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Abstract

This paper introduces tasks into the neoclassical production sector. Competitive firms choose the profit-maximizing amounts of factor-specific tasks that determine their factor demands and output supplies. We show that the effect of factor-augmenting technical change on relative and absolute factor prices can be decomposed into a productivity effect and a task-demand effect of opposite sign. These effects appear since the novel task-based approach distinguishes between the demands for tasks and the demands for factors. This perspective provides a new intuition for the emergence of relative and absolute factor biases and the role of the elasticity of substitution.

JEL-Codes: O330, O410.

Keywords: technical change, factor prices, factor-specific tasks, neoclassical production.

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1 Introduction

The provision of goods and services requires tasks to be accomplished by the factors of production. In other words, what matters in the process of production is not the mere presence of factors but what they do. The present paper accounts for this observation and develops a novel interpretation of the neoclassical production sector where capital and labor perform factor-specific tasks to produce output. A task may be thought of as a unit of work activity that contributes to the production of output.

The task-based interpretation of the neoclassical production sector sheds new light on the intricate relationship between factor-augmenting technical change and relative and absolute factor prices. In particular, we establish that the effect of factor-augmenting technical change on relative and absolute factor prices can be decomposed into a *productivity effect* and an effect on the demand for tasks, henceforth referred to as the *task-demand effect*. These effects appear since the task-based approach distinguishes between the demands for tasks and the demands for factors. This perspective provides a new intuition for the emergence of relative and absolute factor biases and the role of the elasticity of substitution. The findings of the present paper are derived in the following steps.

Section 2 introduces tasks into the neoclassical production sector, defines the competitive equilibrium, and provides a detailed interpretation of the notion and the modelling of tasks. The novel features include a production function defined over factor specific tasks and input coefficients that provide the link between each task and the required amount of the factor that accomplishes it. For expository purposes, we refer to them as capital and labor. Taking tasks as the primitive means that firms maximize profits to find the optimal number of tasks performed by capital and labor. These choices determine their factor demands as well as their output supplies.

Section 3 analyzes the model under the assumption that factor specific tasks are homogeneous in the sense that the productivity of factors in the performance of their respective task is the same. We establish the existence of a unique competitive equilibrium in Section 3.1 and show that the resulting allocation coincides with the one obtained for a neoclassical production sector where firms' production technology is directly defined over their efficient capital and labor inputs. In this sense the suggested interpretation based on tasks provides an intellectual underpinning of the neoclassical production sector used in the literature.

Section 3.2 develops a new intuition for the way in which factor-augmenting technical change affects relative and absolute equilibrium factor prices. Arguably, this is the main result of this paper. We show that the relative factor bias of technical change hinges on the tension between a positive productivity effect and a negative task-demand effect. To grasp the intuition, consider an increase in the relative productivity of capital. Then, according to the productivity effect, the relative price of capital increases since, at a given ratio of tasks, the relative marginal product of capital increases. The task-demand effect reflects the adjustment of the ratio of performed tasks necessary to maintain full employment in both factor markets. Since capital becomes relatively more productive, the relative demand for tasks performed by capital has to increase. To accomplish this with profit-maximizing firms requires a decline in the relative price of capital. Hence, the task-demand effect is negative. The strength of the decline in the relative price of capital hinges on the ease with which factors of production can be substituted. If it is easy enough (difficult) to substitute then the required decline in the relative price of capital is small (high). If firms operate under a CES production function then the measure of the degree of easiness is the elasticity of substitution. This leads to the conclusion, that the productivity effect dominates the task-demand effect if and only if the elasticity of substitution exceeds unity.

Similarly, the effect of a higher capital productivity in the performance of tasks on the equilibrium rental rate of capital can be decomposed into a positive productivity effect and a negative task-demand effect. For a given amount of tasks performed by capital, the productivity effect reflects the higher marginal product of capital. Accordingly, the rental rate of capital increases. The task-demand effect captures adjustments in the factor market for capital. A higher capital productivity requires more tasks to be performed by capital to ensure full employment of capital. As these additional tasks are associated with a lower marginal product, the task-demand effect is negative.

Section 4 introduces heterogeneous tasks. Tasks differ on the input side. Additional tasks have larger input requirements and, therefore, are more expensive to perform. First, we establish the existence of a unique competitive equilibrium in Section 4.1. The implications of task heterogeneity for the effect of factor-augmenting technical change on factor prices is the focus of Section 4.2. We find that heterogeneity adds two important new facets to the analysis.

First, a key determinant of (relative) factor prices is the productivity of the marginal task. As factor-augmenting technical change increases the market size for the tasks to which it applies, additional tasks must be performed to maintain full employment. However, the productivity of the respective factors of production in the marginal task will be lower than for inframarginal tasks. Accordingly, the factor price falls. The second novel feature concerns the market size effect. As input requirements increase across tasks, a given percentage increase in the factor-augmenting productivity parameter leads to a less than proportionate increase in tasks. This weakens the role of the elasticity of substitution for the adjustment of factor prices in response to a change in efficient factor factor supplies. Both features lead to the conclusion that the relative price of capital may decrease in response to a hike in capital-augmenting or labor-augmenting technical change even if the elasticity of substitution exceeds unity. Moreover, factor-augmenting technical progress reduces the absolute price of the factor to which it applies even if the elasticity of substitution exceeds unity.

This paper builds on and contributes to at least two branches of the literature interested in the possible implications of technical change in factor markets. The first branch takes a new look at the old question of factor-price biases of technical change (see, e. g., Acemoglu (2007), Acemoglu (2010)). While technical change is exogenous in the present paper, the results we derive are informative on the conditions under which technical change increases or decreases the relative and the absolute marginal product of factors. As to relative marginal products, our emphasis on tasks provides an intuitive interpretation of the role played by the elasticity of substitution for the sign of these effects in terms of a productivity and a task-demand effect.

Second, we contribute to the literature that studies relevant economic phenomena emphasizing the role of tasks. Examples include Dornbusch, Fischer, and Samuelson (1980) with an application to international trade patterns, Champernowne (1961) and Zeira (1998) with a focus on economic growth and capital-labor substitution, the contribution of Acemoglu and Zilibotti (2001) analyzing productivity differences between advanced and less developed countries, or Acemoglu and Autor (2011) who study employment and earnings profiles across skill groups. Related is also the recent "endogenous growth" literature emphasizing tasks in the context of automation incentives and economic growth (see, e. g., Irmen (2017), Irmen and Tabaković (2017), Acemoglu and Restrepo (2018a), Acemoglu and Restrepo (2018b), Irmen (2020a), or Irmen (2020b)). Unlike this literature, the technology for each task is exogenous in the present paper. This focus allows for a reinterpretation of the neoclassical production sector in terms of tasks. Moreover, it provides new intuitions and new predictions about the effect of exogenous factor-augmenting technical change, e. g., on relative and absolute factor prices.

The remainder of this paper is organized as follows. Sections 2 introduces tasks into the neoclassical production sector. Section 3 derives our main results for homogeneous tasks. Section 4 considers the case of heterogeneous tasks. Section 5 concludes. Proofs are relegated to Section 6, the Appendix.

2 Tasks in the Neoclassical Production Sector

Consider a static economy endowed with capital, $K \in \mathbb{R}_{++}$ and labor, $L \in \mathbb{R}_{++}$. There is one final good produced by a competitive production sector with a continuum of identical firms. We represent this sector by the unit interval and use i to index firms, i. e., $i \in [0, 1]$. The final good serves as numéraire.

