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Hubert Kempf

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Designing the Policy Mix in a Monetary Union

Abstract

This paper studies the design of the policy mix in a monetary union, that is, the institutional arrangement specifying the relationships between the various policymakers present in the union and the extent of their capacity of action. It is assumed that policymakers do not cooperate. Detailing several institutional variants imposed on an otherwise standard macromodel of a monetary union, we prove that there is no Pareto-superior design when cooperation between policymakers is impossible.

JEL-Codes: E580, E620, F450, H760.

Keywords: monetary union, fiscal policy, monetary policy, cooperation, policy mix.

Hubert Kempf
Université Paris-Saclay
École Normale Supérieure
Paris – Saclay / France
hubert.kempf@ens-cachan.fr

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1 Introduction.

Designing the institutions of a multi-national monetary union for stabilization purposes and setting up the policy mix is rife with difficulties[1]. Three main difficulties, specific to a monetary union, must be addressed. First, stabilization objectives are twofold: stabilizing the union's economy and stabilizing the country-specific economies. Second, several of policymakers (more than 2), each with its own macroeconomic preferences, coexist. Third, the (national) economic structures which constitute the union are asymmetrical. In addition, there exists the standard difficulty of the policy mix, that is, no-cooperation or incomplete cooperation between policymakers which also exists in a unified economy.

These complex issues explain why the debates on how to organize a monetary union are rife and intense.[2] Of course the best known example is the European monetary union and its decade-long search for the right 'engine'. Even before its creation, the debates on the statutes of the European system of central banks and the European central bank have been lasting and sometimes acrimonous between national goverments. Since its inception, this union has experimented several institutional settings for framing the fiscal capacities of national Treasuries, modifiying the provisions enclosed in the Maastricht Treaty over time.[3] But the same institutional fragility can be found in any monetary union, as no institutional feature encounters unanimous and permanent consent from governements and political parties.

Therefore the issue is whether there is a design of the relations and responsibilities of the various policymakers, framing the policy mix decisions, which can be seen as Pareto-superior and attracts a large consensus. A striking 'symbiotic' result has been reached by Dixit and Lambertini ([4]): Using a linear-quadratic framework, they show that, when all monetary and fiscal authorities agree on the desirable 'optimal' inflation and output levels, the Nash outcome of a simultaneous non-cooperation game among policymakers (that is, the Nash outcome coincides with the bliss point).[4]

This paper tackles again the design of the policy mix of a monetary union subject to various shocks. It claims that no ranking of settings can be made such that one of them appears to be Pareto-superior in any circumstances. It must be concluded that the design of a monetary union cannot be made according to dogmas or abstract and general principles but must be made according to the peculiarities of the monetary union and its included entities. But even if these specificities are taken into account (implying that they are well informed and known), it may not be possible to define a Pareto-superior design. Disagreements are likely to be inherent to monetary unions.

[1] For a recent survey on the topic, see Foresti ([5]).

[2] Early contributions to the issue are Beetsma and Uhlig ([1]), Beetsma and Bovenberg ([2]), and Uhlig ([10])

[3] On the Growth and Stability Pact, see Buti [3]. On the modification of its provision, refer to https://ec.europa.eu/info/business-economy-euro/economic-and-fiscal-policy-coordination/eu-economic-governance-monitoring-prevention-correction/stability-and-growth-pact/history-stability-and-growth-pact_en

[4] The result obtained by Dixit and Lambertini is even richer: they prove that the outcome of any non-cooperation game is the bliss point. Kempf and Von Thadden ([8, 9]) generalize this result to coalition-based games and prove that the twin assumption of a linear-quadratic framework and unanimous agreement on the desirable configurations is critical to obtain the symbiotic result.

2 A simple model of a monetary union.

In this section we develop a standard reduced form macromodel of a monetary union, when there is a union central bank and country-specific fiscal authorities. Each policymaker is characterized by a loss function which formalizes its preferences as far as macroeconomic stabilization is concerned.[5] An essential feature of the model is its stochastic nature: shocks, both global and idiosyncratic, hit the union's economy. The issue is therefore how and whether there is a best design so as to cope with these shocks.

2.1 The model.

We consider a two-country symmetrical monetary union, hit by a global nominal shock and idiosyncratic country-specific real shocks. In each country there exists a fiscal authority fully and independently controlling a fiscal instrument, denoted by g_i, in reference to its objectives. The union's central bank which is fully independent from any national fiscal authority controls a monetary policy instrument, denoted by π_M.

The country i aggregate product, denoted by y_i, is given by the following equation:

$$y_i = \hat{y} + a g_i + (1-a) g_{-i} + b\left(\pi - \pi^e\right) + u_i \qquad \forall i = 1, 2 \tag{1}$$

where $\hat{y}$ denotes the natural aggregate product in country i, π the inflation rate of the union, π^e the union inflation rate expected by private agents, g_i the fiscal instrument in country i, g_{-i} the fiscal instrument in the other country. u_i is the real shock hitting country i ; it is a white noise i.i.d., of expectation zero and variance σ_u^2. The distribution of real shocks is the same in both countries but the realizations may differ. There exists a cross-border spillover effect: The country i aggregate product depends on both fiscal variables. The magnitude of this effect is stronger when coefficient a is weaker. We assume $a > 1/2$. Finally, the natural aggregate product is the same in both countries. The domestic multiplier of a local impulsion is therefore equal to a and the cross-border multiplier to $(1-a)$.

Inflation is identical in both countries. It is given by the following equation:

$$\pi = c \sum_{j=1}^{2} g_j + \pi_M + \varepsilon \tag{2}$$

where π_M represents the instrument controlled by the central bank of the union and ε the monetary shock affecting uniformly inflation ; ε is a white noise i.i.d., of expectation zero and variance σ_ε^2. Inflation depends on the fiscal instruments and the central bank may be willing to counter this impact through its manipulation of its own monetary instrument π_M.

The various policymakers are characterized by loss functions which formalize their preferences. The loss function of the fiscal policymaker in country i is given by the following formula:

$$L_i = \frac{1}{2}\theta\left(y_i - \tilde{\chi}\right)^2 + \frac{1}{2}\pi^2 \tag{3}$$

[5] The model is a version of the canonical model frequently used to study policy issues, in particular in the context of a monetary union. See for exemple Gros and Hefeker ([6]).

where $\tilde{\chi}$ is the level of aggregate product desired by both fiscal authorities. We assume that they have the same inflation objective (0). θ is the weight given to the product objective relative to the nominal objective. This weight is the same in both countries. Therefore the 'fiscal' loss functions are identical for both countries.

The loss function of the monetary policymaker (the central bank) is given by:

$$L_M = \frac{1}{2}\theta_M \left(\overline{y} - \tilde{\chi}_M\right)^2 + \frac{1}{2}\pi^2 \tag{4}$$

where $\tilde{\chi}_M$ is the level of aggregate product desired by the monetary policymaker and $\overline{y}$ is the mean aggregate product of the union. This product is defined by:

$$\overline{y} = \hat{y} + \frac{1}{2}\left(g_1 + g_2\right) + b\left(\pi - \pi^e\right) + \bar{u} \tag{5}$$

where $\hat{y}$ denotes the mean natural aggregate product in the union, equal to the country natural aggregate product. θ_M is the weight given the central bank to the product objective relative to the nominal objective. The weaker it is, the more concerned about inflation is the central bank.

