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Cloé Garnache, Pierre Mérel

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Environmental Policy in General Equilibrium: New Insights from a Canonical Model

Abstract

This paper derives the incidence of a pollution tax in a stylized general equilibrium framework, building on previous work by Fullerton and Heutel (2007a). Using the CPI as numeraire, we show that tax incidence is a simpler problem than previously thought, and that general insights can be derived without the need to restrict the parameter space. In addition, the counterintuitive possibility that an increase in the tax could increase the pollution level vanishes. The choice of the CPI as numeraire is further justified by the fact that environmental taxes, notably carbon taxes, are typically indexed on inflation.

JEL-Codes: Q520, H230, H220.

Keywords: tax incidence, general equilibrium, relative prices, numeraire.

Cloé Garnache
Department of Economics
University of Oslo / Norway
cloe.garnache@econ.uio.no

Pierre Mérel
Department of Agricultural and Resource Economics, University of California
Davis / USA
merel@primal.ucdavis.edu

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1 Introduction

It has been known since the seminal work of Muth (1964), and later confirmed through the more general results of Heiner (1982) and Braulke (1984), that under fairly general conditions, the law of input demand holds at the industry level even when the prices of outputs or other inputs are allowed to adjust in related markets in response to the input price increase. An interpretation of this law is that an exogenous tax on an industry input reduces the overall use of that input, even if certain firms end up using more of it. That is, demand and "derived demand" both slope down in the aggregate. If the taxed input causes pollution, pollution will unambiguously be reduced by an increase in the tax.[1] This simple economic logic lies at the core of environmental taxation (Baumol and Oates, 1988).

While the literature cited above explicitly allows for the presence of markets besides the one subject to taxation, they rely on partial equilibrium approaches. In seminal contributions to the analysis of the incidence of environmental policy, Fullerton and Heutel (2007a) and Fullerton and Heutel (2010b) show that the desirable effects of environmental taxation on pollution may no longer apply in a general equilibrium context, even in a closed economy and under the assumption of homogenous spending propensities on the

[1]The result holds as long as the supply of the dirty input is less than infinitely inelastic. If it is infinitely inelastic, input use remains constant, but it does not increase.

demand side.[2] That is, an increase in the tax on a polluting input (e.g., carbon) may in some cases lead to more, not less, pollution. Similarly, in a cap-and-trade system, a decrease in the cap on a polluting input could lead to a decrease in the permit price.

Fullerton and Heutel (2007a) suggest that the ambiguity regarding the applicability of the law of input demand to polluting inputs is due to general equilibrium feedback effects.[3] Here, we argue that whenever the law of input demand is violated for pollution, the equilibrium in their model is unstable in the Marshallian sense.[4] The occurrence of such instability, which may seem tangential to the issue of policy incidence, is in fact closely related to it as all comparative statics, not simply that on the pollution outcome, are reversed in this case, notably those pertaining to relative returns to labor and capital. Thus, the inability to rule out equilibrium instability precludes general conclusions to be drawn regarding policy incidence.

The present paper offers a solution to this problem. Specifically, we show that the existence of unstable equilibria is an artefact of the choice of numeraire good. Due to the non-market nature of pollution, the tax on the polluting input is modeled as a nominal, rather than *ad valorem*, tax. As a result, the effects of this nominal tax on goods provision, resource allocation, and relative prices implicitly depend on which good is chosen as numeraire, as the tax increases the price of the polluting input *relative to that of the numeraire*. Said differently, there are as many *varieties* of environmental taxes as choices of numeraire good. Note that the issue is not about the undeterminacy surrounding the overall magnitude of prices, but, since the model is used to *compare* equilibria, that an *increase* in the nominal tax rate increases the price of pollution relative to the price of the numeraire, while potentially decreasing it relative to other prices.[5] This fundamental

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[2]In earlier work, Mieszkowski (1967) relaxes the assumption of identical spending propensities among owners of capital and owners of labor and shows that a series of counterintuitive comparative statics may ensue.

[3]See, for instance, the discussion in footnote 14 of Fullerton and Heutel (2007a). Note that the numerical example provided in that footnote violates the negative semidefiniteness property of the substitution matrix in sector Y. For an example of model parameterization leading to the counterintuitive outcome, see Appendix A.6.

[4]Put simply, Marshallian instability means that if the demand price of a good exceeds the supply price of that good, quantity will move further away from its equilibrium value.

[5]In earlier studies by Bovenberg and Goulder (1997) and De Mooij and Bovenberg (1998) that also consider a pollution tax, capital, the polluting input, and a final consumption good are supplied to/demanded from a small economy in an infinitely elastic fashion, with fixed rates of exchange on the world market. As a result, a tax on the dirty input increases the price of that input relative to that of capital and that of the consumption good. The only price determined endogenously in the model is the wage rate. A government budget constraint imposes that a rise in the pollution tax be matched by a decrease in either the tax on labor or that on capital, giving rise to the possibility of an increase in pollution through an expansion in output. The mechanism behind the increase in pollution in these earlier models is thus distinct from that in

dependency of outcomes on numeraire choice is not present when considering taxation of a market good, as in that case the tax creates a wedge between the equilibrium prices of the good faced by different "users" (e.g., supply vs. demand prices for a tax on a consumption good, or prices of a factor across different industries in the case of a partial factor tax as in Harberger (1962)). While the "users" prices themselves depend on numeraire choice, their relationship to each other does not, nor do their relationships with the prices of other goods in the economy. In contrast, a non-market good such as pollution only has a single "user" and a supply price of zero that does not adjust to changes in numeraire, hence the dependency of real outcomes on numeraire choice.[6]

The question then arises as to which good should be chosen as numeraire when modeling the effects of a tax on a non-market good. An equivalent way to frame that question is to imagine that instead of being expressed in nominal terms, such tax is expressed in *ad valorem* terms relative to the equilibrium value of some market good. Because the prices of market goods respond to changes in numeraire choice in ways that preserve their relationships with each other, anchoring the pollution tax to the price of a market good removes the dependency of real tax outcomes on numeraire choice. Note that instead of being anchored to the price of a single good, the tax could also be anchored to a price index reflecting the average level of prices across a set of goods.[7] In that sense, the choice of which *variety* of environmental taxation to model in general equilibrium may seem overwhelming.

We argue that a reasonable anchoring price index should be one for which the law of input demand is satisfied, that is, an increase in the price of the dirty input relative to the equilibrium value of the index unambiguously leads to a decrease in input use—and attendant pollution. Such requirement amounts to ruling out equilibria that are unstable in the Marshallian sense and thus ensures that meaningful comparative statics can be derived (Samuelson, 1941). With this constraint in mind, we consider two classes of price indices: those defined over the prices of consumption goods (uses side) and those defined over factor prices (sources side). Using the model of Fullerton and Heutel (2007a), we show that within each class, there exists a unique index that satisfies the stability constraint. We express each index using underlying parameters of the economy. On the uses side, the price index is identical to an inflation index, namely the product of the prices of consumption goods raised to their respective expenditure shares. On the sources side, the

Fullerton and Heutel (2007a) and Fullerton and Heutel (2010b).

[6]Of course, the polluting input could be used by more than one industry, but the point is that a tax on that input creates a positive price, that all industries face.