All firms have access to the same production technology. On the one hand, the technology describes the relationship between the accomplishment of tasks and the resulting output. On the other hand, it specifies the input requirements of tasks.

There are two distinct types of tasks, one is performed by capital, the other by labor.[1] Let $m \in \mathbb{R}_+$ denote tasks performed by capital and $n \in \mathbb{R}_+$ tasks performed by labor.

[1]The labels "capital" and "labor" are just for convenience. Mathematically, these two factors may rep-

All firms have access to the same neoclassical production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$, i. e., output of firm i, $Y(i)$, is

$$Y(i) = F\left(M(i), N(i)\right), \tag{2.1}$$

where $M(i)$ and $N(i)$ denote the respective total demands for tasks performed by capital and labor of firm i. In other words, if firm i performs all tasks $m \in [0, M(i)]$ and $n \in [0, N(i)]$ then the maximum output it produces is $F\left(M(i), N(i)\right)$. Since $F\left(M(i), N(i)\right)$ is a neoclassical production function, it exhibits constant returns to scale and positive, yet decreasing marginal products to both types of tasks.[2]

Task m requires $k\left(m\right) = 1/b\left(m\right) > 0$ units of capital. Similarly, task n requires $l\left(n\right) = 1/a(n) > 0$ units of labor. Hence, $b(m) > 0$ is the productivity of capital in the performance of task m whereas $a(n) > 0$ is the productivity of labor in the performance of task n.

2.1 Competitve Behaviour and Equilibrium

Competitive firms take factor prices, i. e., the real rental rate of capital, R, and the real wage, w, as given and maximize profits. Hence, they choose the plan

$$\left(M(i), N(i), \left[k\left(m\right)\right]_{m\in[0,M(i)]}, \left[l\left(n\right)\right]_{n\in[0,N(i)]}\right) \tag{2.2}$$

comprising the total number of tasks performed by either capital or labor and the amount of capital or labor to accomplish each task. These choices are to maximize profits

$$\pi(i) = F\left(M(i), N(i)\right) - C(i), \tag{2.3}$$

where

$$C(i) = R\int_0^{M(i)} k\left(m\right) dm + w\int_0^{N(i)} l\left(n\right) dn \tag{2.4}$$

is firm i's total factor costs comprising the costs of capital and labor incurred to perform $M(i)$, respectively, $N(i)$, tasks.

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resent any pair of distinguishable inputs. Mutatis mutandis, the analytical setup developed here extends readily to more than two inputs each with a corresponding task type. For instance, one may want to replace the single type of tasks performed by labor with two distinct types, one performed by skilled, the other be unskilled labor.

[2] To include the CES production function, we dispense with Inada conditions. However, throughout I assume that the $\lim_{M(i)\to 0} \partial F\left(M(i), N(i)\right)/\partial M(i)$ and $\lim_{N(i)\to 0} \partial F\left(M(i), N(i)\right)/\partial N(i)$ are sufficiently high to deliver an interior solution to the profit-maximization problem of firms.

Let

$$K(i) = \int_0^{M(i)} k\,(m)\,dm \quad \text{and} \quad L(i) = \int_0^{N(i)} l\,(n)\,dn \tag{2.5}$$

denote firm i's demand for capital and labor.

Factor endowments are inelastically supplied, i. e., capital owners and workers supply the services of their factors to the firms that offer the highest remuneration. Therefore, all firms must pay the same factor prices to attract inputs. Moreover, capital and labor are perfectly divisible across tasks of the respective type. Then, the factor market clearing conditions are

$$\begin{aligned} R \geq 0, \quad \int_0^1 K(i)di \leq K, \quad \text{and} \quad R\left(\int_0^1 K(i)di - K\right) = 0, \\ w \geq 0, \quad \int_0^1 L(i)di \leq L \quad \text{and} \quad w\left(\int_0^1 L(i)di - L\right) = 0. \end{aligned} \tag{2.6}$$

They account for the fact that an excess supply is only possible if the respective factor price is zero. The competitive equilibrium is then defined as follows.

Definition 1 ***(Competitive Equilibrium)***

The competitive equilibrium is a set of plans (2.2) for all firms $i \in [0,1]$ and factor prices $(R^*, w^*) \geq 0$ such that these plans maximize profits (2.3) for given factor prices, and the factor market clearing conditions (2.6) hold.

2.2 The Notion of Tasks - An Interpretation

The notion and the modelling of tasks introduced in the present paper differs substantially from the existing literature where tasks often bear a resemblance to differentiated intermediate goods (see, e. g., Acemoglu and Zilibotti (2001) and Acemoglu and Restrepo (2018a)). The following paragraphs provide a clarifying interpretation of the analytical strategy that leads to Definition 1.

The production function $Y(i) = F\,(M(i), N(i))$ treats tasks, $M(i)$ and $N(i)$, in the same way as capital and labor are treated in the textbook version of the neoclassical production function (see, e. g., Acemoglu (2009), Chapter 2). This means that tasks of a given type are perfect substitutes in the sense that two different tasks, say, m and m', are interchangeable, i. e., performing task m' instead of taks m has no effect on output. There is, however, a first feature of task heterogeneity captured by the "law of a diminishing marginal product." Accordingly, the (marginal) contribution to output of a given task depends on its position on the real line. This feature of heterogeneity would not be present if the production function was, e. g., of the form $Y(i) = M(i) + N(i)$.

Moreover, tasks of different types are complements. Adding a new task performed by labor increases the (marginal) contribution to output of all tasks performed by capital, i. e., $\partial^2 F\left(M(i), N(i)\right)/\partial M(i)\partial N(i) > 0$. This is a direct consequence of the law of a diminishing marginal product in conjunction with constant returns to scale.

On the input side, tasks may be *homogeneous* or *heterogeneous*. Tasks are homogeneous if the productivity of capital and the productivity of labor in the performance of the respective task type is constant, i. e., $b(m) = b$ and $a(n) = a$ (see Assumption 1 below). Then, the performance of each task m requires the same amount of capital. Likewise, each task n requires the same amount of labor. To fix ideas, think of a task as a unit of work activity, i. e., something that needs to be done. Then, $1/b$ has an interpretation as the time a machine needs to perform any task m. Similarly, $1/a$ is the time a worker needs to perform any task n. Accordingly, the factor demands, $K(i)$ and $L(i)$ of equation (2.5), state firm i's demand for hours of machine services and for hours of labor services, respectively.

If tasks are homogeneous then an intuitive adaptation of "activity analysis" may be used to characterize a firm's *task input requirement set*.[3] Heuristically, this works as follows. Each firm has access to different techniques. Each technique specifies a necessary mix of tasks to be performed by capital and labor to produce one unit of output. An arbitrary replication of these techniques is possible so that any amount of output can be produced with a single technique. With two available techniques, their convex combinations give rise to a convex task input requirement set. With many (efficient) techniques and their respective convex combinations a piecewise linear isoquant with finitely many kinks appears. The corresponding task input requirement set may then be approximated with a "'smoothed" task input requirement set. Its bound is a strictly convex isoquant in $(N(i), M(i))$-space.