The loss functions of the fiscal authorities differ from the one of the central bank. The differences are due to differing product objectives and weight coefficients: $\tilde{\chi}_M \neq \tilde{\chi}$ and $\theta_M \neq \theta$. *In fine* the parameter vector is $\{a, b, c, \hat{y}, \sigma_u^2, \sigma_\varepsilon^2, \tilde{\chi}_M, \tilde{\chi}, \theta_M, \theta\}$. In the following computations we retain a unique difference, $\tilde{\chi}_M \neq \tilde{\chi}$.[6]

2.2 Institutional variants.

The relationships between the central bank and the fiscal policymakers rely on institutional rules designing the way authorities act with respect to one another. In particular, a rule may constrain national fiscal Treasuries to balance their budget. Another rule may specify how fiscal authorities cooperate or the type of hierarchical dependance relationships between policymakers. In this paper, we focus on rules constraining fiscal or monetary actions. It is interesting to study a constraint limiting the discretionary capacity of fiscal authorities given the numerous discussions on the opportunity to limit the extent of public deficits in monetary unions, or even to impose a balanced budget rule whatever the circumstances. Here we retain the following constraint on public budgets: $g_j = 0$. Conversely, it is often suggested that a central bank should not adopt a discretionary policy and react to shocks. In the present setting, this amounts to impose $\pi_M = 0$

The model is static: there exist no lagged terms in this economy. The endogenous variables of the model are functions of the realizations of shocks and the objectives of policymakers. The timing of the sequence of decisions is the following:

1. Private agents form their inflation expectations.

2. A state of nature is realized: the values of shocks are known.

[6] As we shall see, this is enough to make the computations complex.

3. The various policymakers make their decisions in accordance to the set of restrictions imposed on them: the values taken by the instrument variables are set.

4. Aggregate products and inflation are obtained. Losses are computed for the various policymakers.

Decisions made at stage 3 depend on the relationship between the various policymakers and on the constraints importance of rules for the policy mix in a monetary union. We distinguish three options. In all cases, we assume that policymakers do not cooperate.

1. The first option corresponds to *constrained fiscal policies/unconstrained monetary policy* (Option *1*). The fiscal Treasuries are unable to have a discretionay fiscal policy: $g_j = 0, \ \forall i = 1, 2$. We can interpret this constraint as the imposition of a strict balanced budget constraint. In such an option, the sole policy decision is made by the central bank which takes the responsibility of stabilization issues based solely on its preferences.

2. The second option corresponds to *a balanced policy mix* (Option *2*). In this option, there is neither quantitative constraint nor hierarchical relationship. Policymakers make their decisions in stage 3 simultaneously and non-cooperatively.

3. The third option corresponds to *unconstrained fiscal policies/constrained monetary policy* (Option *3*). In this option we assume a totally inactive central bank. Formally, we assume $\pi_m = 0$. Inflation results entirely on fiscal decisions (taken non-cooperatively). The option is the reverse case of Option *1*.

These options reflect recurring controversies on the capacity of the central bank and the balanced budget constraint, in particular in the context of the eurozone.

2.3 Expected losses.

Combining the various equations of the model, the following expressions for the losses of the various policymakers obtain:

$$L_i = \frac{1}{2}\theta\left(\left(a+bc\right)g_i + \left(1+bc-a\right)g_{-i} + b\left(\pi_M + \varepsilon - \pi^e\right) + u_i - \chi\right)^2 + \frac{1}{2}\left(c\left(g_1+g_2\right) + \pi_M + \varepsilon\right)^2 \tag{6}$$

$$L_M = \frac{1}{2}\theta\left(\left(\frac{1}{2}+bc\right)\left(g_1+g_2\right) + b\left(\pi_M + \varepsilon - \pi^e\right) + \bar{u} - \chi_M\right)^2 + \frac{1}{2}\left(c\left(g_1+g_2\right) + \pi_M + \varepsilon\right)^2 \tag{7}$$

with $\chi_M = \tilde{\chi}_M - \hat{y}$, $\chi = \tilde{\chi} - \hat{y}$. $\bar{u}$ is the mean real shock. We normalize the problem assuming $\chi_M = 0$. We shall (loosely) interpret χ as the difference between product objectives of the various authorities. To compare the various options, we use as indicators the *expected* losses of the policymakers $E\left(L_i\right)$ and $E\left(L_M\right)$ obtained for the different options.

2.3.1 The case of constrained fiscal policies/unconstrained monetary policy (Option 1).

In the case when national fiscal authorities are unable to manipulate their instrument ($g_i = 0, \forall i$), the losses are immediately given by:

$$L_i = \frac{1}{2}\left[\theta\left(\chi + b\left(\pi_M + \varepsilon\right) + u_i\right)^2 + \left(\pi_M + \varepsilon\right)^2\right] \tag{8}$$

$$L_M = \frac{1}{2}\left[\theta\left(\chi_M + b\left(\pi_M + \varepsilon\right) + \bar{u}\right)^2 + \left(\pi_M + \varepsilon\right)^2\right] \tag{9}$$

The sole policy decision is taken by the central bank which chooses π^M. As the objective of fiscal authorities plays no role in this variant and, given the properties of shocks, the inflation anticipations of private agents are obviously:

$$\pi^e = 0 \tag{10}$$

The optimal solution, denoted by π_M^{1*}, is given by the following equation (see Appendix)[7]:

$$\pi_M^{1*} = -\frac{\theta b}{(1+\theta b^2)}\bar{u} - \varepsilon \tag{11}$$

The central bank's decision depends negatively on nominal and real shocks. Insofar as it is the sole acting authority using a single instrument, it is immediate that it aims at compensating their real and inflationary impact, given that its real and nominal objectives are nul. The central bank reacts one-to-one to the nominal shock. In addition, it augments its counteracting action so as to fight the impact of the mean real shock. Thanks to its instrument, it is able to nullify the impact of the nominal shock but it lacks supplementary instruments so as to also neutralize the real shock.

The expected loss of the monetary authority in this configuration, which we denote by $E\left(L_M^{1*}\right)$, is given by:

$$E\left(L_M^{1*}\right) = \frac{1}{2}\left[\theta\left(\frac{1}{(1+\theta b^2)}\right)^2 + \left(\frac{\theta b}{(1+\theta b^2)}\right)^2\right]\sigma_u^2 = \frac{\theta}{2\left(1+\theta b^2\right)}\sigma_u^2 \tag{12}$$

The expected loss of a fiscal authority, denoted by π_M^{1*}, is given by:

$$E\left(L_i^{1*}\right) = \frac{1}{2}\left(\frac{(\theta b)^2}{1+\theta b^2} + \theta\right)\sigma_u^2 + \frac{\theta}{2}\chi^2 \tag{13}$$

Their expected losses depend on their product objective χ since fiscal authorities are inactive and they are unable to reach it. Conversely, the expected loss of the central bank does not depend on it: the central bank solely cares about its own objective (0). The higher the variance of the real shock, the higher the expected losses of the fiscal and monetary policymakers. Fiscal authorities are more affected by it than the central bank.

[7] The solving is based on the undetermined coefficients method applied to games with uncertainty. It has already been used by Kempf ([7]).

2.3.2 Balanced policy mix (Option 2).

For this option, the losses are equal to:

$$L_i = \frac{1}{2}\theta\left(\left(a+bc\right)g_i + \left(1+bc-a\right)g_{-i} + b\left(\pi_M+\varepsilon\right) + u_i - \chi\right)^2 + \frac{1}{2}\left(c\left(g_1+g_2\right)+\pi_M+\varepsilon\right)^2 \tag{14}$$

$$L_M = \frac{1}{2}\theta\left(\left(\frac{1}{2}+bc\right)\left(g_1+g_2\right) + b\left(\pi_M+\varepsilon\right) + \bar{u}\right)^2 + \frac{1}{2}\left(c\left(g_1+g_2\right)+\pi_M+\varepsilon\right)^2 \tag{15}$$

Solving the optimization programmes of the various policymakers leads to the following decision rules:

$$g_1 = \frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi - \frac{a}{2a-1}u_1 + \frac{1-a}{2a-1}u_2 \tag{16}$$

$$g_2 = \frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi + \frac{1-a}{2a-1}u_1 - \frac{a}{2a-1}u_2 \tag{17}$$

$$\pi_M = -\frac{\left(2c+\theta b\right)\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi + c\left(u_1+u_2\right) - \varepsilon \tag{18}$$

The nominal shock is entirely nullified by the central bank's decision and thus does not affect the decision of the fiscal authorities. These authorities concentrate on the real shocks and leave the central bank tackling the nominal one. The central bank's decision is affected by the fiscal authorities's positive product objective, higher than its own product objective: in a non-cooperative logic, the central bank attempts to compensate by means of a restrictive action the temptation of fiscal authorities to stimulate the economy so as to reach their product objective. Finally the central bank reacts positively to real shocks and not negatively as in the previously studied option. This is due to the fact that fiscal authorities respond too strongly in countering the impact of real shocks: acting non-cooperatively, they do not take into account the cross-border fiscal spillovers of their own action. The central bank which aims at stabilizing the entire union, is induced to correct this downward bias by reacting positively to the mean real shock.