[7]A similar point is made in Baylis et al. (2014, pg. 63).

price index is equal to the product of factor prices raised to their respective income shares. Importantly, none of these indices depend on behavioral parameters such as substitution elasticities, either in production or consumption.[8] Therefore, a government could anchor the pollution tax to either index without unrealistic informational requirements. For instance, the pollution tax could be tied to the Consumer Price Index (CPI). With a pollution tax so designed, not only is the effect of a rise in the tax in accordance with the law of input demand and the intent of environmental taxation, but the tax incidence results discussed by Fullerton and Heutel (2007a) in the context of specific parameter values are also shown to hold generally.[9] Furthermore, because our analysis does not require focussing on special cases, we are able to provide general results regarding incidence on the uses side that were left unexplored in the original study.

Specifically, we make explicit the conditions under which the pollution tax may lead to a reduction in the price of the dirty good relative to that of the clean good, a possibility that has been recognized, but not elucidated, in Fullerton and Heutel (2010b), and overlooked in much of the literature. Indeed, incidence studies focussing on the uses side generally take it as a premise that the price of the dirty good will rise relative to that of the clean good, and then compare impacts across groups purchasing these goods in varying proportions (Grainger and Kolstad, 2010; Cronin et al., 2019; Fullerton and Muehlegger, 2019). But if the price of the dirty good decreases relative to that of the clean good, the burden will fall disproportionately on those who purchase relatively less, not more, of the dirty good. We show that this counterintuitive result can arise if the factor used more intensively by the clean industry (say capital) is a better substitute for pollution than the other factor (say labor), and the price of the former rises relative to that of the latter as a result of the pollution tax.

Although fairly recent, the contribution of Fullerton and Heutel (2007a) has had an undeniable influence in the environmental economics literature. Their model has been used in Fullerton and Heutel (2007b), Fullerton and Heutel (2010b), Fullerton and Heutel (2010a), and Fullerton et al. (2011). It has been modified or extended in further contributions by Fullerton and Monti (2013), Baylis et al. (2013), Rivers (2013), Baylis et al. (2014), Rausch and Schwarz (2016), and Goulder et al. (2016). Their analysis has also provided

[8] Our model assumes homothetic, but not necessarily Cobb-Douglas, consumer preferences.

[9] The special cases analyzed in Fullerton and Heutel (2007a) and Fullerton and Heutel (2010b) are quite restrictive. Fullerton and Heutel (2007a) focus on the case where the clean and dirty industries have equal factor intensities and the case where labor and capital are equally substitutable for pollution in the dirty industry. In their study of a pollution cap, Fullerton and Heutel (2010b) analyze the special case of equal factor intensities, along with the case of zero substitution across inputs in the dirty sector (see their appendix).

needed clarity on important drivers of the incidence of environmental policy in studies that involve more complex and realistic models of the economy, e.g., Rausch et al. (2011), Rausch and Mowers (2014), or Marten et al. (2019), and in policy discussions, e.g., Morris and Munnings (2013).

Our paper makes four contributions to the literature. First, we show that well-designed pollution taxes need not have ambiguous effects on equilibrium pollution levels in general equilibrium, and that counterintuitive pollution outcomes identified in previous literature imply unstable equilibria. Second, we ask whether there exist anchoring prices that avoid equilibrium instability altogether. We show that the answer is yes, and that, reassuringly perhaps, the set of candidate indices is limited. On the uses side, the only index satisfying the stability constraint for all parameter values is equivalent to the CPI. On the sources side, it is equivalent to a producer price index for primary production factors. Third, using these indices, we demonstrate that tax incidence results previously derived in special cases hold in fact quite generally. We also provide new and general results pertaining to incidence on the uses side. Fourth, we empirically demonstrate that the choice of numeraire in general equilibrium models influences comparative statics with respect to nominal taxes on non-market goods like pollution. Notably, using a model calibrated to the US economy, we show that predicted pollution impacts may differ by up to 40% depending on the choice of anchoring price, and that this choice acts as an essential driver of predicted incidence on the uses side. The fact that our model is extremely stylized suggests that even larger discrepancies could occur in richer models where the number of candidate numeraire goods is larger.

In terms of policy implications, we note that many recent policy proposals for a U.S. carbon tax already choose to index the tax to inflation, albeit for reasons likely unrelated to the results discussed here (Marron et al., 2015; Aldy, 2016; Vail and Burtraw, 2016; Metcalf, 2018; Climate Leadership Council, 2019; H.R. 763, 2019; H.R. 3966, 2019; H.R. 4058, 2019; S. 1128, 2019; S. 2368, 2018; S. 2284, 2019).[10] Furthermore, carbon taxes indexed to inflation have already been implemented in a number of countries, including Denmark and Sweden (Withana et al., 2013), Iceland (PMR, 2017), Norway (Haites et al., 2018), Chile, Colombia, and Mexico (UN, 2019), the Netherlands (Andersen et al., 2007), South Africa (Act No. 15, 2019), and the United Kingdom (National Audit Office, 2007).[11] The arguments laid out in the present paper serve to strengthen the case for such indexation. Our simulation results

[10]Inflation adjustments would be on top of any ramping up of the tax meant to increase pollution reduction incentives over time.

[11]California and Québec operate a cap-and-trade program rather than a carbon tax. Yet, the auction price floor and price ceiling are both indexed to inflation (ICAP, 2019b,a).

further suggest that accurately modeling the effects of such policies in general equilibrium requires special attention to the choice of anchoring price.

Although our general equilibrium analysis is of direct relevance to environmental policy, our findings on the comparative statics of tax incidence do not rely on any assumption regarding the actual external effects of the “dirty” input on technology or consumer utility; all that is required for our results is that this input be a non-market good available to the economy in sufficiently large quantity. As such, the framework conceptually applies to other settings, for instance per-period restrictions on the extraction of a natural resource for the purposes of conservation or sustainability.

2 Model and notation

We borrow the assumptions of Fullerton and Heutel (2007a), which build upon the seminal model of Harberger (1962). The model is parsimonious and aims to capture the essential drivers and overall magnitude of environmental tax incidence, rather than provide accurate predictions for a given economy, as would a more detailed computable general equilibrium approach. A notable advantage is that it can be solved analytically, providing a “model of the model” (Fullerton and Heutel, 2010a). Relatedly, Fullerton and Ta (2019) demonstrate that a stylized and analytically solvable model of the US economy delivers quantitative predictions on the effects of a carbon tax that are not far from those obtained from the detailed CGE model developed by Goulder and Hafstead (2018), and can help in understanding the drivers of these effects.

There are two economic sectors, X and Y, that use labor (L) and capital (K) as inputs. The dirty sector (Y) also uses a dirty input (Z, pollution). In each sector, production displays constant returns to scale.[12] The economy is endowed with fixed quantities of labor and capital allocated across the two sectors, and with an unspecified quantity of dirty input. The use of the dirty input by sector Y causes pollution, which a tax on input Z aims to reduce. The effect of pollution on utility is left implicit, which means that the model takes the tax as an exogenous policy parameter without attempting to derive its socially optimal level. Instead, the purpose of the model is to predict the effect of a change in the environmental tax on equilibrium prices and quantities.