Tasks are heterogeneous on the input side if the productivities $b(m)$ and $a(n)$ vary across tasks. In addition to the (output side) heterogeneity associated with the law of a diminishing marginal product, this introduces a second dimension of heterogeneity. Moreover, the question of how $b(m)$ and $a(n)$ are distributed needs to be addressed. In Section 4 I study the case where both productivities decline at the same exponential rate (see Assumption 2). This means that tasks with a higher factor productivity also have a higher marginal product. Finally, observe that task heterogeneity on the input side does not change the interpretation of firm i's factor demands, $K(i)$ and $L(i)$.

[3] See, e. g., Varian (1992), Chapter 1, for a traditional textbook treatment of "activity analysis." In my analytical framework a firm's input requirement set comprises all vectors of tasks that allow for the production of a given amount of output. Therefore, I refer to this set as the *task input requirement set*.

3 Homogeneous Tasks

Let tasks be homogenous in the sense that the productivity of capital and labor in the performance of their respective tasks is independent of m and n.

Assumption 1 ***(Homogeneous Tasks)*** *$b(m) = b > 0$ and $a(n) = a > 0$.*

3.1 The Competitive Equilibrium

With Assumption 1, firm i's factor costs (2.4) are

$$C(i) = \frac{RM(i)}{b} + \frac{wN(i)}{a}. \tag{3.1}$$

Using the latter in (2.3) leads to the first-order (sufficient) conditions for an interior solution

$$\begin{aligned} \frac{\partial \pi(i)}{\partial M(i)} &= \frac{\partial F\left(M(i), N(i)\right)}{\partial M(i)} - \frac{R}{b} = 0, \\ \frac{\partial \pi(i)}{\partial N(i)} &= \frac{\partial F\left(M(i), N(i)\right)}{\partial N(i)} - \frac{w}{a} = 0, \end{aligned} \tag{3.2}$$

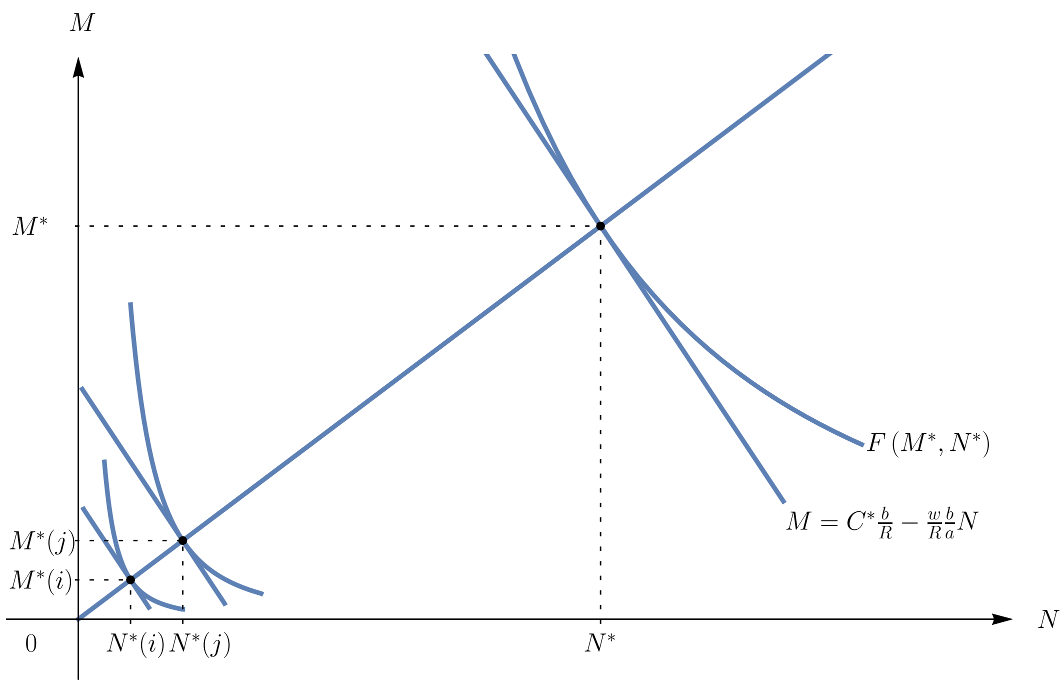
for all $i \in [0, 1]$. Hence, the marginal value product of task $M(i)$ is equal to its marginal cost. Since performing a task with capital requires $1/b$ units of capital, the latter is equal to R/b. A similar interpretation applies to labor.

Proposition 1 ***(Competitive Equilibrium with Homogeneous Tasks)***

The competitive equilibrium involves aggregate quantities of tasks, $M^* = bK$, $N^* = aL$, and aggregate output, $F\left(M^*, N^*\right)$. The equilibrium task intensity of all firms is equal to $\left(M(i)/N(i)\right)^* = M^*/N^* = bK/(aL)$. The scale of each firm remains indeterminate. Factor prices are strictly positive and given by $R^* = b \cdot \partial F\left(M^*, N^*\right)/\partial M(i) > 0$ and $w^* = a \cdot \partial F\left(M^*, N^*\right)/\partial N(i) > 0$.

Proposition 1 shows that the economy's efficient amount of capital and labor, i. e., bK and aL, coincides with the respective equilibrium amount of tasks performed by capital and labor. Hence, in equilibrium, the respective task productivity b and a is factor-augmenting. Moreover, the equilibrium task-intensity $\left(M(i)/N(i)\right)^*$ of all firms is equal to the economy's efficient capital intensity, $bK/(aL)$. Since firms operate under constant returns to scale, their scale of production remains indeterminate. Figure 3.1 provides an illustration of the equilibrium configuration.

Figure 3.1: **The Competitive Equilibrium in (N, M)-Space.**



Note: In equilibrium, all firms produce with the same task intensity $(M(i)/N(i))^* = M^*/N^*$. However, $M^*(i)$ and $N^*(i)$ remain indeterminate. Possible scales for two firms i and j are shown. For the production sector as a whole, production occurs at minimum costs, C^*. The corresponding isocost curve is tangential to the isoquant for the output level $Y = F(M^*, N^*)$.

Three remarks are in order. First, an intuitive explanation for the scale indeterminacy of the competitive equilibrium comes from "activity analysis" introduced above. A firm's profit-maximizing production plan involves the convex combination of techniques that minimizes unit costs for given costs per task. This combination will then be scaled up to produce the desired amount of output. In equilibrium, the cost-minimizing convex combination of techniques will be such that for each firm $M(i)/N(i) = M(i')/N(i') = M^*/N^*$. Accordingly, the scale indeterminacy of the competitive equilibrium just means that some firms scale up the optimal mix of techniques by more than others.