It is then possible to compute the expected losses of the various policymakers. For the fiscal authorities this expected loss is equal to:

$$E\left(L_{3i}^*\right) = \frac{1}{2}\frac{\theta\left(\theta b - 2\left(1-c\right)\left(b\theta\left(a+bc\right)-c\right)\right)^2 + \left(\theta^2 b\left(a+bc\right)\right)^2}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]^2}\chi^2 + \frac{1}{2}\theta\left[\frac{2abc}{2a-1}\right]^2\sigma_u^2 \tag{19}$$

Similarly, we get for the central bank:

$$E\left(L_{3M}^*\right) = \frac{1}{2}\chi^2\frac{\left[c\theta\left(a+bc\right)\left(1+bc\right)\right]^2 + \left[\theta^2 b\left(a+bc\right)\right]^2}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]^2} \tag{20}$$

The expected loss of the central bank is solely affected by the difference in product objectives. Not only its action has neutralized the nominal shock but in addition the conjunction of actions reacting to real shocks implies that their mean has no impact on the union's product level and thus on inflation. However, the impact of the difference is product objectives is due to the interaction between policy actions: as the central bank reacts to the actions of fiscal authorities which depend on their positive product objective, it is induced to react to this objective.

The expected loss of a fiscal authority is not affected by the variance of the nominal shock thanks to the action of the central bank. However, it is affected by the variance of the real shock, contrarily to the central bank. This is due to the cross-border spillover effects of an idiosyncratic shock: the no-cooperation configuration does not allow to optimally take them into account. Similarly, again because of the absence of cooperation between policymakers, they cannot reach their own product objective since fiscal authorities react to the action of the central bank which has a different product objective. Thus their expected losses depend on this latter objective.

2.3.3 The case of unconstrained fiscal policies/constrained monetary policy (Option 3).

This option is characterized by $\pi_M = 0$. The losses are equal to:

$$L_i = \frac{1}{2}\left[\theta_i\left(\tilde{\chi}_i + \sum_{j=1}^{n} a_{ij}g_j + b_i\left(c\sum_{j=1}^{n} g_j + \varepsilon\right) + u_i\right)^2 + \left(c\sum_{j=1}^{n} g_j + \varepsilon\right)^2\right] \tag{21}$$

$$L_M = \frac{1}{2}\left[\theta_M\left(\tilde{\chi}_M + \frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}g_j + \bar{b}\left(c\sum_{j=1}^{n} g_j + \varepsilon\right) + \bar{u}\right)^2 + \left(c\sum_{j=1}^{n} g_j + \varepsilon\right)^2\right] \tag{22}$$

The fiscal authorities's decisions are given by the following equations:

$$\begin{aligned} g_1 = &\frac{\theta(a+bc)}{\theta(a+bc)+2c^2}\chi - \frac{\theta(a+bc)^2+c^2}{\left[\theta(a+bc)(1+2bc)+2c^2\right](2a-1)}u_1 \\ &+\frac{\theta(a+bc)(1+bc-a)+c^2}{\left[\theta(a+bc)(1+2bc)+2c^2\right](2a-1)}u_2 - \frac{(\theta(a+bc)b+c)}{\theta(a+bc)(1+2bc)+2c^2}\varepsilon \end{aligned} \tag{23}$$

$$\begin{aligned} g_2 = &\frac{\theta(a+bc)}{\theta(a+bc)+2c^2}\chi - \frac{\theta(a+bc)^2+c^2}{\left[\theta(a+bc)(1+2bc)+2c^2\right](2a-1)}u_2 \\ &+\frac{\theta(a+bc)(1+bc-a)+c^2}{\left[\theta(a+bc)(1+2bc)+2c^2\right](2a-1)}u_1 - \frac{(\theta(a+bc)b+c)}{\theta(a+bc)(1+2bc)+2c^2}\varepsilon \end{aligned} \tag{24}$$

As the central bank does not intervene, the fiscal authorities cannot rely on its action, useful against the nominal shock. They have to react to this shock. The corresponding expected loss for the fiscal authorities is given by:

$$E\left(L_{5i}^{*}\right) = c^2\left(\frac{2\theta c^2 + (\theta(a+bc))^2}{(\theta(a+bc)+2c^2)^2}\right)\chi^2 + \frac{c^2+\theta(a+bc)^2}{(\theta(a+bc)(1+2bc)+2c^2)^2}\left(2c^2\sigma_u^2+\sigma_\varepsilon^2\right) \tag{25}$$

and for the central bank:

$$E\left(L_{5M}^{*}\right)=\frac{1}{2}\frac{\theta^{2}\left(a+bc\right)^{2}\left(\left(1+2bc\right)^{2}+4c^{2}\right)}{\left[\theta\left(a+bc\right)+2c^{2}\right]^{2}}\chi^{2}$$

$$+\frac{\left(\frac{1+2bc}{2}\right)^{2}\left[\theta\left(a+bc\right)\left(2bc-1\right)+2c^{2}\right]^{2}+\left(2c\theta\left(a+bc\right)\right)^{2}}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}\right]^{2}}\sigma_{u}^{2}$$

$$\frac{\left(\frac{1+2bc}{2}\right)^{2}\left[b\theta\left(2bc-1\right)+2c\left(c-1\right)\right]^{2}+\theta^{2}\left(a+bc\right)^{2}}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}\right]^{2}}\sigma_{\varepsilon}^{2} \tag{26}$$

The central bank, constrained to inaction, experiences an expected loss linked to the product objective of the fiscal authorities. This is is the consequence of the fact that the decisions of the latter authorities, depending on its objective, affect the effective product. The central bank's expected loss is also affected by the variances of all shocks. In particular by the variance of the monetary shock against which it is unable to act.

The expected loss of a fiscal authority also depends on the product objective which is not reached and on the variance of shocks because the cross-border spillovers of country-specific fiscal actions are not properly managed. More precisely, as the fiscal authorities control two instruments when faced by three shocks, this contributes to the inability to perfectly manage the union's economies. In particular, the inexistence of an active monetary policy exposes the fiscal authorities to be impacted by the nominal shock as in option 2, as proved above.

2.4 Comparing variants.

The following table summarizes the obtained results for the three options we studied:

Expected losses		
	Fiscal authority i	*Central bank*
Option 1	$\frac{\theta}{2}\chi^{2}+\frac{1}{2}\left(\frac{(\theta b)^{2}}{1+\theta b^{2}}+\theta\right)\sigma_{u}^{2}$	$\frac{1}{2}\frac{\theta}{1+\theta b^{2}}\sigma_{u}^{2}$
Option 2	$\frac{1}{2}\frac{\theta(\theta b-2(1-c)(b\theta(a+bc)-c))^{2}+\left(\theta^{2}b(a+bc)\right)^{2}}{[\theta(a+bc)(1+2b(1-c))+2c(1-c)-\theta b]^{2}}\chi^{2}$ $+\frac{1}{2}\theta\left[\frac{2abc}{2a-1}\right]^{2}\sigma_{u}^{2}$	$\frac{[c\theta(a+bc)(1+bc)]^{2}+\left[\theta^{2}b(a+bc)\right]^{2}}{2[\theta(a+bc)(1+2b(1-c))+2c(1-c)-\theta b]^{2}}\chi^{2}$
Option 3	$c^{2}\left(\frac{2\theta c^{2}+(\theta(a+bc))^{2}}{(\theta(a+bc)+2c^{2})^{2}}\right)\chi^{2}$ $+\frac{c^{2}+\theta(a+bc)^{2}}{(\theta(a+bc)(1+2bc)+2c^{2})^{2}}\left(2c^{2}\sigma_{u}^{2}+\sigma_{\varepsilon}^{2}\right)$	$\frac{\theta^{2}(a+bc)^{2}\left((1+2bc)^{2}+4c^{2}\right)}{2[\theta(a+bc)+2c^{2}]^{2}}\chi^{2}$ $+\frac{\left(\frac{1+2bc}{2}\right)^{2}\left[\theta(a+bc)(2bc-1)+2c^{2}\right]^{2}+(2c\theta(a+bc))^{2}}{[\theta(a+bc)(1+2bc)+2c^{2}]^{2}}\sigma_{u}^{2}$ $+\frac{\left(\frac{1+2bc}{2}\right)^{2}[b\theta(2bc-1)+2c(c-1)]^{2}+\theta^{2}(a+bc)^{2}}{[\theta(a+bc)(1+2bc)+2c^{2}]^{2}}\sigma_{\varepsilon}^{2}$

Given this table, we can make several methodological remarks.