The model allows for pre-existing *ad valorem* taxes on all goods, but since the focus of our discussion is on environmental taxes (i.e., the tax on input Z), we follow Fullerton and Heutel (2007a) and ignore them in the discussion. That is, the only exogenous change is

[12]In sector Y, the constant returns to scale are with respect to all three inputs L, K, and Z.

a rise in the tax on the dirty input, and all other taxes are assumed to remain constant. As in Harberger (1962), government is assumed to use additional tax revenue to purchase the two goods in the same proportion as would households under the initial prices. A consequence of this assumption is that the change in the relative aggregate demands for consumption goods only depends on the change in their relative prices. An alternative set of assumptions would be that tax revenue is redistributed to households, who have identical homothetic preferences.

We denote by L_X (resp. L_Y) the quantity of labor employed in sector X (resp. sector Y), K_X (resp. K_Y) the quantity of capital employed in sector X (resp.Y), p_X (resp. p_Y) the price of good X (resp. good Y), w (resp. r) the price of labor (resp. capital), and p_Z the price of Z (that is, the tax per unit of Z). Small relative changes in equilibrium variables are denoted with a "hat." For instance, $\hat{p}_Y \equiv \frac{dp_Y}{p_Y}$.

The model is solved by differentiating equilibrium conditions pertaining to production, consumption, and resource availability, yielding the following set of linear equations:[13]

$$\begin{aligned}
\hat{L}_X + \gamma_L \hat{L}_Y &= 0 && (1)\\
\hat{K}_X + \gamma_K \hat{K}_Y &= 0 && (2)\\
\hat{X} - \theta_{XL}\hat{L}_X - \theta_{XK}\hat{K}_X &= 0 && (3)\\
\hat{Y} - \theta_{YL}\hat{L}_Y - \theta_{YK}\hat{K}_Y - \theta_{YZ}\hat{Z} &= 0 && (4)\\
\hat{p}_X - \theta_{XL}\hat{w} - \theta_{XK}\hat{r} &= 0 && (5)\\
\hat{p}_Y - \theta_{YL}\hat{w} - \theta_{YK}\hat{r} &= \theta_{YZ}\hat{p}_Z && (6)\\
\hat{L}_X - \hat{K}_X + \sigma_X\hat{w} - \sigma_X\hat{r} &= 0 && (7)\\
\hat{L}_Y - \hat{Z} - \theta_{YL}(e_{LL} - e_{LZ})\hat{w} - \theta_{YK}(e_{LK} - e_{KZ})\hat{r} &= \theta_{YZ}(e_{LZ} - e_{ZZ})\hat{p}_Z && (8)\\
\hat{K}_Y - \hat{Z} - \theta_{YL}(e_{LK} - e_{LZ})\hat{w} - \theta_{YK}(e_{KK} - e_{KZ})\hat{r} &= \theta_{YZ}(e_{KZ} - e_{ZZ})\hat{p}_Z && (9)\\
\hat{X} - \hat{Y} + \sigma_u\hat{p}_X - \sigma_u\hat{p}_Y &= 0 && (10)
\end{aligned}$$

where $\gamma_L \equiv \frac{L_Y}{L_X} > 0$ and $\gamma_K \equiv \frac{K_Y}{K_X} > 0$ denote the resources allocated to sector Y relative to sector X, $\theta_{XL} \equiv \frac{wL_X}{p_X X}$ denotes the cost share of labor in sector X (and similarly for the parameters θ_{XK}, θ_{YL}, θ_{YK}, and θ_{YZ}), $\sigma_u \geq 0$ is the elasticity of substitution in consumption between X and Y, $\sigma_X \geq 0$ is the elasticity of substitution between labor and capital in sector X, and the parameters e_{ij}, for $i, j \in \{L, K, Z\}$, represent Allen elasticities of substitution defined as $e_{ij} = \frac{a_{ij}}{\theta_{Yj}}$, where a_{ij} is the conditional input demand elasticity for input i with respect to the price of input j in sector Y. Microeconomic theory places restrictions on the

[13] The formal derivations are shown in Appendix A.2.

acceptable values of the θ_{Xj}, θ_{Yj}, and e_{ij} parameters that are described in Appendix A.1.

Equations (1) and (2) pertain to resource use and imply that a change in a resource allocated to one sector must be offset by a change in the resource allocated to the other sector. Equations (3)–(6) are a consequence of profit maximization and constant returns to scale in each sector: effects on output are directly related to effects on inputs through the cost shares, for both quantities and prices. Equation (7) relates the change in the ratio of input demands in sector X to the change in the ratio of input prices using the substitution elasticity. Equations (8) and (9) represent the generalization of this relationship to the three inputs in sector Y, and thus feature the Allen substitution elasticities. Finally, Equation (10) relates the change in the ratio of goods consumed to the change in the ratio of their prices using the elasticity of substitution in consumption.

Given an exogenous change $\hat{p}_Z$, the system describing equilibrium displacement thus has 10 equations for 11 unknowns. Choosing a numeraire good adds the missing relationship, but changing the numeraire also changes the nature of the tax increase and therefore the variety of environmental policy considered. In the next section, we express the tax on pollution, $\hat{p}_Z$, relative to an explicit price index, thereby eliminating the dependency of comparative static results on numeraire choice.

3 Anchoring the environmental tax to a price index

Consider the following Cobb-Douglas price index:

$$\mathcal{P} = p_X^{\alpha_X} p_Y^{\alpha_Y} w^{\alpha_L} r^{\alpha_K}$$

with positive exponents and $\alpha_X + \alpha_Y + \alpha_L + \alpha_K = 1$ due to homogeneity of degree one. (Since prices are determined up to a multiplicative constant, the price index so defined is also determined up to the same multiplicative constant.) The relative change in $\mathcal{P}$ can then be expressed as

$$\hat{\mathcal{P}} = \alpha_X \hat{p}_X + \alpha_Y \hat{p}_Y + \alpha_L \hat{w} + \alpha_K \hat{r}.$$

If the price index $\mathcal{P}$ is used to anchor the pollution tax, then $p_Z = \mathcal{P}\tau_Z$, where τ_Z is now interpretable as an *ad valorem* tax relative to $\mathcal{P}$, and therefore

$$\hat{p}_Z = \hat{\mathcal{P}} + \hat{\tau}_Z.$$

Note that due to the equilibrium relationship $\hat{p}_X = \theta_{XL}\hat{w} + \theta_{XK}\hat{r}$, including p_X in the price index $\mathcal{P}$ is redundant. That is, any change in the weight on p_X can be exactly offset by changes in the weights on w and r, leaving $\hat{\mathcal{P}}$ unchanged. This is not the case for p_Y because $\hat{p}_Y = \theta_{YL}\hat{w} + \theta_{YK}\hat{r} + \theta_{YZ}\hat{\tau}_Z$, that is, unlike $\hat{p}_X$, $\hat{p}_Y$ implicitly includes $\hat{\tau}_Z$ independently of $\hat{w}$ and $\hat{r}$. Without loss of generality, we can therefore focus on price indices of the form

$$\mathcal{P} = p_Y^{\beta}\left(w^{\alpha} r^{1-\alpha}\right)^{1-\beta} \tag{11}$$

where $0 \leq \alpha, \beta \leq 1$. If $\beta = 0$ then the index reflects an average of factor prices, that is, the price of a combined labor-capital input. If $\alpha = \theta_{XL}$, then due to Equation (5) the index reflects an average of prices of consumption goods. Note that our specification of the price index includes as a special case the normalization made by Fullerton and Heutel (2007a), that is, $\mathcal{P} = p_X$ ($\beta = 0$, $\alpha = \theta_{XL}$).[14] It also includes as special cases the choices of p_Y ($\beta = 1$), w ($\beta = 0$, $\alpha = 1$), or r ($\beta = 0$, $\alpha = 0$) as alternative numeraire goods.