Second, observe that the competitive equilibrium of Proposition 1 delivers the same allocation that would be obtained if firms had direct access to the production function $F(bK(i), aL(i))$, with $K(i)$ and $L(i)$ denoting firm i's demand for capital and labor. This follows since the factor-market clearing condition (2.6) applies, and profits are $\pi(i) =$

$F(bK(i), aL(i)) - RK(i) - wL(i)$. Accordingly, firm i's first-order conditions read

$$\begin{aligned} \frac{\partial \pi(i)}{\partial K(i)} &= \frac{\partial F(bK(i), aL(i))}{\partial K(i)} - R = 0, \\ \frac{\partial \pi(i)}{\partial L(i)} &= \frac{\partial F(bK(i), aL(i))}{\partial L(i)} - w = 0. \end{aligned} \tag{3.3}$$

These conditions coincide with those of (3.2) since

$$\frac{\partial F(bK(i), aL(i))}{\partial K(i)} = bF_1(bK(i), aL(i)) = b \cdot \left.\frac{\partial F(M(i), N(i))}{\partial M(i)}\right|_{\substack{M(i)=bK(i) \\ N(i)=aL(i)}},$$

$$\frac{\partial F(bK(i), aL(i))}{\partial L(i)} = aF_2(bK(i), aL(i)) = a \cdot \left.\frac{\partial F(M(i), N(i))}{\partial N(i)}\right|_{\substack{M(i)=bK(i) \\ N(i)=aL(i)}}.$$

Third, notice that the allocation given in Proposition 1 also obtains if we use the representative-firm shortcut where a single competitive firm, equipped with a production function $F(M, N)$ chooses the plan $\left(M, N, [k(m)]_{m\in[0,M]}, [l(n)]_{n\in[0,N]}\right)$ in conjunction with the factor-market clearing conditions $M/b - K = 0$ and $N/a - L = 0$, where M/b and N/a are the respective factor demands for capital and labor. In what follows, we shall apply this shortcut.[4]

3.2 Tasks, Technology, and Factor Prices

This section draws on Proposition 1 and shows that the equilibrium effect of technological change on relative factor prices as well as on absolute factor prices reflects the tension between a *productivity effect* and a *task-demand effect* of opposite sign. This finding appears in the task-based interpretation of the neoclassical production sector expounded in the previous sections because it distinguishes between the demands for tasks and the demands for factors. The task-demand effect leads to a substitution of tasks. It will therefore depend on the elasticity of substitution whereas the productivity effect does not. To highlight this distinction, let us consider a competitive economy with a representative firm equipped with a CES production function

$$F(M, N) = \left[\gamma_M M^{\frac{\sigma-1}{\sigma}} + \gamma_N N^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \tag{3.4}$$

where $\sigma \in (0, \infty]$ is the elasticity of substitution between tasks and $0 < \gamma_M, \gamma_N < 1$ are distribution parameters satisfying $\gamma_M + \gamma_N = 1$.

[4]Little would change in Proposition 1 and its proof if we replaced the production function of (2.1) by any homothetic function $\tilde{F} : \mathbb{R}^2_+ \to \mathbb{R}_+$ where $\tilde{F}(M(i), N(i)) = h(F(M(i), N(i)))$ with $h'(\cdot) > 0 > h''(\cdot)$. In fact, the equilibrium allocation and equilibrium factor price are simply obtained by replacing F by $\tilde{F}$.

3.2.1 Relative Factor Prices

Profit-maximization with respect to M and N leads to the following condition on the relative price of capital

$$\frac{R}{w} = \gamma \left(\frac{b}{a}\right) \left(\frac{M}{N}\right)^{\frac{-1}{\sigma}}, \tag{3.5}$$

or, equivalently,

$$\ln\left(\frac{R}{w}\right) = \ln\gamma + \ln\left(\frac{b}{a}\right) - \frac{1}{\sigma}\ln\left(\frac{M}{N}\right), \tag{3.6}$$

where $\gamma \equiv \gamma_M/\gamma_N$. Both expressions reflect the substitution effect that changing factor prices exert on the relative task intensity chosen by the representative firm. We interpret (3.6) as the (log-linear) inverse of the relative demand for tasks performed by capital. The elasticity of substitution determines the responsiveness of the relative price of capital to changes in M/N, i. e.,

$$\frac{\partial \ln(R/w)}{\partial \ln(M/N)} = -\frac{1}{\sigma}. \tag{3.7}$$

Intuitively, the substitution of tasks depends on the varying conditions the firm faces in the factor markets and on its production function. The former is parameterized by the change in R/w, the latter by σ. A given percentage decline in R/w, induces a large percentage increase in M/N if the substitution is technically easy to accomplish, i. e., if σ is large.

Next, consider the factor market equilibrium for capital and labor. According to Proposition 1, these conditions determine the equilibrium task intensity $M^*/N^* = bK/(aL)$, or equivalently

$$\ln\left(\frac{M^*}{N^*}\right) = \ln\left(\frac{b}{a}\right) + \ln\left(\frac{K}{L}\right). \tag{3.8}$$

Hence, in light of (3.6) and (3.8) the following result concerning the effect of a change in the relative productivity of capital on the equilibrium relative price of capital follows.

Proposition 2 ***(Tasks, Technology, and Relative Factor Prices with Homogeneous Tasks)***

Consider the competitive equilibrium of Proposition 1. Then, it holds that

$$\frac{d\ln(R^*/w^*)}{d\ln(b/a)} = \underbrace{\frac{\partial \ln(R^*/w^*)}{\partial \ln(b/a)}}_{\textit{Productivity Effect}} + \underbrace{\frac{\partial \ln(R^*/w^*)}{\partial \ln(M/N)} \cdot \frac{\partial \ln\left(M^*/N^*\right)}{\partial \ln\left(b/a\right)}}_{\textit{Task-Demand Effect}}. \tag{3.9}$$

According to Proposition 2 the total effect of changing the relative productivity of capital, b/a, on equilibrium relative factor prices, R^*/w^*, can be decomposed into a positive *productivity effect* and a negative *task-demand effect*. This is illustrated in Figure 3.2.

From (3.6) the productivity effect is

$$\frac{\partial \ln(R^*/w^*)}{\partial \ln(b/a)} = 1. \tag{3.10}$$

Given M^*/N^*, the productivity effect captures that an increase in b/a leads to an equi-proportionate increase in the relative marginal product of tasks performed by capital. Accordingly, an equi-proportionate increase in R^*/w^* is called for so that profit-maximizing firms still desire to perform M^*/N^* tasks following the increase in b/a. Observe that the productivity effect leaves the position of the isocost curve shown in Figure 3.1 unchanged. Indeed, one readily verifies that the isocost curve associated with the CES production function of (3.4) is

$$M = \left[\gamma_M^\sigma + \gamma_N^\sigma \left(\frac{bw}{aR}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} - \left(\frac{bw}{aR}\right) N.$$

Hence, adjustments that satisfy (3.10) leave $bw/(aR)$ unchanged.