1. No policy mix option appears systematically preferable, that is, generating lower expected losses for all policymakers than other option, for any vector of parameters. It is even difficult to spot for which subset of the parameter space one option happens to be Pareto-superior to the other ones.

2. It is possible to show that option 1, where fiscal authorities are very strongly constrained and systematically inactive, is Pareto-superior to the other two for some parameter configurations. But, inversely, we can find configurations for which the Pareto-superior option is option 2 or option 3.

3. In general the interests of the various authorities are opposed. The preferred option for the central bank is not preferred by the country fiscal authorities. If the various assumptions made so as to simplify computations are relaxed, the formulas given above get more complex and the reasons for conflicts are increased.

4. The nature of shocks and their distribution laws play a crucial role in the computation of expected losses. Here the sole dissymmetry between the two countries comes from the fact that country shocks are idiosyncratic even though they are generated by the same distribution law. If this assumption is relaxed, the conflicts between authorities are likely to be increased.

5. More generally, we have assumed symmetric countries and identical objectives for the national fiscal authorities. If we introduce structural asymmetries and differing macroeconomic preferences, the following conclusion is reinforced:

   Except under very specific conditions, there is no institutional design defining the relations between the fiscal authorities and the central bank which is Pareto-superior to other designs.[8]

   In other terms, one cannot expect that a institutional design of the links between the various policymakers is supported by a general consensus.

6. We retained an assumption of no-cooperation between the various policymakers. The results which have been obtained reflect this absence of cooperation. Given the cross-border spillover effects existing in a monetary union, each authority takes as given the decisions of other authorities when defining its reaction function, that is, the formula by which its decision is made (for example in the case of option 2, equations (33) and (34), given in the appendix) without internalizing the consequences of its decision on the other policymakers's decisions. The three options we retained are constrained by this no-cooperation feature. It is this absence of cooperation which explains why, except for some exceptional cases, it is impossible to reach a decision unanimously preferred.

What singularizes the policy mix in a monetary union is the presence of a large number of policymakers (typically $N+1$) which must manage a heterogenenous situation due to different objectives of the policymakers, differing structures and shocks and in addition, cross-border spillover effects with differing impacts. This complexity is accrued by the configuration of powers given to the different fiscal authorities and the mandate of the central bank. In the present paper, these constraints are captured by various limits on the use of instruments and decision sequences which are imposed on the relationships between policymakers. These constraints affect the outcome of the policy mix in complex ways. This explains why it is not possible to define a ranking of these designs and recommend in general one monetary union

[8] The conditions leading to the symbiotic result are such exceptions.

design. The evaluation of the 'good' design of the policy mix in a monetary union must be made according to circumstances and structural features.

3 Conclusion.

The main result of this paper is that there is no design of the interactions between non-cooperating policymakers acting in a monetary union which is Pareto-dominant in any circumstances (ruling out the conditions leading to the symbiotic result). This result has been obtained using a canonical linear-quadratic stochastic model of a monetary union with various fiscal policymakers and a unique monetary authority. Even for a given configuration of a monetary union (structural characteristics, loss functions of the policymakers, nature of shocks being given), it is in general impossible to find a design which is Pareto-superior and ensures that the policy mix reaches an outcome unanimously preferred by all policymakers. This is due to the presence of cross-border spillover effects within the monetary union which affect differently the various coexisting policymakers involved in the policy mix. The economic complexity of a monetary union makes almost impossible to define an institutional framework organizing the relationships between policymakers and the constraints imposed on them desirable in any circumstance, and not even for a given set of circumstances.

Other options of institutional designs of a monetary union could be envisioned. The issue of leadership in the policy mix is particularly important as it may be related to the issue of the fiscal theory of the price level and the distinction between passive and active policies. Another issue is the merits of a federal fiscal authority for the policy mix: it is unlikely that such an authority would represent an improvement in the stabilization of a monetary union in any circumstances but applying the method used in the present article should allow us to highlight under which circumstances (i.e. for which type of spillovers) the institutional design involving such a policymaker would be Pareto-superior.[9] This is left for further research.

In other words, a monetary union is rife with conflicts of interest due to spillover effects and asymmetries of various origins which cannot be dealt with through institutional fixing of non-cooperating policymakers. The immediate consequence is that a bargaining process between authorities and focusing on solving these conflicts is essential for the viability of a monetary union.

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Appendix.

The case of unconstrained fiscal policies/constrained monetary policy.

The optimization programme of the monetary authority (unique policymaker) in option *1* is:

$$min_{\pi_M} L_M = \frac{1}{2}\left[\theta\left(b\left(\pi_M+\varepsilon\right)+\bar{u}\right)^2+\left(\pi_M+\varepsilon\right)^2\right]. \tag{27}$$

The first-order condition is:

$$\frac{\partial L_M}{\partial \pi_M} = \theta b\left(b\left(\pi_M+\varepsilon\right)+\bar{u}\right)+\left(\pi_M+\varepsilon\right)=0$$

which implies the following expression for the optimal solution π_M^{1*}:

$$\pi_M^{1*} = -\frac{\theta b}{(1+\theta b^2)}\bar{u}-\varepsilon. \tag{28}$$

Using this expression in (27), combined with the constraint $g_i = 0, \forall i$, the loss for the monetary authority is equal to:

$$L_M^1 = \frac{1}{2}\left[\theta\left(\frac{1}{1+\theta b^2}\bar{u}\right)^2+\left(\frac{\theta b}{(1+\theta b^2)}\bar{u}\right)^2\right]$$

and its expected loss $E\left(L_M^{1*}\right)$:

$$E\left(L_M^{1*}\right)=\frac{1}{2}\left[\theta\left(\frac{1}{(1+\theta b^2)}\right)^2\sigma_u^2+\left(\frac{\theta b}{(1+\theta b^2)}\right)^2\sigma_u^2\right]=\frac{\theta}{2\left(1+\theta b^2\right)}\sigma_u^2. \tag{29}$$

Let us now compute the loss of a fiscal authority for this option:

$$L_i^1=\frac{1}{2}\left[\theta\left(b\left(\pi_M^1+\varepsilon\right)+u_i-\chi\right)^2+\left(\pi_M^1+\varepsilon\right)^2\right].$$

Given the expression for π_M^{1*}, we get :

$$L_i^1=\frac{1}{2}\left[\theta\left(b\left(-\frac{\theta b}{(1+\theta b^2)}\bar{u}\right)+u_i-\chi\right)^2+\left(-\frac{\theta b}{(1+\theta b^2)}\bar{u}\right)^2\right].$$

Thus the expected loss is:

$$E\left(L_i^{1*}\right)=\frac{1}{2}\left(\frac{(\theta b)^2}{1+\theta b^2}+\theta\right)\sigma_u^2+\frac{\theta}{2}\chi^2. \tag{30}$$

Option 2. The case of a balanced policy mix.