In what follows, we focus on price indices that reflect prices either on the uses side ($\alpha = \theta_{XL}$) or the sources side ($\beta = 0$). Although we could analyze each case separately, the price index in Equation (11) allows us to handle both cases within a single framework. We also focus on price indices that can be constructed from the observation of an initial equilibrium allocation and do not require knowledge of substitution elasticities, either in production or consumption. The idea is that since the anchoring price index corresponds to a particular policy choice, it is desirable for implementability to restrict the search to indices that can be designed with readily available economic information.

Note that the restriction to price indices of the Cobb-Douglas form is made here without loss of generality. Consider for instance a CES price index on the uses side

$$\mathcal{P} = \left[\beta p_X^{\frac{\sigma-1}{\sigma}} + (1-\beta) p_Y^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}$$

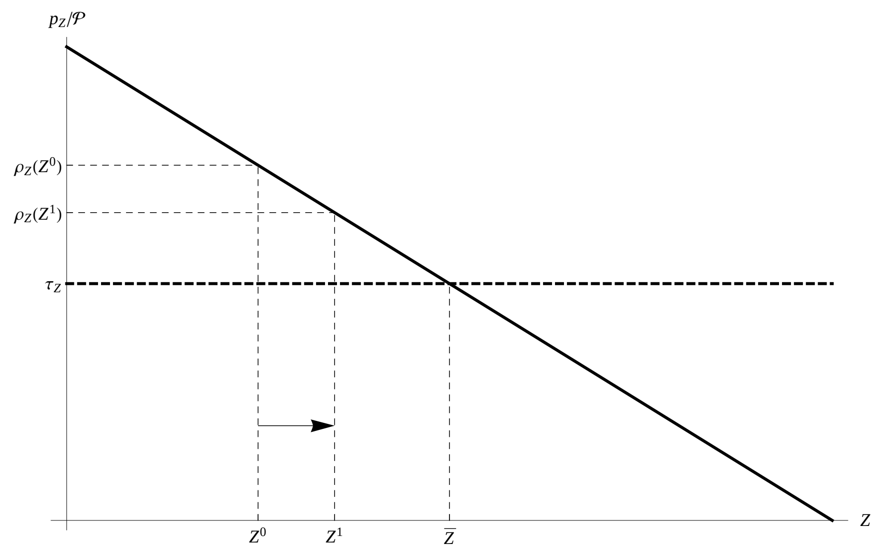
where $0 \leq \beta \leq 1$ and $\sigma \geq 0$. It is easy to show that

$$\hat{\mathcal{P}} = B\hat{p}_X + (1-B)\hat{p}_Y$$

where $B \equiv \frac{\beta p_X^{\frac{\sigma-1}{\sigma}}}{\beta p_X^{\frac{\sigma-1}{\sigma}} + (1-\beta) p_Y^{\frac{\sigma-1}{\sigma}}}$. Therefore, a CES index would not meaningfully expand the set of acceptable indices, as the relative change in the value of the CES index is still reducible to a convex combination of relative price changes. As such, any restriction on the CES parameters β and σ to ensure a downward-sloping demand for the polluting input would necessarily be channelled through the Cobb-Douglas share B.

[14] Fullerton and Heutel (2007a) choose the normalization $\hat{p}_X = 0$ while setting $p_Z = \tau_Z$ (that is, τ_Z is a nominal tax), which given the equilibrium condition (5) is equivalent to setting $\beta = 0$ and $\alpha = \theta_{XL}$.

Figure 1: The Marshallian adjustment process in the market for the dirty input



3.1 Stability of the competitive equilibrium with pollution tax

As explained in Samuelson (1941), "the problem of stability of equilibrium is intimately tied up with the problem of deriving fruitful theorems in comparative statics." The concept of equilibrium stability requires a definition of the dynamics of the economy. An equilibrium is said to be stable (in the small) if for sufficiently small displacements, all the variables approach their equilibrium values as time goes to infinity. Here we use the concept of Marshallian stability, as explained for instance in Samuelson (1941) or Plott and George (1992). Marshallian stability mandates that quantity adjusts to the difference between the demand and the supply prices of a good.[15]

We consider the following Marshallian adjustment process: if at any point in time t, the quantity of polluting input Z_t is such that the demand price for that quantity (expressed relative to the price index $\mathcal{P}$) exceeds the supply price (that is, the pollution tax τ_Z), then the quantity must increase. The opposite holds in the case when the demand price is less than the supply price. Figure 1 shows a graphical representation of such adjustment process in the normal case of a downward-sloping derived demand, assuming that in the initial condition the quantity is less than the equilibrium value. Because the derived demand for the input slopes down, the equilibrium is stable: the adjustment process brings quantity closer to its equilibrium value. Had the derived demand sloped up, the

[15]In contrast, Walrasian stability mandates that price adjusts to excess demand, that is, the difference between quantity demanded and quantity supplied. In our model with horizontal supply of polluting input, the quantity of polluting input supplied is not a well-defined function of price, which precludes the use of Walrasian stability.

equilibrium would have been Marshallian unstable.

A simple mathematical representation of the Marshallian process posits the following dynamic relationship:

$$\frac{dZ}{dt} = \rho_Z(Z) - \tau_Z$$

where $\rho_Z(Z)$ represents the inverse demand function for the polluting input, expressed relative to the price index $\mathcal{P}$. Following Samuelson (1941), we can use a first-order expansion of the function $\rho_Z(Z)$ around the equilibrium point $\bar{Z}$ to obtain

$$\frac{dZ}{dt} = \rho_Z'(\bar{Z})\left(Z - \bar{Z}\right)$$

where we have made use of $\rho_Z(\bar{Z}) = \tau_Z$. The solution to this differential equation, together with the initial condition $Z = Z^0$, is simply $Z(t) = \bar{Z} + (Z^0 - \bar{Z})e^{\rho_Z'(\bar{Z})t}$, implying that $\lim_{t\to\infty} Z(t) = \bar{Z}$ if and only if $\rho_Z'(\bar{Z}) \leq 0$, that is, the demand for pollution slopes down. This result establishes the fact that whenever the derived demand for pollution slopes up, the equilibrium cannot be stable in the Marshallian sense. In what follows, we characterize price indices $\mathcal{P}$ that ensure that the derived demand for pollution slopes down, that is, the equilibrium is stable in the Marshallian sense.