The second summand on the right-hand side of (3.9) is the *task-demand effect*. It is a general equilibrium effect and the product of two factors. The second factor reflects the adjustment in M/N necessary to maintain full employment of capital and labor following the change in b/a. From (3.8), the implied substitution of tasks is equi-proportionate, i.e., $\partial \ln(M^*/N^*)/\partial \ln(b/a) = 1$. The first factor shows how this substitution of tasks must be supported by changes in relative factor prices so that profit-maximizing firms are willing to implement it. In accordance with (3.7), the more difficult the technical substitution is, i.e., the lower the elasticity of substitution, the stronger is the required adjustment in relative factor prices. Hence, the task-demand effect is equal to

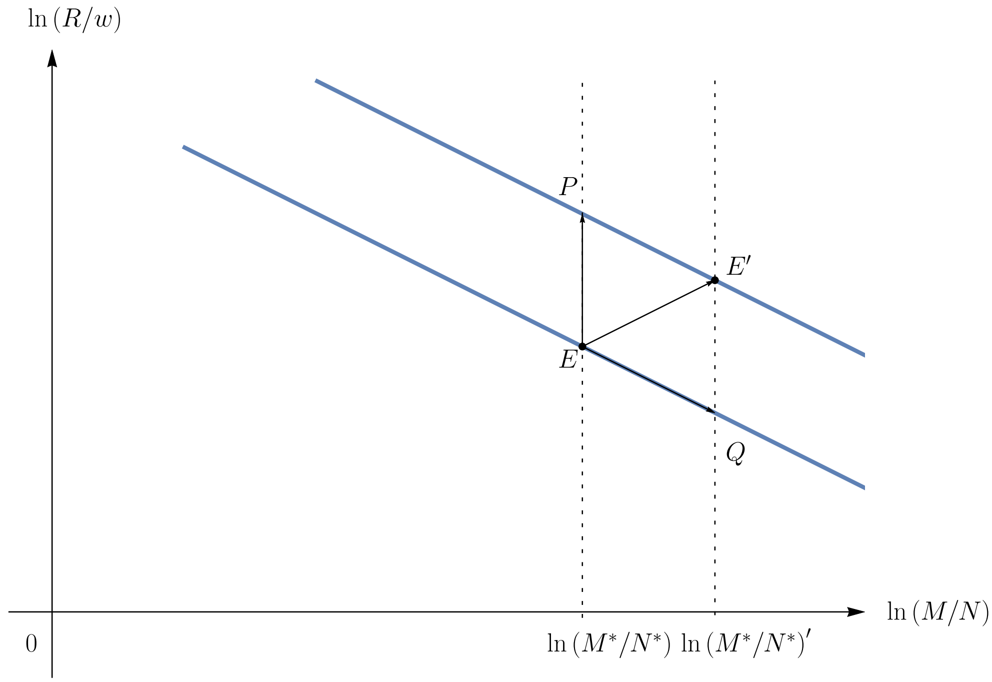
$$\frac{\partial \ln(R^*/w^*)}{\partial \ln(M/N)} \cdot \frac{\partial \ln(M^*/N^*)}{\partial \ln(b/a)} = -\frac{1}{\sigma}. \tag{3.11}$$

In light of (3.10) and (3.11) the total effect is then equal to

$$\frac{d \ln(R^*/w^*)}{d \ln(b/a)} = 1 - \frac{1}{\sigma} \tag{3.12}$$

and reflects the tension between the positive productivity and the negative task-demand effect. While the former leads to an equi-proportionate increase in the relative price of capital, the latter implies a decline in the relative price of capital that may be more or less than equi-proportionate depending on whether the substitution of tasks is difficult ($\sigma < 1$) or easy ($\sigma > 1$). The particularity of the Cobb-Douglas production function is that $\sigma = 1$. In this case, the task-demand effect implies an equi-proportionate decline in

Figure 3.2: **Relative Factor Prices and an Increase in b/a - the Case of $\sigma > 1$.**



Note: The log-linear inverse of the relative demand for tasks performed by capital is shown for two ratios b/a and $(b/a)'$ where $(b/a)' > b/a$. Since $\sigma > 1$ the slope of these relative demands is smaller than 1 in absolute value. The production sector starts at point E with b/a and moves to point E' with $(b/a)'$. Due to the productivity effect the relative demand for tasks shifts upwards from E to P. The task-demand effect implies a downward move along the initial relative demand from E to Q. The total effect is the sum of these effects, i. e., the move from E to E'. Since $\sigma > 1$ we obtain $(R^*/w^*)' > R^*/w^*$.

the relative price of capital such that the total effect of the relative productivity change on relative factor prices is zero.

To sum up, the task-based interpretation of the neoclassical production sector proposed here provides an explicit intuition for why an increase in b/a is biased towards labor if $\sigma < 1$ and towards capital if $\sigma > 1$, i. e.,

$$\frac{d\ln(R^*/w^*)}{d\ln(b/a)} \gtreqless 0 \quad \Leftrightarrow \quad \sigma \gtreqless 1. \tag{3.13}$$

The reason is that σ determines the strength of the response of R^*/w^* to the substitution of tasks necessary to maintain full employment of capital and labor. This response may or may not dominate the productivity effect of a change in b/a.[5]

[5] All results of Section 3.2.1 go through if we allow for decreasing returns to scale and replace the CES

It is worth contrasting this intuition with the one proposed in the canonical neoclassical production sector where a representative competitive firm has access to the CES defined over factors in efficiency units (see, e. g., Acemoglu (2009), Chapter 15). If

$$F(bK, aL) = \left[\gamma_M (bK)^{\frac{\sigma-1}{\sigma}} + \gamma_N (aL)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

then in equilibrium the equivalent of (3.5) is $R^*/w^* = \gamma (b/a)^{(\sigma-1)/\sigma} (K/L)^{-1/\sigma}$. Here, the intuition goes that relative capital-augmenting technical change is biased towards labor if $\sigma < 1$ since an increase of b/a implies a decline in the relative demand of labor. In other words, a fixed amount of capital and labor will find employment only if R/w falls. However, this intuition remains elusive on the economics of why σ determines the sign of $d(R/w)/d(b/a)$. The task-based interpretation introduced in the present paper fills this gap.

3.2.2 Absolute Factor Prices

Next, consider the effect of factor productivity on the absolute level of equilibrium factor prices.[6] Here, we confine attention to the rental rate of capital. An analogous argument applies to the equilibrium real wage. From (3.2) and factor market clearing, the equilibrium rental rate of capital satisfies

$$R^* = b \cdot \frac{\partial F(M^*, N^*)}{\partial M}, \quad \text{where } M^* = bK \text{ and } N^* = aL. \tag{3.14}$$

Hence, the effect of changing the productivity of capital or the one of labor on R^* is

$$\begin{aligned} \frac{dR^*}{db} &= \frac{\partial F(M^*, N^*)}{\partial M} + b \cdot \frac{\partial^2 F(M^*, N^*)}{\partial M^2} \frac{\partial M^*}{\partial b} \gtreqless 0, \\ \frac{dR^*}{da} &= b \cdot \frac{\partial^2 F(M^*, N^*)}{\partial M \partial N} \cdot \frac{\partial N^*}{\partial a} > 0. \end{aligned} \tag{3.15}$$

The first line suggests that the effect of b on R^* can be decomposed into two effects of opposite sign. The first is a positive productivity effect. As b increases, the effect on R^* is equal to the marginal product of task M^*, which is positive. The second is a task-demand effect. As b increases, the demand for tasks performed by capital must increase to maintain full employment. Since these tasks are associated with a lower marginal product, the task-demand effect is negative. For the CES of (3.4) it is straightforward to show that the task-demand effect may dominate the productivity effect for values of the

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of (3.4) by $\tilde{F}(M, N) = [F(M, N)]^{\nu}$, where $0 < \nu < 1$. Since $\tilde{F}(M, N)$ is homothetic, Proposition 1 continues to hold (see, Footnote 4).

[6] See Irmen (2014) for an analysis of the effect of factor-augmenting technical change on factor prices in the canonical neoclassical production sector.

elasticity of substitution that are sufficiently smaller than one. More precisely, it holds that

$$\frac{dR^*}{db} \gtreqless 0 \quad \Leftrightarrow \quad \sigma \gtreqless \frac{1}{\gamma\left(\frac{M^*}{N^*}\right)^{\frac{\sigma-1}{\sigma}}+1}. \tag{3.16}$$

For instance, if $\gamma_M = \gamma_N = \sigma = 1/2$ we have $dR^*/db < 0$ for $M^*/N^* > 1$.