In option 2, the loss functions are:

$$L_i=\frac{1}{2}\theta\left(\left(a+bc\right)g_i+\left(1+bc-a\right)g_{-i}+b\left(\pi_M+\varepsilon-\pi^e\right)+u_i-\chi\right)^2+\frac{1}{2}\left(c\left(g_1+g_2\right)+\pi_M+\varepsilon\right)^2 \tag{31}$$

$$L_M=\frac{1}{2}\theta\left(\left(\frac{1}{2}+bc\right)\left(g_1+g_2\right)+b\left(\pi_M+\varepsilon-\pi^e\right)+\bar{u}\right)^2+\frac{1}{2}\left(c\left(g_1+g_2\right)+\pi_M+\varepsilon\right)^2. \tag{32}$$

The programme of fiscal authority i is:

$$max_{g_i}L_i=\frac{1}{2}\theta\left(\left(a+bc\right)g_i+\left(1+bc-a\right)g_{-i}+b\left(\pi_M+\varepsilon-\pi^e\right)+u_i-\chi\right)^2+\frac{1}{2}\left(c\left(g_1+g_2\right)+\pi_M+\varepsilon\right)^2.$$

Thus the first-order condition writes:

$$\frac{\partial L_i}{\partial g_i}=\theta\left(\left(a+bc\right)g_i+\left(1+bc-a\right)g_{-i}+b\left(\pi_M+\varepsilon-\pi^e\right)+u_i-\chi\right)\left(a+bc\right)+c\left(c\left(g_1+g_2\right)+\pi_M+\varepsilon\right)=0$$

from which the reaction function of this authority derives:

$$g_i=-\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)}{\left(\theta\left(a+bc\right)+c^2\right)}g_{-i}-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\pi_M-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\varepsilon$$
$$-\frac{\theta\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}u_i+\frac{\theta\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\chi+\frac{\theta b\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\pi^e. \tag{33}$$

The programme of the monetary authority is:

$$max_{\pi_M} L_M = \frac{1}{2}\theta\left(\left(\frac{1}{2}+bc\right)(g_1+g_2)+b\left(\pi_M+\varepsilon-\pi^e\right)+\bar{u}\right)^2+\frac{1}{2}\left(c\left(g_1+g_2\right)+\pi_M+\varepsilon\right)^2.$$

The first-order condition writes:

$$\frac{\partial L_i}{\partial \pi_M} = \theta b\left(\left(\frac{1}{2}+bc\right)(g_1+g_2)+b\left(\pi_M+\varepsilon-\pi^e\right)+\bar{u}\right)+c\left(g_1+g_2\right)+\pi_M+\varepsilon=0$$

from which the reaction function of this authority derives:

$$\pi_M = -\frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)}{\left(1+\theta b^2\right)}\left(g_1+g_2\right)+\frac{\theta b^2}{1+\theta b^2}\pi^e-\frac{\theta b}{\left(1+\theta b^2\right)}\bar{u}-\frac{\left(1+\theta b^2\right)}{\left(1+\theta b^2\right)}\varepsilon. \tag{34}$$

The solutions g_1^*, g_2^*, π_M^* must satisfy:

$$g_1 = -\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)}{\left(\theta\left(a+bc\right)+c^2\right)}g_2-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\pi_M-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\varepsilon$$

$$-\frac{\theta\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}u_1+\frac{\theta\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\chi+\frac{\theta b\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\pi^e \tag{35}$$

$$g_2 = -\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)}{\left(\theta\left(a+bc\right)+c^2\right)}g_1-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\pi_M-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\varepsilon$$

$$-\frac{\theta\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}u_2-\frac{\theta\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\chi+\frac{\theta b\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\pi^e \tag{36}$$

$$\pi_M = \frac{\theta b^2}{1+\theta b^2}\pi^e-\frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)}{\left(1+\theta b^2\right)}\left(g_1+g_2\right)-\frac{\theta b}{\left(1+\theta b^2\right)}\bar{u}-\varepsilon. \tag{37}$$

Let us assume that:

$$g_1 = f_{1\chi}\chi+f_{11}u_1+f_{12}u_2+f_{1\varepsilon}\varepsilon \tag{38}$$

$$g_2 = f_{2\chi}\chi+f_{21}u_1+f_{22}u_2+f_{2\varepsilon}\varepsilon \tag{39}$$

$$\pi_M = f_{M\chi}\chi+f_{M1}u_1+f_{M2}u_2+f_{M\varepsilon}\varepsilon \tag{40}$$

$$\pi^e = f_{e\chi}\chi. \tag{41}$$

We know that:

$$\pi^e = cE\left(g_1+g_2\right)+E\left(\pi_M\right).$$

Hence:

$$f_{e\chi} = c\left(f_{1\chi}+f_{2\chi}\right)+f_{M\chi}.$$

Remark that:

$$g_1+g_2 = \left(f_{1\chi}+f_{2\chi}\right)\chi+\left(f_{11}+f_{21}\right)u_1+\left(f_{12}+f_{22}\right)u_2+\left(f_{1\varepsilon}+f_{2\varepsilon}\right)\varepsilon.$$

We get from (35) and (38):

$$f_{1\chi}\chi + f_{11}u_1 + f_{12}u_2 + f_{1\varepsilon}\varepsilon = -\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}\left(f_{2\chi}\chi + f_{21}u_1 + f_{22}u_2 + f_{2\varepsilon}\varepsilon\right)$$

$$-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left(\theta\left(a+bc\right)+c^2\right)}\left(f_{M\chi}\chi + f_{M1}u_1 + f_{M2}u_2 + f_{M\varepsilon}\varepsilon\right) - \frac{\left(\theta\left(a+bc\right)b+c\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}\varepsilon$$

$$+\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}\chi - \frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}u_1 + \frac{\theta b\left(a+bc\right)}{\left(\theta\left(a+bc\right)+c^2\right)}f_{e\chi}\chi.$$

Hence:

$$f_{1\chi} = \frac{\theta\left(a+bc\right) - \left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{2\chi} - \left(\theta\left(a+bc\right)b+c\right)f_{M\chi} + \theta b\left(a+bc\right)f_{e\chi}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{11} = -\frac{\theta\left(a+bc\right) + \left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{21} + \left(\theta\left(a+bc\right)b+c\right)f_{M1}+}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{12} = -\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{22} + \left(\theta\left(a+bc\right)b+c\right)f_{M2}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{1\varepsilon} = -\frac{\left(\theta\left(a+bc\right)b+c\right)\left(1+f_{M\varepsilon}\right) + \left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{2\varepsilon}}{\left[\theta\left(a+bc\right)^2+2c^2\right]}$$

and:

$$f_{2\chi} = \frac{\theta\left(a+bc\right) - \left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{1\chi} - \left(\theta\left(a+bc\right)b+c\right)f_{M\chi} + \theta b\left(a+bc\right)f_{e\chi}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{21} = -\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{11} + \left(\theta\left(a+bc\right)b+c\right)f_{M1}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{22} = -\frac{\theta\left(a+bc\right) + \left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{12} + \left(\theta\left(a+bc\right)b+c\right)f_{M2}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{2\varepsilon} = -\frac{\left(\theta\left(a+bc\right)b+c\right) + \left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{1\varepsilon} + \left(\theta\left(a+bc\right)b+c\right)f_{M\varepsilon}}{\left[\theta\left(a+bc\right)^2+c^2\right]}.$$

As the two countries are symmetric, we state:

$$f_{1\chi} = f_{2\chi}, f_{11} = f_{22}, f_{1\varepsilon} = f_{2\varepsilon}, f_{12} = f_{21}.$$

From the expression for π_M, we also get:

$$f_{M\chi}\chi + f_{M1}u_1 + f_{M2}u_2 + f_{M\varepsilon}\varepsilon = \left(\frac{\theta b^2}{\left(1+\theta b^2\right)}f_{e\chi} - \frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)}{\left(1+\theta b^2\right)}2f_{1\chi}\right)\chi$$

$$-\left(\frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)(f_{11}+f_{21})}{(1+\theta b^2)}+\frac{\theta b}{2\left(1+\theta b^2\right)}\right)(u_1+u_2)-\frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)2f_{1\varepsilon}+(1+\theta b^2)}{(1+\theta b^2)}\varepsilon$$

hence:

$$f_{M\chi}=\frac{\theta b^2}{1+\theta b^2}f_{e\chi}-\frac{(\theta b\,(1+2bc)+2c)}{(1+\theta b^2)}f_{1\chi}$$

$$f_{M1}=-\left(\frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)(f_{11}+f_{21})}{(1+\theta b^2)}+\frac{\theta b}{2\left(1+\theta b^2\right)}\right)$$

$$f_{M2}=-\left(\frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)(f_{11}+f_{21})}{(1+\theta b^2)}+\frac{\theta b}{2\left(1+\theta b^2\right)}\right)$$

$$f_{M\varepsilon}=-\frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)2f_{1\varepsilon}+(1+\theta b^2)}{(1+\theta b^2)}.$$