3.2 Desirable properties of a price index

We are looking for weights α, β on the price index in Equation (11) such that the following three conditions are satisfied:

Condition 1 *The pollution demand elasticity* $\frac{\hat{Z}}{\hat{\tau}_z}$ *is defined for all parameter values, that is, the determinant of the equilibrium system is never zero and thus does not change sign.*

Condition 2 *The weights* α *and* β *only depend on the observable parameters* γ_K, γ_L, θ_{XK}, θ_{YK}, *and* θ_{YL}, *or combinations thereof.*

Condition 3 *The pollution tax has the intended effect for all parameter values, that is,* $\frac{\hat{Z}}{\hat{\tau}_z} \leq 0$.

3.3 Derivation of the price indices

Our main results regarding the existence and uniqueness of price indices that avoid violations of the law of input demand derive from the following proposition.

Proposition 1 *Condition 1 and Condition 2 imply the following restriction:*

$$\left[\beta\theta_{YK} + (1-\beta)(1-\alpha)\right]\theta_{XL}(1+\gamma_L) = \left[\beta\theta_{YL} + (1-\beta)\alpha\right]\theta_{XK}(1+\gamma_K), \qquad (12)$$

while Condition 3 is automatically satisfied as long as Condition 1 is.

The formal proof can be found in Appendix A.3.[16] There, we show that the choice of (α, β) affects the elasticity $\frac{\hat{Z}}{\hat{\tau}_z}$ only through its effect on the determinant of the equilibrium system. Equation (12) is shown to be both necessary and sufficient for this determinant to have a constant sign for all possible parameter values. Specifically, we show that if σ_X and σ_u are small enough, values of the Allen elasticities of substitution (e_{LL}, e_{KK}, e_{LK}) may lead to a sign reversal, even when the restrictions from theory are imposed, unless Equation (12) holds. Importantly, the sign reversal requires labor and capital to be complementary in the production of good Y ($e_{LK} < 0$), irrespective of the choice of anchoring price index. Consequently, there cannot be a sign reversal, and thus a violation of the law of input demand, in a model with a single clean input, as in that case $e_{LK} = 0$.[17] Note that when it happens, the reversal in the sign of the determinant of the equilibrium system causes all comparative static results, not just the effect on pollution, to be reversed relative to the normal case.

Equation (12) is violated in Fullerton and Heutel (2007a)'s model where $\beta = 0$ and $\alpha = \theta_{XL}$, except in the special case of equal factor intensities in the two sectors ($\gamma_L = \gamma_K$). Consequently, there exists a nonempty subset of the parameter space for which all comparative statics are reversed. Such a subset is depicted in Figure 2 for the special case where $\sigma_X = \sigma_u = 0$, $\gamma_L = 1$, and $e_{LL} = -1$. In both panels, the set of values of $(\gamma_K, e_{KK}, e_{LK})$ supporting the sign reversal is shown as the region located above the blue surface and below the orange surface.[18] The choice $e_{LL} = -1$ is a normalization that only affects the scale of e_{KK} and e_{LK}. As such, the values of e_{KK} and e_{LK} can be reinterpreted as those of $\frac{e_{KK}}{|e_{LL}|}$ and $\frac{e_{LK}}{|e_{LL}|}$, respectively. The figure shows that the sign reversal is more likely to happen with either small values of the capital intensity γ_K paired with large magnitudes of the Allen elasticities e_{KK} and e_{LK} relative to e_{LL} (red dot), or large values of γ_K paired with small magnitudes of e_{KK} and e_{LK} relative to e_{LL} (brown dot). The blue dot corresponds to a numerical example given in Appendix A.6, where γ_K, $|e_{KK}|$, and $|e_{LK}|$ are all close to (but different than) one. Similarly shaped violation regions exist for alternative values of γ_L. Although the region depicted in Figure 2 assumes $\sigma_X = \sigma_u = 0$,[19] there also exist

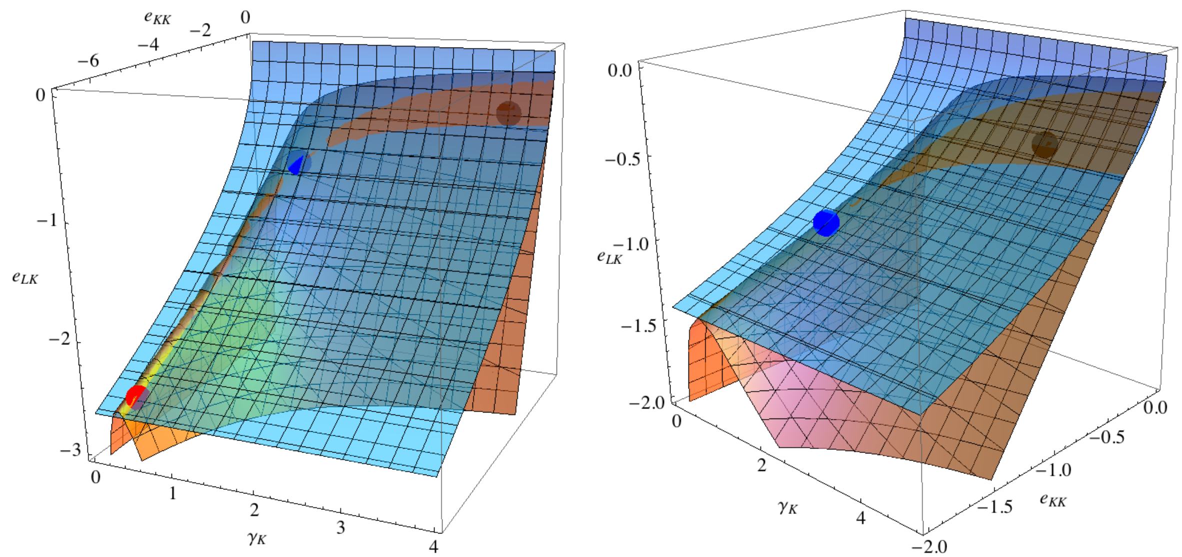
[16] Equation (12) is similar to Equation (A-17).

[17] See Appendix B for a formal proof.

[18] Using the notation of Appendix A.3, this parameter subspace supports $\Delta = C_3 < 0$. Note that when $\beta = 0$ and $\alpha = \theta_{XL}$, $B_L = \theta_{XL}$ and $B_K = \theta_{XK}$. The sign of Δ is then independent of the values of the cost shares, conditional on the choices of γ_L and γ_K.

[19] If we allow either σ_X or σ_u to be nonzero, we can no longer represent the violation region on a three-dimensional graph, because the parameters θ_{YL}, θ_{YK}, σ_X, and σ_u also determine the sign of the equilibrium

Figure 2: Violations of the law of input demand with p_X as numeraire



(a) (b)

Note: We set $\sigma_X = \sigma_u = 0$, $\gamma_L = 1$, $e_{LL} = -1$, and assume $e_{LK} \leq 0$. Panels (a) and (b) depict the same region over different parameter ranges. All points located below the blue surface violate the restrictions from theory (the negative semidefiniteness of the Slutsky matrix implies that $e_{LL}e_{KK} \geq e_{LK}^2$). Within the relevant subspace located above the blue surface, the parameter region located below the orange surface supports violations of the law of input demand.

parameter values that lead to a sign reversal when $\sigma_X > 0$ and $\sigma_u > 0$. An example is given in Appendix A.6. Appendix A.7 depicts the sign reversal region when p_Y, rather than p_X, is used as the anchoring price.

We may now specialize the restriction in Equation (12) to the case of indices on the uses side.