The second line of (3.15) shows that the effect of a on R^* works only through a positive task-demand effect. A higher productivity of labor means that N^* must increase to maintain full employment of labor. Then, the complementarity of both tasks in the neoclassical production function implies that R^* unequivocally increases.[7]

4 Heterogeneous Tasks

Let us relax Assumption 1 and allow for tasks that are heterogeneous with respect to their input requirements.

Assumption 2 ***(Heterogeneous Tasks)*** *$b(m) = be^{-\delta m}$ and $a(n) = ae^{-\delta n}$, where $\delta > 0$.*

Assumption 2 has at least three economic implications. First, tasks are ordered so that low values of m and n correspond to a high factor productivity in the performance of the respective task. In other words, tasks with a high marginal product are those where the factor productivity is high. Second, the parameter δ determines how fast these input requirements increase in m or n. Hence, δ is a measure of the degree of heterogeneity in terms of diminishing returns including the case of homogeneous tasks with $b(m) = b$ and $a(n) = a$ for $\delta = 0$. Third, increasing b or a means a higher productivity of the respective factor across all tasks. Hence, b or a are factor-augmenting with the caveat that their productivity enhancing effect is higher for tasks in which the respective factor is already more productive, i. e., $\partial^2 b(m)/\partial b \partial m < 0$ and $\partial^2 a(n)/\partial a \partial n < 0$.

4.1 The Competitive Equilibrium

Firm i's factor costs (2.4) are now equal to

$$C(i) = \frac{R}{b}\int_0^{M(i)} e^{\delta m} dm + \frac{w}{a}\int_0^{N(i)} e^{\delta n} dn. \tag{4.1}$$

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[7]With slight modifications, the results of Section 3.2.2 go through under decreasing returns to scale as suggested in Footnote 5. As to the sign of dR^*/db, we have to replace γ on the right hand side of (3.16) by $\tilde{\gamma} \equiv \nu\gamma$. The sign of dR^*/da in (3.15) is preserved as long as $\sigma < 1/(1-\nu)$.

Using the latter in (2.3) leads to the first-order (sufficient) conditions for an interior solution

$$\begin{aligned} \frac{\partial \pi(i)}{\partial M(i)} &= \frac{\partial F\left(M(i), N(i)\right)}{\partial M(i)} - \frac{R}{b} e^{\delta M(i)} = 0, \\ \frac{\partial \pi(i)}{\partial N(i)} &= \frac{\partial F\left(M(i), N(i)\right)}{\partial N(i)} - \frac{w}{a} e^{\delta N(i)} = 0, \end{aligned} \tag{4.2}$$

for all $i \in [0, 1]$ which extends (3.2) to the case of $\delta > 0$. Intuitively, the cost of the respective marginal task increases in $M(i)$ and $N(i)$ reflecting higher input requirements for the marginal than for inframarginal tasks. In view of Definition 1 the following proposition obtains.

Proposition 3 ***(Competitive Equilibrium with Heterogeneous Tasks)***

There is a unique competitive equilibrium involving aggregate quantities of tasks

$$M^* = \frac{\ln(1+\delta bK)}{\delta} \quad \text{and} \quad N^* = \frac{\ln(1+\delta aL)}{\delta},$$

strictly positive factor prices

$$R^* = \frac{b}{e^{\delta M^*}} \left(\frac{\partial F\left(M^*, N^*\right)}{\partial M} \right) > 0 \quad \text{and} \quad w^* = \frac{a}{e^{\delta N^*}} \left(\frac{\partial F\left(M^*, N^*\right)}{\partial N} \right) > 0,$$

and aggregate output equal to

$$\frac{F\left(\ln(1+\delta bK), \ln(1+\delta aL)\right)}{\delta}.$$

All firms produce at the same scale, i. e., $M^*(i) = M^*$ and $N^*(i) = N^*$.

Proposition 3 exploits the fact that the marginal cost of tasks increases under Assumption 2. Therefore, $\pi(i)$ is strictly concave in $(M(i), N(i))$, and the equilibrium scale of each firm can be pinned down. Since a higher δ increases the input requirements of both factors, we have $\partial M^*/\partial\delta < 0$ and $\partial N^*/\partial\delta < 0$. As a consequence, aggregate output increases as δ declines with $\lim_{\delta\to 0} F\left(\ln(1+\delta bK), \ln(1+\delta aL)\right)/\delta = F\left(bK, aL\right)$.[8]

The equilibrium allocation of Proposition 3 coincides with the one obtained if firms had direct access to the production function $F\left(\ln\left(1 + bK(i)\right), \ln\left(1 + aL(i)\right)\right)/\delta$, where $K(i)$ and $L(i)$ denote firm demands for capital and labor. However, here the efficient amount of capital (labor) of the economy is equal to the equilibrium amount of tasks performed by capital (labor) and no longer equal to bK or aL.[9]

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[8]Observe that $F\left(\ln(1+\delta bK), \ln(1+\delta aL)\right)/\delta$ is not homothetic in (bK, aL). Therefore, the findings on technical change and factor prices are quite different from and have different intuitions than those stated in the Footnotes 4 - 7 for a CES with decreasing returns to efficient factor supplies.

[9]Notice that the 'number' of firms will affect aggregate output and factor prices since capital and labor exhibit diminishing returns in the accomplishment of tasks. A larger number of firms implies that each firm operates at a smaller scale. The equilibrium productivity of capital and labor will then be higher and aggregate output increases. In what follows we neglect this complication to facilitate the comparison with the results derived in Section 3.

4.2 Tasks, Technology, and Factor Prices

This section studies the relationship between factor prices and the technology represented by the parameters b and a. The results are then compared to those derived for homogeneous tasks in Section 3. To facilitate this comparison we focus on the CES of (3.4).

4.2.1 Relative Factor Prices

Evaluated at the competitive equilibrium of Proposition 3, the first-order conditions (4.2) deliver the relative price of capital in equilibrium as

$$\begin{aligned} \frac{R^*}{w^*} &= \gamma \left(\frac{b\,(M^*)}{a\,(N^*)} \right) \left(\frac{M^*}{N^*} \right)^{\frac{-1}{\sigma}}, \\ &= \gamma \left(\frac{b}{a} \right) \left(\frac{e^{-\delta M^*}}{e^{-\delta N^*}} \right) \left(\frac{M^*}{N^*} \right)^{\frac{-1}{\sigma}}. \end{aligned} \tag{4.3}$$

Hence, the relative price of capital is the product of the distribution parameter, the relative productivity of the *marginal* tasks, and the factor reflecting the substitutability of tasks. Equation (4.3) generalizes equation (3.5) derived for homogeneous tasks to the case where $\delta > 0$. The comparison of these expressions gives rise to two observations. First, it is the relative productivity of the marginal tasks, $b\,(M^*)\,/a\,(N^*)$, that matters. If $\delta = 0$ this ratio boils down to b/a. Second, the heterogeneity of tasks does not affect the firm's willingness to substitute tasks of the two types, i. e., the factor $(M^*/N^*)^{-1/\sigma}$ is the same as in (3.5). It will, however, influence the effect of changes in b and a on the adjustment of M^* and N^* through the task-demand effect. Indeed, with Proposition 3 one readily verifies that

$$\frac{\partial \ln M^*}{\partial \ln b} < 1 \quad \text{and} \quad \frac{\partial \ln N^*}{\partial \ln a} < 1. \tag{4.4}$$

Unlike for homogeneous tasks, the response of M^* and N^* to percentage changes of the respective productivity parameters is less than proportionate reflecting increasing factor requirements of the additional tasks necessary to maintain full employment.