For the coefficients linked to ε, we get:

$$f_{1\varepsilon}=-\frac{(\theta\,(a+bc)\,b+c)\,(1+f_{M\varepsilon})}{[\theta\,(a+bc)\,(1+2bc)+2c^2]}.$$

Combining expressions for $f_{M\varepsilon}$ and $f_{1\varepsilon}$, we get:

$$f_{M\varepsilon}=-\frac{\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)2f_{1\varepsilon}+(1+\theta b^2)}{(1+\theta b^2)}$$

or equivalently:

$$f_{M\varepsilon}\left(\left(1+\theta b^2\right)\left[\theta\,(a+bc)\,(1+2bc)+2c^2\right]-\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)(\theta\,(a+bc)\,b+c)\right)$$
$$=\left(\theta b\left(\frac{1}{2}+bc\right)+c\right)(\theta\,(a+bc)\,b+c)-\left(1+\theta b^2\right)\left[\theta\,(a+bc)\,(1+2bc)+2c^2\right].$$

Therefore:

$$f_{M\varepsilon}=-1,\qquad f_{2\varepsilon}=f_{1\varepsilon}=0.$$

For coefficients linked to χ, we know:

$$\left[\theta\,(a+bc)^2+c^2\right]f_{1\chi}=\theta\,(a+bc)-\left(\theta\,(a+bc)\,(1+bc-a)+c^2\right)f_{2\chi}-(\theta\,(a+bc)\,b+c)\,f_{M\chi}+\theta b\,(a+bc)\,f_{e\chi}$$

and:

$$\left(1+\theta b^2\right)f_{M\chi}=\theta b^2 f_{e\chi}-(\theta b\,(1+2bc)+2c)\,f_{1\chi}.$$

Combining this equation and the equation for $f_{e\chi}$, as $f_{1\chi}=f_{2\chi}$, we get:

$$\left(1+\theta b^2\right)f_{M\chi}=\theta b^2\,(2cf_{1\chi}+f_{M\chi})-(\theta b\,(1+2bc)+2c)\,f_{1\chi}$$

hence:

$$f_{M\chi}=\left(2c\left(\theta b^2-1\right)-\theta b\left(1+2bc\right)\right)f_{1\chi}=-\left(2c+\theta b\right)f_{1\chi}.$$

Then, using the equality $f_{e\chi}=c\left(f_{1\chi}+f_{2\chi}\right)+f_{M\chi}=2cf_{1\chi}+f_{M\chi}$, we get:

$$\left[\theta\left(a+bc\right)^2+c^2\right]f_{1\chi}=\theta\left(a+bc\right)-\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{2\chi}-cf_{M\chi}+\theta b\left(a+bc\right)\left(f_{e\chi}-f_{M\chi}\right)$$

$$\left[\theta\left(a+bc\right)^2+c^2\right]f_{1\chi}=\theta\left(a+bc\right)-\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{1\chi}+\left(2c+\theta b\right)f_{1\chi}+2c\theta b\left(a+bc\right)f_{1\chi}.$$

Hence:

$$\left[\theta\left(a+bc\right)^2+c^2+\theta\left(a+bc\right)\left(1+bc-a\right)+c^2-\left(2c+\theta b\right)-2c\theta b\left(a+bc\right)\right]f_{1\chi}=\theta\left(a+bc\right)$$

$$\left[\theta\left(a+bc\right)\left(1+2b-2c\theta b\right)+2c^2-2c-\theta b\right]f_{1\chi}=\theta\left(a+bc\right)$$

that is:

$$f_{1\chi}=\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}=f_{2\chi}$$

$$f_{M\chi}=-\frac{\left(2c+\theta b\right)\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}$$

$$f_{e\chi}=-\frac{\theta^2 b\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}.$$

If $c=1$, we get:

$$f_{1\chi}=\frac{a+b}{a}\qquad f_{M\chi}=\theta b\frac{a+b}{a}.$$

About f_{ii} , f_{ij} and f_{Mi}, we know:

$$f_{11}=-\frac{\theta\left(a+bc\right)+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{21}+\left(\theta\left(a+bc\right)b+c\right)f_{M1}}{\left[\theta\left(a+bc\right)^2+c^2\right]}.$$

We know that:

$$f_{M1}=f_{M2}=-\left(\frac{\left(\theta b\left(1+2bc\right)+2c\right)\left(f_{11}+f_{21}\right)+\theta b}{2\left(1+\theta b^2\right)}\right)$$

$$f_{12}=-\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{22}+\left(\theta\left(a+bc\right)b+c\right)f_{M2}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{11}=-\frac{\theta\left(a+bc\right)+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{21}+\left(\theta\left(a+bc\right)b+c\right)f_{M1}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{21}=-\frac{\theta\left(a+bc\right)+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{11}+\left(\theta\left(a+bc\right)b+c\right)f_{M1}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{22}=-\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{12}+\left(\theta\left(a+bc\right)b+c\right)f_{M2}}{\left[\theta\left(a+bc\right)^2+c^2\right]}.$$

As:

$$-\left[\theta\left(a+bc\right)^{2}+c^{2}\right]\left(f_{11}+f_{21}\right)=\theta\left(a+bc\right)+\left(\theta\left(a+bc\right)b+c\right)f_{M1}+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^{2}\right)f_{21}$$

we get:

$$f_{11}+f_{21}=-1.$$

It is deduced:

$$f_{M1}=f_{M2}=c.$$

Hence:

$$f_{11}=\frac{-\theta\left(a+bc\right)+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^{2}\right)\left(1+f_{11}\right)-\left(\theta\left(a+bc\right)b+c\right)c}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}=-\frac{a}{2a-1}.$$

Finally:

$$f_{21}=\frac{a}{2a-1}-1=\frac{1-a}{2a-1}.$$

Thus:

$$g_{1}=\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi-\frac{a}{2a-1}u_{1}+\frac{1-a}{2a-1}u_{2}$$

$$g_{2}=\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi+\frac{1-a}{2a-1}u_{1}-\frac{a}{2a-1}u_{2}$$

$$\pi_{M}=-\frac{\left(2c+\theta b\right)\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi+cu_{1}+cu_{2}-\varepsilon.$$

The reaction functions are now fully determined.

Expected losses

As

$$\varphi_{i}=\left(a+bc\right)g_{i}+\left(1+bc-a\right)g_{-i}+b\left(\pi_{M}+\varepsilon-\pi^{e}\right)+u_{i}-\chi$$

from previous results we get:

$$\varphi_{1}=\left(a+bc\right)\left(\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi-\frac{a}{2a-1}u_{1}+\frac{1-a}{2a-1}u_{2}\right)$$

$$+\left(1+bc-a\right)\left(\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi+\frac{1-a}{2a-1}u_{1}-\frac{a}{2a-1}u_{2}\right)$$

$$+b\left(-\frac{2c\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi+cu_{1}+cu_{2}-\varepsilon\right)$$

$$+b\varepsilon+u_{i}-\chi$$

that is:

$$\varphi_1 = \left(\frac{-a\left(a+bc\right)+\left(1-a\right)\left(1+bc-a\right)}{2a-1}+bc+1\right)u_1$$
$$+\left(\frac{\left(a+bc\right)\left(1-a\right)-a\left(1+bc-a\right)}{2a-1}+bc\right)u_2$$
$$+\left(\frac{\theta\left(a+bc\right)\left(a+bc+1+bc-a-2bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}-1\right)\chi$$
$$=\left(-1-bc+bc+1\right)u_1+\frac{2abc}{2a-1}u_2+\frac{\theta b-2\left(1-c\right)\left(b\theta\left(a+bc\right)-c\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi$$
$$=\frac{2abc}{2a-1}u_2+\frac{\theta b-2\left(1-c\right)\left(b\theta\left(a+bc\right)-c\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi.$$