Corollary 1 *On the uses side ($\alpha = \theta_{XL}$), Equation* (12) *implies that either $\gamma_L = \gamma_K$ or $\beta = \frac{\gamma_L \gamma_K}{\gamma_L \gamma_K + \gamma_K \theta_{YL} + \gamma_L \theta_{YK}} = \frac{p_Y Y}{p_X X + p_Y Y}$, that is, the desired price index is*

$$\mathcal{P} = p_X^{\theta_X} p_Y^{\theta_Y} \tag{13}$$

where $\theta_X \equiv \frac{p_X X}{p_X X + p_Y Y}$ represents the initial expenditure share on good X and $\theta_Y = 1 - \theta_X$.

No other choice of weights will guarantee that the law of input demand holds for all parameter values. In particular, neither p_X nor p_Y as choices of anchoring indices would allow one to rule out upward-sloping demand for the dirty input. Thus, the choice of numeraire matters when modeling a nominal tax on pollution, as the comparative statics with respect to $\hat{p}_Z$ may have different signs (and different magnitudes) across different normalizations. This dependency of comparative statics on numeraire choice breaks down if the pollution tax is anchored to an explicit price or price index. But then the choice of anchoring price matters, in the same way that the choice of numeraire matters for the nominal pollution tax.

The following corollary addresses the case of price indices on the sources side.

Corollary 2 *On the sources side ($\beta = 0$), Equation* (12) *implies that $\alpha = \frac{\theta_{XL}(1+\gamma_L)}{\theta_{XL}(1+\gamma_L)+\theta_{XK}(1+\gamma_K)} = \frac{w\bar{L}}{w\bar{L}+r\bar{K}}$, that is, the price index is*

$$\mathcal{R} = w^{\theta_L} r^{\theta_K} \tag{14}$$

where $\theta_L \equiv \frac{w\bar{L}}{w\bar{L}+r\bar{K}}$ represents the initial contribution of labor to national income and $\theta_K = 1 - \theta_L$.

Again, these weights are uniquely defined.[20]

3.4 Equivalence between our price indices and inflation indices

Inflation is typically measured using a price index that reflects the overall cost of a reference basket of goods under varying prices. Consider that the reference basket is the

system determinant.

[20]More generally, if one is willing to consider price indices that involve prices on both the uses and sources side (that is, p_X, p_Y, w, and r), it can be shown that any price index of the form $\mathcal{P}^{\phi}\mathcal{R}^{1-\phi}$, with $\phi \in [0, 1]$, will also satisfy Equation (12).

consumption basket before the change in the pollution tax. Our preferred pollution tax is anchored to the price index $\mathcal{P} = p_X^{\theta_X} p_Y^{\theta_Y}$ where θ_X and θ_Y are the reference budget shares. When prices change in the economy, the change in our price index is:

$$
\begin{aligned}
\hat{\mathcal{P}} &= \theta_X \hat{p}_X + \theta_Y \hat{p}_Y \\
&= \frac{p_X X}{p_X X + p_Y Y} \frac{dp_X}{p_X} + \frac{p_Y Y}{p_X X + p_Y Y} \frac{dp_Y}{p_Y} \\
&= \frac{X dp_X + Y dp_Y}{p_X X + p_Y Y} \\
&= \frac{d\text{CPI}}{\text{CPI}} \\
&= \widehat{\text{CPI}}
\end{aligned}
$$

where $\text{CPI} \equiv p_X X + p_Y Y$ represents the value of the reference basket. Therefore, expressing the pollution tax as an *ad valorem* tax anchored to $\mathcal{P}$ is equivalent to adjusting a nominal tax on pollution for inflation as measured by the CPI.

A similar argument can be used to demonstrate that our price index on the sources side represents a producer price index, where the weights on factor prices correspond to the reference shares of each primary factor in national income.

4 Simple tax incidence

4.1 Sources side

The following proposition, proven in Appendix A.4, addresses tax incidence on the sources side.

Proposition 2 *Whenever Equation* (12) *holds,*

$$
sign\left\{\frac{\hat{w} - \hat{r}}{\hat{\tau}_Z}\right\} = sign\left\{\sigma_u(\gamma_K - \gamma_L) + \gamma_L(1+\gamma_K)e_{LZ} - \gamma_K(1+\gamma_L)e_{KZ}\right\}.
$$

Which sector bears proportionately more of the tax burden therefore depends on the consumption elasticity σ_u, the resource allocation parameters γ_L and γ_K, and the partial substitution elasticities e_{LZ} and e_{KZ}, but not on the cost shares in either sector or the substitution elasticity in the untaxed sector.[21]

---

[21] The size of $\frac{\hat{w}-\hat{r}}{\hat{\tau}_Z}$ itself depends on the full set of model parameters and the choice of anchoring index, see Section 5 and Appendix C.

These few parameters affect the direction of the change in the price of labor relative to capital in an intuitive yet subtle way. If labor and capital are equally substitutable for pollution ($e_{LZ} = e_{KZ} > 0$), then the pollution tax places a disproportionate burden on capital (i.e., $\hat{w} - \hat{r} > 0$) when the polluting sector is capital intensive ($\gamma_K > \gamma_L$) as long as goods are sufficiently substitutable in consumption (i.e., $\sigma_u > e_{LZ} = e_{KZ}$). If not ($\sigma_u < e_{LZ}$), then the input substitution effect dominates and the demand for capital in the capital-intensive polluting sector rises so that the return to capital increases relative to the wage rate. If both sectors are equally capital-intensive (i.e., $\gamma_L = \gamma_K$), then capital bears disproportionately more of the tax burden if and only if it is less substitutable for the polluting input than is labor.

These economic insights were already discussed, albeit for specific classes of model parameterizations, in the original contribution of Fullerton and Heutel (2007a). Indeed, the authors were prevented from drawing general conclusions by the fact that they could not sign the main equilibrium system denominator except in special cases, which as we have shown is a direct consequence of modeling the pollution tax increase relative to the price of good X.

4.2 Uses side

The following proposition, proven in Appendix A.5, addresses tax incidence on the uses side.

Proposition 3 *Whenever Equation* (12) *holds,*

$$\begin{aligned} sign\left\{\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z}\right\} \quad = \quad & sign\,\{\sigma_X(1+\gamma_L\theta_{XL}+\gamma_K\theta_{XK}) + (\theta_{YL}\gamma_K(1+\gamma_L)+\theta_{YK}\gamma_L(1+\gamma_K))\,e_{LK} \\ & +\gamma_L(1+\gamma_K)(\theta_{XK}-\theta_{YK})e_{LZ} + \gamma_K(1+\gamma_L)(\theta_{XL}-\theta_{YL})e_{KZ}\}\,. \end{aligned}$$

As for incidence on the sources side, this expression has an ambiguous sign.[22] However, we show in Appendix A.5 that in the case of equal factor intensities ($\gamma_L = \gamma_K$), $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} > 0$, that is, users of Y share proportionately more tax burden than users of X irrespective of the values of the substitution elasticities.