To study the effect of b and a on the relative price of capital, we take logs on both sides of (4.3) and obtain

$$\ln\left(\frac{R^*}{w^*}\right) = \ln\gamma + \ln\left(\frac{b}{a}\right) - \delta\,(M^* - N^*) - \frac{\ln\,(M^*/N^*)}{\sigma}. \tag{4.5}$$

Due to the presence of diminishing returns in the accomplishment of tasks, the inverse of the relative demand for tasks performed by capital is no longer log-linear. Nevertheless, there is a relationship between R^*/w^* and the productivity levels b and a.

Proposition 4 ***(Tasks, Technology, and Relative Factor Prices with Heterogeneous Tasks)***

Consider the competitive equilibrium of Proposition 3. Then, it holds that

$$\begin{aligned}\frac{d\,(R^*/w^*)}{d\ln b} &= \left(1-\delta M^*\frac{\partial \ln M^*}{\partial \ln b}\right)-\frac{1}{\sigma}\frac{\partial \ln M^*}{\partial \ln b},\\ \frac{d\,(R^*/w^*)}{d\ln a} &= \left(-1+\delta N^*\frac{\partial \ln N^*}{\partial \ln a}\right)+\frac{1}{\sigma}\frac{\partial \ln N^*}{\partial \ln a}.\end{aligned} \tag{4.6}$$

Moreover, there are critical values $\bar{\sigma}_b = \bar{\sigma}_b(\delta) > 1$ with $d\bar{\sigma}_b/d\delta > 0$ and $\bar{\sigma}_a = \bar{\sigma}_a(\delta) > 1$ with $d\bar{\sigma}_a/d\delta > 0$ such that

$$\begin{aligned}\frac{d\ln(R^*/w^*)}{d\ln b} &\gtreqless 0 \qquad \Leftrightarrow \qquad \sigma \gtreqless \bar{\sigma}_b,\\ \frac{d\ln(R^*/w^*)}{d\ln a} &\gtreqless 0 \qquad \Leftrightarrow \qquad \sigma \lesseqgtr \bar{\sigma}_a.\end{aligned} \tag{4.7}$$

The intuition behind the first finding of Proposition 4 comes in terms of the productivity and the task-demand effect identified before. To see this consider the two effects of b on R^*/w^* as stated in (4.6). The productivity effect is given by the expression in parentheses and has now two components. The first component is a proportionate effect since b increases the productivity of capital in all performed tasks. The second component captures the interaction between the task productivity and the task-demand effects for tasks performed by capital. For each task performed by capital a higher b reduces the capital input. Hence, given M^* the demand for capital falls short of its supply. In other words, to maintain full employment of capital the amount of tasks performed by capital has to increase. Accordingly, the productivity of capital in the marginal task declines and so does R^*. The second term in the first line of (4.6) reflects the task-demand effect that leads to a substitution of tasks. In line with (4.4), this effect is weaker than under homogeneous tasks since M^* increases less than proportionately. Mutatis mutandis, this intuition also explains the effect of a on R^*/w^*.

The results stated in (4.7) show that the simple rule (3.13) must be modified to account for task heterogeneity.[10] In fact, with heterogeneous tasks, a percentage change in b may decrease R^*/w^* even if $\sigma > 1$ since $\bar{\sigma}_b > 1$. Moreover, the range of σ for which this is possible increases in the degree of heterogeneity parameterized by δ. This suggests that the response to changes in δ of the relative factor productivity across tasks is more pronounced than the one through the task-demand effect. In the same vein, a percentage

[10] Observe that $d\ln(R/w)/d\ln(b/a) = d\ln(R/w)/d\ln b = -d\ln(R/w)/d\ln a$ since $d\ln(b/a) = d\ln b - d\ln a$. Therefore, the rule stated in (3.13) also applies to $d\ln(R/w)/d\ln b$ and with opposite sign to $d\ln(R/w)/d\ln a$.

increase in a may increase R^*/w^* even for $\sigma > 1$. Finally, taking the limit $\delta \to 0$ in (4.6) delivers

$$\left.\frac{d\ln\left(R^*/w^*\right)}{d\ln b}\right|_{\delta=0} = 1 - \frac{1}{\sigma} \quad \text{and} \quad \left.\frac{d\ln\left(R^*/w^*\right)}{d\ln a}\right|_{\delta=0} = -1 + \frac{1}{\sigma}, \tag{4.8}$$

i. e., for $\delta > 0$ but sufficiently close to zero (3.13) remains a good approximation.

4.2.2 Absolute Factor Prices

To elicit the effect of factor productivity on the absolute level of equilibrium factor prices consider the first-order conditions (4.2) and factor market clearing. The equilibrium rental rate of capital satisfies

$$R^* = b(M^*) \cdot \frac{\partial F\left(M^*, N^*\right)}{\partial M}. \tag{4.9}$$

Hence, the effect of changing b or a on R^* is

$$\begin{aligned} \frac{dR^*}{db} &= \left(\frac{\partial b(M^*)}{\partial b} + \frac{\partial b(M^*)}{\partial M} \cdot \frac{\partial M^*}{\partial b}\right) \cdot \frac{\partial F\left(M^*, N^*\right)}{\partial M} + b(M^*) \cdot \frac{\partial^2 F\left(M^*, N^*\right)}{\partial M^2} \cdot \frac{\partial M^*}{\partial b} \gtreqless 0, \\ \frac{dR^*}{da} &= b(M^*) \cdot \frac{\partial^2 F\left(M^*, N^*\right)}{\partial M \partial N} \cdot \frac{\partial N^*}{\partial a} > 0. \end{aligned} \tag{4.10}$$

As for homogeneous tasks, the total effect of b on R^* can be decomposed into a productivity effect and a task-demand effect of opposite sign. The comparison with (3.15) reveals a new channel that dampens the productivity effect: as M^* increases in response to a higher b, the productivity of the marginal task declines, hence, R^* falls. For the CES of (3.4) this leads to

$$\frac{dR^*}{db} \gtreqless 0 \quad \Leftrightarrow \quad \sigma \gtreqless \frac{\partial \ln M^*}{\partial \ln b}\left(\frac{1}{\gamma\left(\frac{M^*}{N^*}\right)^{\frac{\sigma-1}{\sigma}} + 1} + \delta\right), \tag{4.11}$$

which extends (3.16) to $\delta > 0$. This expression is hard to sign in general. However, since $\partial \ln M^* / \partial \ln b < 1$ it becomes feasible that an increase in b decreases R^* even if $\sigma > 1$. For instance, if $\gamma_M = \gamma_N = aL = 1$, $bK = 2$, and $\delta \geq 2$ then there is a critical value $\hat{\sigma}_b > 1$ such that $dR^*/db < 0$ for all $\sigma < \hat{\sigma}_b$. For this set of parameters Table 1 gives some critical values $\hat{\sigma}_b$ for varying values of δ. It suggests that a higher degree of task heterogeneity makes it easier for $dR^*/db < 0$ to occur even for $\sigma > 1$. However, taking the limit $\delta \to 0$ in (4.11) delivers (3.16), i. e., the latter remains a good approximation for $\delta > 0$ but sufficiently close to zero.