Similarly:

$$\phi_i = c\left(g_1+g_2\right)+\pi_M+\varepsilon$$
$$=c\left(g_1+g_2\right)-\frac{\left(2c+\theta b\right)\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi+cu_1+cu_2-\varepsilon+\varepsilon$$
$$=c\left(2\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi-u_1-u_2\right)-\frac{\left(2c+\theta b\right)\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi$$
$$+c\left(u_1+u_2\right)$$
$$=-\frac{\theta^2 b\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi.$$

Hence:

$$E\left(L_{3i}^*\right)=\frac{1}{2}\theta E\left(\varphi_i\right)^2+\frac{1}{2}E\left(\phi_i\right)^2$$
$$=\frac{1}{2}\frac{\theta\left(\theta b-2\left(1-c\right)\left(b\theta\left(a+bc\right)-c\right)\right)^2+\left(\theta^2 b\left(a+bc\right)\right)^2}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]^2}\chi^2+\frac{1}{2}\theta\left[\frac{2abc}{2a-1}\right]^2\sigma_u^2. \tag{42}$$

We also get:

$$\varphi_M=\left(\frac{1}{2}+bc\right)\left(g_1+g_2\right)+b\left(\pi_M+\varepsilon-\pi^e\right)+\bar{u}$$
$$=\left(\frac{1}{2}+bc\right)\left(g_1+g_2\right)+b\left(-\frac{2c\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi+cu_1+cu_2-\varepsilon+\varepsilon\right)+\bar{u}$$
$$=\left(\frac{1}{2}+bc\right)\left(\frac{2\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi-\left(u_1+u_2\right)\right)$$
$$-\frac{2cb\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi+\left(\frac{1}{2}+bc\right)\left(u_1+u_2\right)$$
$$=\left\{\frac{c\theta\left(a+bc\right)\left(1+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\right\}\chi$$
$$\phi_M=\phi_i=-\frac{\theta^2 b\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]}\chi.$$

Hence:

$$E\left(L_{3M}^{*}\right)=\frac{1}{2}\theta\left(\varphi_{M}\right)^{2}+\frac{1}{2}\left(\phi_{M}\right)^{2}=\frac{1}{2}\chi^{2}\frac{\left[c\theta\left(a+bc\right)\left(1+bc\right)\right]^{2}+\left[\theta^{2}b\left(a+bc\right)\right]^{2}}{\left[\theta\left(a+bc\right)\left(1+2b\left(1-c\right)\right)+2c\left(1-c\right)-\theta b\right]^{2}}. \quad (43)$$

Option 3. The case of unconstrained fiscal policies/constrained monetary policy.

In this option the central bank is inactive: $\pi_M = 0$. The optimization programme of a fiscal authority, combined with $\pi_M = 0$, is:

$$max_{g_i}E\left(L_i\right)=E\left[\frac{1}{2}\theta\left(\left(a+bc\right)g_i+\left(1+bc-a\right)g_{-i}+b\varepsilon-b\pi^{e}+u_i-\chi\right)^{2}+\frac{1}{2}\left(c\left(g_1+g_2\right)+\varepsilon\right)^{2}\right]. \quad (44)$$

L_i can be written as:

$$L_i=\frac{1}{2}\theta\left(\varphi_i\right)^{2}+\frac{1}{2}\left(\phi_i\right)^{2}.$$

The first-order condition is:

$$\frac{\partial L_i}{\partial g_i}=\theta\left(a+bc\right)\left(\left(a+bc\right)g_i+\left(1+bc-a\right)g_{-i}+b\varepsilon-b\pi^{e}+u_i-\chi\right)+c\left(c\left(g_1+g_2\right)+\varepsilon\right)=0$$

so that:

$$\left[\theta\left(a+bc\right)^{2}+c^{2}\right]g_i=-\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^{2}\right)g_{-i}+b\theta\left(a+bc\right)\pi^{e}-\left(\theta\left(a+bc\right)b+c\right)\varepsilon$$
$$+\theta\left(a+bc\right)\chi-\theta\left(a+bc\right)u_i.$$

Hence:

$$g_1=-\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^{2}\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}g_2-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}\varepsilon$$
$$+\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}\chi-\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}u_1+\frac{b\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}\pi^{e}$$
$$g_2=-\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^{2}\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}g_1-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}\varepsilon$$
$$+\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}\chi-\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}u_2+\frac{b\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^{2}+c^{2}\right]}\pi^{e}.$$

Let us assume:

$$g_1=f_{1\chi}\chi+f_{11}u_1+f_{12}u_2+f_{1\varepsilon}\varepsilon$$
$$g_2=f_{2\chi}\chi+f_{21}u_1+f_{22}u_2+f_{2\varepsilon}\varepsilon$$
$$\pi^{e}=f_{1e}\chi.$$

We notice that:

$$g_1+g_2=\left(f_{1\chi}+f_{2\chi}+\frac{2b\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}f_{1e}\right)\chi+\left(f_{11}+f_{21}\right)u_1+\left(f_{12}+f_{22}\right)u_2+\left(f_{1\varepsilon}+f_{2\varepsilon}\right)\varepsilon.$$

As:

$$\pi=c\left(g_1+g_2\right)+\varepsilon$$

we get:

$$\pi^e=E\left(\pi\right)=c\left(g_1+g_2\right)$$

and:

$$f_{1e}=2cf_{1\chi}.$$

Hence:

$$f_{1\chi}\chi+f_{11}u_1+f_{12}u_2+f_{1\varepsilon}\varepsilon=-\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}\left(f_{2\chi}\chi+f_{21}u_1+f_{22}u_2+f_{2\varepsilon}\varepsilon\right)$$

$$-\frac{\left(\theta\left(a+bc\right)b+c\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}\varepsilon+\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}\chi-\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}u_1+\frac{b\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}f_{1e}\chi.$$

Therefore:

$$f_{1\chi}=\frac{\theta\left(a+bc\right)+b\theta\left(a+bc\right)f_{1e}-\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{2\chi}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{11}=-\frac{\theta\left(a+bc\right)+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{21}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{12}=-\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}f_{22}$$

$$f_{1\varepsilon}=-\frac{\left(\theta\left(a+bc\right)b+c\right)+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{2\varepsilon}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

and:

$$f_{2\chi}=\frac{\theta\left(a+bc\right)+b\theta\left(a+bc\right)f_{1e}-\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{1\chi}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{21}=-\frac{\theta\left(a+bc\right)-b\theta\left(a+bc\right)f_{1e}+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{11}}{\left[\theta\left(a+bc\right)^2+c^2\right]}$$

$$f_{22}=-\frac{\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)}{\left[\theta\left(a+bc\right)^2+c^2\right]}f_{12}$$

$$f_{2\varepsilon}=-\frac{\left(\theta\left(a+bc\right)b+c\right)+\left(\theta\left(a+bc\right)\left(1+bc-a\right)+c^2\right)f_{1\varepsilon}}{\left[\theta\left(a+bc\right)^2+c^2\right]}.$$

As the two countries are symmetric, we state:

$$f_{1\chi} = f_{2\chi}, f_{11} = f_{22}, f_{1\varepsilon} = f_{2\varepsilon}, f_{12} = f_{21}.$$

As $f_{e\chi} = 2cf_{1\chi}$:

$$f_{1\chi} = \frac{\theta\left(a+bc\right) + b\theta\left(a+bc\right) f_{e\chi}}{\theta\left(a+bc\right)\left(1+2bc\right)+2c^2} = f_{2\chi}$$

we get:

$$\left(\theta\left(a+bc\right)+2c^2\right) f_{1\chi} = \theta\left(a+bc\right).$$

Hence:

$$f_{1\chi} = \frac{\theta\left(a+bc\right)}{\theta\left(a+bc\right)+2c^2} = f_{2\chi}$$

$$f_{11} = -\frac{\theta\left(a+bc\right)^2 + c^2}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^2\right]\left(2a-1\right)}$$

$$f_{12} = \frac{\theta\left(a+bc\right)\left(1+bc-a\right)+c^2}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^2\right]\left(2a-1\right)} = f_{21}$$

$$f_{1\varepsilon} = -\frac{\left(\theta\left(a+bc\right)b+c\right)}{\theta\left(a+bc\right)\left(1+2bc\right)+2c^2}.$$