[22]This ambiguity comes in contrast to the effect of a partial factor tax in the standard Harberger model, which unambiguously leads to a relative increase in the price of the good produced by the taxed sector (Mieszkowski, 1967). The replacement of the partial factor tax by a tax on a third, non-market input explains this difference. Note that the ambiguity here is unrelated to the choice of numeraire. Fullerton and Heutel (2007a) do not discuss it explicitly, because the special cases they analyze ($\gamma_L = \gamma_K$ and $e_{LK} = e_{LZ} = e_{KZ}$) imply unequivocal increases in the price of good Y relative to that of good X.

Although one may legitimately expect the price of the dirty good to rise relative to that of the clean good when the pollution tax increases (Rausch et al., 2011; Fullerton and Muehlegger, 2019), the opposite may happen depending on the patterns of factor intensity and input substitution in the two industries. For instance, consider the following parameter values, which lead to $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} < 0$: $\theta_{YL} = 0.5$, $\theta_{YK} = 0.4$, $\gamma_L = 1$, $\gamma_K = 0.25$, $e_{LK} = 1$, $e_{LZ} = -1.9$, $e_{KZ} = 6$, and $\sigma_X = 0.5$. In that case, capital is a better substitute for pollution in the dirty industry ($e_{KZ} > e_{LZ}$). This causes the price of capital to rise relative to that of labor when the pollution tax increases ($\hat{r} - \hat{w} > 0$). Sector X being capital-intensive relative to sector Y ($\frac{\gamma_K}{\gamma_L} = 0.25$), and having more limited substitution possibilities ($\sigma_X = 0.5$ compared to the large magnitudes of e_{LK}, e_{LZ}, and e_{KZ}[23]), the price of good X rises relative to that of good Y ($\hat{p}_X - \hat{p}_Y > 0$). Note that demand conditions (σ_u) play no role in determining the sign of $\hat{p}_Y - \hat{p}_X$, although they play a role in determining the magnitude of the effect.[24] Also note that it is not necessary for one of the cross-price Allen elasticities of substitution to be negative for the price of good X to rise relative to that of good Y; for instance, the following set of parameter values generates comparable effects: $\theta_{YL} = 0.5$, $\theta_{YK} = 0.4$, $\gamma_L = 4$, $\gamma_K = 1$, $e_{LK} = 1$, $e_{LZ} = 1$, $e_{KZ} = 8$, and $\sigma_X = 0.5$. In Appendix A.5, we provide additional depictions of the parameter region that supports $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} \leq 0$.

The following proposition, proven in Appendix A.5, generalizes the intuition developed in the examples above, by stating that instances whereby p_Y decreases relative to p_X require (i) the factor used more intensively in sector X to be a better substitute for pollution than the other factor, and (ii) the price of the factor used more intensively in sector X to increase relative to the price of the other factor. Hence, this proposition makes an explicit link between incidence on the sources side and incidence on the uses side.[25]

Proposition 4 *If $\gamma_L > \gamma_K$, then $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} < 0 \Rightarrow \left\{e_{KZ} > e_{LZ} \text{ and } \frac{\hat{w} - \hat{r}}{\hat{\tau}_Z} < 0\right\}$. Similarly, if $\gamma_L < \gamma_K$, then $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} < 0 \Rightarrow \left\{e_{LZ} > e_{KZ} \text{ and } \frac{\hat{w} - \hat{r}}{\hat{\tau}_Z} > 0\right\}$.*

Importantly, Proposition 4 does not imply that incidence on the sources side dictates incidence on the uses sides, as the implications are unidirectional. For instance, if sector

[23]The values of e_{LK}, e_{LZ}, and e_{KZ} are not independent due to restrictions pertaining to the negative semidefiniteness of the Slutsky matrix. However, the overall magnitude of theses elasticities is independent of that of σ_X.

[24]Similarly, substitution possibilities in sector X (σ_X) play no role in determining the sign of the incidence on the sources side.

[25]Although Proposition 4 may seem intuitive, the proof is far from trivial. In Appendix A.5, we actually show a slightly stronger statement than that reported in the proposition, namely that if $\gamma_L > \gamma_K$, then $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} < 0 \Rightarrow \left\{e_{KZ} > \frac{\gamma_L(1+\gamma_K)}{\gamma_K(1+\gamma_L)} e_{LZ} \text{ and } \frac{\hat{w} - \hat{r}}{\hat{\tau}_Z} < 0\right\}$. That latter statement is stronger because $\frac{\gamma_L(1+\gamma_K)}{\gamma_K(1+\gamma_L)} > 1$ for $\gamma_L > \gamma_K$.

Table 1: Model parameters

Parameter	Value
e_{LK}	1
σ_u	1
σ_X	1
γ_L	0.25
γ_K	0.25
θ_{XL}	0.60
θ_{YL}	0.45
θ_{YK}	0.30

Note: Once we have set γ_L, γ_K, θ_{YL}, and θ_{YK}, θ_{XL} is determined by $\theta_{XL} = \frac{\gamma_K \theta_{YL}}{\gamma_K \theta_{YL} + \gamma_L \theta_{YK}}$.

Y is labor-intensive, p_Y may only decrease relative to p_X if w decreases relative to r, but this latter condition alone is not sufficient, even if $\sigma_X = 0$. The proposition does imply, however, that whenever the price of Y decreases relative to that of X, incidence on the sources side is entirely determined by the relative factor intensity.

5 Does the choice of anchoring price matter in practice?

In the previous sections, we have formally shown how comparative statics for an *ad valorem* pollution tax depend, from an analytical standpoint, on the choice of anchoring price. Consequently, comparative statics for a nominal tax, as in Fullerton and Heutel (2007a), depend on the choice of numeraire. Appendix A.6 also provides an example whereby using p_X as anchoring price leads to the prediction that an increase in the pollution tax increases pollution, whereas the use of our anchoring indices would lead to a decrease in pollution. Importantly, the signs of the predicted tax incidence on the sources and uses sides would also be inconsistent for that parameterization between the two choices of anchoring prices (i.e., p_X versus our proposed price indices). Intuitively, cases whereby a rise in the pollution tax relative to p_X (or any other price) lead to reversed comparative statics precisely correspond to cases whereby the value of the pollution tax actually decreases relative to our proposed price indices.

Here, we investigate whether the choice of anchoring price index (or, for a nominal tax, the choice of numeraire) matters in practice, that is, for reasonable model parameterizations reflecting existing economies. We use the model parameterization of Fullerton and Heutel (2007a) for the US economy, which is close to that used in Fullerton and Heutel (2010b). In these papers, the polluting sector is defined by selecting polluting industries

Table 2: Pollution effect of a 10% increase in the nominal pollution tax (%)

e_{KZ}	e_{LZ}	$\hat{Z}$					
		p_X ($\mathcal{R}$)	p_Y	$\mathcal{P}$	w	r	st. dev.
0.0	0.0	-2.00	-2.67	-2.13	-2.00	-2.00	0.24
0.5	0.0	-3.58	-4.77	-3.81	-3.54	-3.63	0.43
1.0	0.0	-5.10	-6.80	-5.44	-5.00	-5.26	0.62
-0.5	0.5	-2.70	-3.60	-2.88	-2.76	-2.62	0.33
0.0	0.5	-4.38	-5.83	-4.67	-4.42	-4.31	0.54
0.5	0.5	-6.00	-8.00	-6.40	-6.00	-6.00	0.73
1.0	0.5	-7.58	-10.10	-8.08	-7.50	-7.69	0.92
-0.5	1.0	-4.97	-6.63	-5.31	-5.14	-4.75	0.62
0.0	1.0	-6.70	-8.93	-7.15	-6.84	-6.50	0.83
0.5	1.0	-8.38	-11.17	-8.93	-8.46	-8.25	1.02
1.0	1.0	-10.00	-13.33	-10.67	-10.00	-10.00	1.22