Finally observe that the direction of the effect of a on R^* remains positive and driven by the complementarity of tasks in F. The degree of heterogeneity may affect the strength but not the direction of this effect.

Table 1: Critical values $\hat{\sigma}_b$ for changing values of δ.

δ	2	3	4	5	8	10
$\hat{\sigma}_b$	1.23	1.53	1.8	2.07	2.8	3.27

5 Concluding Remarks

This paper introduces tasks in a straightforward way into the neoclassical production sector. Section 2 develops the general framework. Its application leads to novel insights on the intricate relationship between factor-augmenting technical progress and (relative and absolute) factor prices. The focus on tasks suggests that factor-augmenting technical progress gives rise to a tension between a positive productivity effect and a negative task-demand effect. In common parlance, the former is a direct effect, the latter a general equilibrium effect.

If tasks are homogeneous and the production function is of the CES-type then profit-maximizing behavior delivers a log-linear (inverse) relative demand for tasks, and factor markets imply a log-linear relationship between the equilibrium relative demand for tasks and relative efficient factor supplies. Then, the productivity effect is equi-proportionate whereas the elasticity of substitution determines whether the task-demand effect is more or less than equi-proportionate. The tension between a productivity and a task-demand effect also drives the effect of technical progress on absolute factor prices. Here, an increase in the productivity of one factor may reduce the factor price of this factor if the elasticity of substitution is sufficiently smaller than unity.

If tasks are heterogeneous and the production function is of the CES-type then the (inverse) relative demand for tasks is no longer log-linear. Nevertheless, the tension between the productivity effect and the task-demand effect remains alive. The productivity effect hinges on the relative productivity of the marginal tasks. Adjustments in factor markets that change the equilibrium number of tasks will affect absolute and relative equilibrium factor prices. Factor-augmenting technical change requires a less than proportionate increase in the equilibrium number of tasks since the per-task input requirements increase. As a consequence, the relative price of capital may decrease in response to an increase in the factor augmenting productivity of capital (labor) even if the elasticity of substitution is greater than unity. The tension between a productivity and a task-demand effect also drives the effect of technical progress on absolute factor prices. Here, an increase in the productivity of one factor may reduce the factor price of this factor if the elasticity of substitution is sufficiently smaller than unity.

6 Appendix: Proofs

6.1 Proof of Proposition 1

Consider firm i's demand for capital. With (2.5), we have $K(i) = M(i)/b$. Hence, for $R > 0$, (2.6) delivers $\int_0^1 K(i)di = \int_0^1 M(i)/b = K$. Hence, $\int_0^1 M(i)di = bK = M^*$. Similarly, one derives for labor and $w > 0$ that $\int_0^1 N(i)di = aL = N^*$. Next consider the first-order conditions (3.2). Since the derivatives $\partial F(M(i), N(i))/\partial M(i)$ and $\partial F(M(i), N(i))/\partial N(i)$ are homogeneous of degree zero, these conditions imply $M(i)/N(i) = M(i')/N(i')$ for all $i, i' \in [0,1]$ and $i \neq i'$. Therefore, it holds that $M(i)/N(i) = [M(i) + M(i')] / [N(i) + N(i')]$ and

$$\left(\frac{M(i)}{N(i)}\right)^* = \frac{\int_0^1 M(i)di}{\int_0^1 N(i)di} = \frac{bK}{aL} = \frac{M^*}{N^*}. \tag{6.1}$$

Accordingly, the equilibrium task intensity, the aggregate equilibrium output, and the equilibrium factor prices are as stated in Proposition 1. Since M^* and N^* are positive and finite, we have $R^* > 0$ and $w^* > 0$. Since $\pi(i)$ is homogenous of degree one in $(M(i), N(i))$, the absolute levels $M^*(i)$ and $N(i)^*$ remain indeterminate. ■

6.2 Proof of Proposition 2

In equilibrium (3.6) becomes

$$\ln\left(\frac{R^*}{w^*}\right) = \ln\gamma + \ln\left(\frac{b}{a}\right) - \frac{1}{\sigma}\ln\left(\frac{M^*}{N^*}\right).$$

In light of (3.8) the proposition follows. ■

6.3 Proof of Proposition 3

We start proving that $M(i) = M(j) = M^*$ and $N(i) = N(j) = N^*$ for all $i, j \in [0,1]$ and $i \neq j$ holds in equilibrium. Since $C(i) = C(M(i), N(i))$ is strictly convex, profits $\pi(i) = \pi(M(i), N(i))$ are strictly concave. Assume to the contrary that two firms i and j choose $(M(i), N(i)) \neq (M(j), N(j))$. This is only possible if $\pi(M(i), N(i)) = \pi(M(j), N(j))$. Now, consider $(M, N) = \mu(M(i), N(i)) + (1-\mu)(M(j), N(j))$ for some $\mu \in (0,1)$. Since profits are strictly concave we have $\pi(M, N) > \mu\pi(M(i), N(i)) + (1-\mu)\pi(M(j), N(j))$. Therefore, $\pi(M, N) > \pi(M(i), N(i)) = \pi(M(j), N(j))$ which is a contradiction. Hence, for all $i \in [0,1]$ we must have $(M(i), N(i)) = (M, N)$.

As a consequence, the aggregate demand for capital of all firms is $\int_0^M e^{\delta m} dm/b = \left(e^{\delta M} - 1\right)/(\delta b)$. Hence, for $R > 0$, (2.6) delivers M^* as indicated in the proposition. Similarly, one derives N^*. The first-order conditions (4.2) deliver R^* and w^* as indicated. Since $F(\cdot)$ has constant returns to scale, $F(M^*, N^*) = F(\ln(1+\delta bK), \ln(1+\delta aL))/\delta$. ■

6.4 Proof of Proposition 4

The statements of equation (4.6) are immediate from (6.2). Computing the respective effects delivers with $z \equiv \delta b K$

$$\delta M^* \frac{\partial \ln M^*}{\partial \ln b} = \frac{z}{1+z} \quad \text{and} \quad \frac{1}{\sigma} \frac{\partial \ln M^*}{\partial \ln b} = \frac{1}{\sigma} \left(\frac{z}{1+z} \right) \left(\frac{1}{\ln(1+z)} \right). \tag{6.2}$$

It follows that $\bar{\sigma}_b = z/(\ln(1+z)) > 1$ which is increasing in z, hence, also in δ. As to labor, let $x \equiv \delta a L$. Then,

$$\delta N^* \frac{\partial \ln N^*}{\partial \ln a} = \frac{x}{1+x} \quad \text{and} \quad \frac{1}{\sigma} \frac{\partial \ln N^*}{\partial \ln a} = \frac{1}{\sigma} \left(\frac{x}{1+x} \right) \left(\frac{1}{\ln(1+x)} \right). \tag{6.3}$$

It follows that $\bar{\sigma}_a = x/(\ln(1+x)) > 1$ which is increasing in x, hence, also in δ. ■

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