar{w}/\partial R + (\partial\bar{w}/\partial q^a)(\partial\bar{q}^a/\partial R)$, where $\partial\bar{q}^a/\partial R = g(R) < 0$. Since $\bar{q}^a$ maximizes the wage $\partial\bar{w}/\partial\bar{q}^a = 0$ which establishes (6.5). Since $V(\bar{q}^a, L; R)$ is linear in L, $\bar{w}$ does not depend on it. ■

8.12 Proof of Proposition 10

Noting that $\tau_w(w) > 0$ the existence and uniqueness of $\bar{\kappa} > 0$ follows from the same argument as set out in the proof of Theorem 1. Total differentiation of (6.6) delivers (6.7). ■

8.13 Proof of Proposition 11

Given in the main text of Section 6.3.

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