Notice that:

$$g_1 + g_2 = 2f_{1\chi}\chi + b\theta\left(a+bc\right) f_{1e} + \left(f_{11}+f_{21}\right)\left(u_1+u_2\right) + 2f_{1\varepsilon}\varepsilon$$

$$\left(a+bc\right) f_{11} + \left(1+bc-a\right) f_{21} + 1 =$$

$$\frac{\theta\left(a+bc\right)\left[\left(1+2bc\right)\left(1-2a\right)\right] + \left(1-2a\right)c^2 + \left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^2\right]\left(2a-1\right)}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^2\right]\left(2a-1\right)}$$

$$= \frac{c^2}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^2\right]}$$

and:

$$\left(a+bc\right) f_{12} + f_{22}\left(1+bc-a\right) = \frac{c^2}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^2\right]}.$$

Finally:

$$f_{11} + f_{21} = -\frac{\theta\left(a+bc\right)}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^2\right]}$$

and:

$$2cf_{1\varepsilon} + 1 = \frac{\theta\left(a+bc\right)}{\theta\left(a+bc\right)\left(1+2bc\right)+2c^2}.$$

Therefore:

$$g_1 + g_2 = 2\frac{\theta\left(a+bc\right)}{\theta\left(a+bc\right)+2c^2}\chi + \frac{\theta\left(a+bc\right)}{\theta\left(a+bc\right)\left(1+2bc\right)+2c^2}\left(u_1+u_2\right) + 2f_{1\varepsilon}\varepsilon.$$

Thus the decisions of the fiscal authorities are given by the following equations:

$$g_1 = \frac{\theta(a+bc)}{\theta(a+bc)+2c^2}\chi - \frac{\theta(a+bc)^2+c^2}{[\theta(a+bc)(1+2bc)+2c^2](2a-1)}u_1$$

$$+\frac{\theta(a+bc)(1+bc-a)+c^2}{[\theta(a+bc)(1+2bc)+2c^2](2a-1)}u_2 - \frac{(\theta(a+bc)b+c)}{\theta(a+bc)(1+2bc)+2c^2}\varepsilon$$

$$g_2 = \frac{\theta(a+bc)}{\theta(a+bc)+2c^2}\chi - \frac{\theta(a+bc)^2+c^2}{[\theta(a+bc)(1+2bc)+2c^2](2a-1)}u_2$$

$$+\frac{\theta(a+bc)(1+bc-a)+c^2}{[\theta(a+bc)(1+2bc)+2c^2](2a-1)}u_1 - \frac{(\theta(a+bc)b+c)}{\theta(a+bc)(1+2bc)+2c^2}\varepsilon.$$

From the definition de φ_i, we know:

$$\varphi_i = (f_{1\chi}-1)\chi + ((a+bc)f_{11}+(1+bc-a)f_{21}+1)u_1$$

$$+((1+2bc)f_{1\varepsilon}+b)\varepsilon + ((a+bc)f_{12}+f_{22}(1+bc-a))u_2$$

Hence:

$$\varphi_i = \left(\frac{\theta(a+bc)-\theta(a+bc)-2c^2}{\theta(a+bc)+2c^2}\right)\chi + \frac{c^2}{\theta(a+bc)(1+2bc)+2c^2}u_1$$

$$-\frac{c}{\theta(a+bc)(1+2bc)+2c^2}\varepsilon + \frac{c^2}{\theta(a+bc)(1+2bc)+2c^2}u_2$$

equivalently:

$$\varphi_i = \frac{-2c^2}{\theta(a+bc)+2c^2}\chi + \frac{c^2}{\theta(a+bc)(1+2bc)+2c^2}u_1$$

$$-\frac{c}{\theta(a+bc)(1+2bc)+2c^2}\varepsilon + \frac{c^2}{\theta(a+bc)(1+2bc)+2c^2}u_2.$$

Moreover:

$$\phi_i = c(g_1+g_2)+\varepsilon = c(2f_{1\chi}\chi + (f_{11}+f_{21})(u_1+u_2)) + (2cf_{1\varepsilon}+1)\varepsilon.$$

Hence:

$$\phi_i = \frac{2c\theta(a+bc)}{\theta(a+bc)+2c^2}\chi - \frac{c\theta(a+bc)}{\theta(a+bc)(1+2bc)+2c^2}(u_1+u_2)$$

$$+\frac{\theta(a+bc)}{\theta(a+bc)(1+2bc)+2c^2}\varepsilon$$

and thus:

$$E(\varphi_i)^2 = \left(\frac{2c^2}{\theta(a+bc)+2c^2}\right)^2\chi^2 + 2\left(\frac{c^2}{\theta(a+bc)(1+2bc)+2c^2}\right)^2\sigma_u^2$$

$$+\left(\frac{c}{\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}}\right)^{2}\sigma_{\varepsilon}^{2}$$

$$E\left(\phi_{i}\right)^{2}=\left(\frac{2c\theta\left(a+bc\right)}{\theta\left(a+bc\right)+2c^{2}}\right)^{2}\chi^{2}+2\left(\frac{c\theta\left(a+bc\right)}{\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}}\right)^{2}\sigma_{u}^{2}$$

$$+\left(\frac{\theta\left(a+bc\right)}{\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}}\right)^{2}\sigma_{\varepsilon}^{2}.$$

Expected losses are equal to:

$$E\left(L_{5i}^{*}\right)=\frac{1}{2}\theta E\left(\varphi_{i}\right)^{2}+\frac{1}{2}E\left(\phi_{i}\right)^{2}$$

$$=c^{2}\left(\frac{2\theta c^{2}+\left(\theta\left(a+bc\right)\right)^{2}}{\left(\theta\left(a+bc\right)+2c^{2}\right)^{2}}\right)\chi^{2}+\frac{c^{2}+\left(\theta\left(a+bc\right)\right)^{2}}{\left(\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}\right)^{2}}\left(2c^{2}\sigma_{u}^{2}+\sigma_{\varepsilon}^{2}\right). \qquad (45)$$

Similarly we get for $E\left(L_{M}^{*}\right)$ (assuming $\chi_{M}=0$).

$$L_{M}=\frac{1}{2}\theta\left(\left(\frac{1}{2}+bc\right)\left(g_{1}+g_{2}\right)+b\varepsilon+\bar{u}\right)^{2}+\frac{1}{2}\left(c\left(g_{1}+g_{2}\right)+\varepsilon\right)^{2}=\frac{1}{2}\theta\left(\varphi_{M}\right)^{2}+\frac{1}{2}\left(\phi_{M}\right)^{2}$$

with:

$$\varphi_{M}=\frac{\theta\left(a+bc\right)\left(1+2bc\right)}{\theta\left(a+bc\right)+2c^{2}}\chi+\left(\frac{1+2bc}{4}\right)\frac{\left[\theta\left(a+bc\right)\left(2bc-1\right)+2c^{2}\right]}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}\right]}\left(u_{1}+u_{2}\right)$$

$$+\left(\frac{1+2bc}{2}\right)\frac{b\theta\left(2bc-1\right)+2c\left(c-1\right)}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}\right]}\varepsilon$$

and:

$$\phi_{M}=c\left(g_{1}+g_{2}\right)+\varepsilon=\phi_{i}.$$

Therefore:

$$E\left(L_{5M}^{*}\right)=\frac{\theta^{2}\left(a+bc\right)^{2}\left(\left(1+2bc\right)^{2}+4c^{2}\right)}{\left[\theta\left(a+bc\right)+2c^{2}\right]^{2}}\chi^{2}+\frac{\left(\frac{1+2bc}{2}\right)^{2}\left[\theta\left(a+bc\right)\left(2bc-1\right)+2c^{2}\right]^{2}+\left(2c\theta\left(a+bc\right)\right)^{2}}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}\right]^{2}}\sigma_{u}^{2}$$

$$+\frac{\left(\frac{1+2bc}{2}\right)^{2}\left[b\theta\left(2bc-1\right)+2c\left(c-1\right)\right]^{2}+\theta^{2}\left(a+bc\right)^{2}}{\left[\theta\left(a+bc\right)\left(1+2bc\right)+2c^{2}\right]^{2}}\sigma_{\varepsilon}^{2}. \qquad (46)$$