Note: Since $\gamma_L = \gamma_K$, $\theta_{XL} = \theta_L$ and therefore the index p_X leads to the same results as the index $\mathcal{R}$. The standard deviation is calculated accounting for both indices.

based on the EPA's Toxic Release Inventory for 2002. We do not limit our analysis to a comparison of our price indices $\mathcal{P}$ and $\mathcal{R}$ to theirs (p_X). Instead, we broaden the scope of the analysis by also considering the following indices: p_Y, w, r. Other indices could be defined, however we believe that the set of chosen indices affords sufficient insights into the empirical question. Baseline parameter values are given in Table 1. All parameter values are fixed, except for the Allen cross-price elasticities e_{KZ} and e_{LZ} which are allowed to take on the values $\{-0.5, 0.0, 0.5, 1.0\}$. We exclude pairs of elasticities that lead to violations of the negative semidefiniteness of the Slutsky matrix in sector Y. Because it is assumed that $\gamma_L = \gamma_K$, $\theta_{XL} = \theta_L$ and therefore the index p_X leads to the exact same results as the index $\mathcal{R}$. This assumption is relaxed in additional simulations reported in Appendix C.1.

Results in Table 2 show that the choice of numeraire matters for the calculation of the predicted effect of the nominal tax on pollution, even if there is no reversal in sign.[26] For a given model parameterization, the standard deviation of predicted pollution impacts across numeraire choices lies in excess of 10% of the impact estimates obtained with the CPI ($\mathcal{P}$) or the factor price index ($\mathcal{R}$). The largest discrepancies are found when comparing the use of the wage rate or the rental on capital to that of the dirty good (good Y) as numeraires. In such comparisons, predicted pollution effects ($\hat{Z}$) differ by up to about 40%, and the choice of numeraire leads to variations in predictions that are often

[26] As indicated in Section 3.3, a reversal in sign is precluded by the assumption that $e_{LK} > 0$, that is, labor and capital are substitutes in the production of good Y.

Table 3: Incidence effects of a 10% increase in the nominal pollution tax (%)

e_{KZ}	e_{LZ}	Sources side: $\hat{w}-\hat{r}$						Uses side: $\hat{p}_Y-\hat{p}_X$					
		p_X ($\mathcal{R}$)	p_Y	$\mathcal{P}$	w	r	st. dev.	p_X ($\mathcal{R}$)	p_Y	$\mathcal{P}$	w	r	st. dev.
0.0	0.0	0.00	0.00	0.00	0.00	0.00	0.00	2.50	3.33	2.67	2.50	2.50	0.30
0.5	0.0	-0.26	-0.35	-0.28	-0.26	-0.26	0.03	2.50	3.33	2.67	2.47	2.54	0.30
1.0	0.0	-0.51	-0.68	-0.54	-0.50	-0.53	0.06	2.50	3.33	2.67	2.45	2.58	0.30
-0.5	0.5	0.53	0.71	0.56	0.54	0.51	0.07	2.50	3.33	2.67	2.55	2.42	0.31
0.0	0.5	0.26	0.35	0.28	0.26	0.26	0.03	2.50	3.33	2.67	2.53	2.46	0.31
0.5	0.5	0.00	0.00	0.00	0.00	0.00	0.00	2.50	3.33	2.67	2.50	2.50	0.30
1.0	0.5	-0.25	-0.34	-0.27	-0.25	-0.26	0.03	2.50	3.33	2.67	2.48	2.54	0.30
-0.5	1.0	0.79	1.05	0.84	0.81	0.75	0.10	2.50	3.33	2.67	2.58	2.39	0.31
0.0	1.0	0.52	0.69	0.55	0.53	0.50	0.06	2.50	3.33	2.67	2.55	2.43	0.31
0.5	1.0	0.25	0.34	0.27	0.26	0.25	0.03	2.50	3.33	2.67	2.53	2.46	0.31
1.0	1.0	0.00	0.00	0.00	0.00	0.00	0.00	2.50	3.33	2.67	2.50	2.50	0.30

Note: Since $\gamma_L = \gamma_K$, $\theta_{XL} = \theta_L$ and therefore the index p_X leads to the same results as the index $\mathcal{R}$. The standard deviation is calculated accounting for both indices.

comparable to those arising from alternative sets of cross-price elasticities. Predictions obtained using p_X as numeraire are relatively close to those obtained with the index $\mathcal{P}$.

Note that the choice of p_Y as the anchoring price could seem natural to a modeler. Indeed, when modeling the effects of taxes on market goods, e.g., the capital tax considered in Harberger (1962), the default choice of anchoring price would be the market price of the good subject to taxation (*ad valorem* tax on the own price). In the absence of a market price for the dirty input, one could easily be tempted to anchor the pollution tax to the price of the dirty good instead. The results of Table 2 caution against such a choice if the policy being modeled is one that indexes the pollution price on, say, the CPI.

Effects on incidence mirror those on pollution effects, although the size of the effects themselves is smaller. Still, for some model parameterizations, the incidence of the nominal tax is shown to differ qualitatively between a model that uses w or r as numeraire and one that uses p_Y. For instance, for $e_{KZ} = -0.5$ and $e_{LZ} = 1.0$, using r yields a predicted increase in $\frac{w}{r}$ (resp. $\frac{p_Y}{p_X}$) of 0.75% (resp. 2.39%), versus an increase of 1.05% (resp. 3.33%) when using p_Y. On the uses side, the largest source of variation in predicted effects is the choice of numeraire, not the choice of Allen cross-price substitution elasticities.

An intuitive explanation as to why the choice of p_Y leads to pollution and incidence effects that are larger in magnitude than those from other prices, notably $\mathcal{P}$, is that the tax causes p_Y to rise relative to $\mathcal{P}$ ($\hat{p}_Y > \hat{\mathcal{P}}$). As a result, a given rise in the *ad valorem* tax rate will have larger real effects if the anchoring price is p_Y rather than $\mathcal{P}$. Mathematically, with p_Y as anchoring price the relative change in the pollution price is $\hat{p}_Z = \hat{\tau}_Z + \hat{p}_Y$, versus $\hat{p}_Z = \hat{\tau}_Z + \hat{\mathcal{P}}$ with $\mathcal{P}$ as anchoring price. Relative to $\mathcal{P}$, the pollution price thus rises by $\hat{\tau}_Z + \hat{p}_Y - \hat{\mathcal{P}}$ in the first instance versus $\hat{\tau}_Z + \hat{\mathcal{P}} - \hat{\mathcal{P}} = \hat{\tau}_Z$ in the second one. Note, however, that p_Y need not rise relative to $\mathcal{P}$, even if it does so for the parameterizations presented here. As explained in Section 4.2, p_Y may actually *decrease* relative to p_X with the pollution tax. Whenever this happens, it is also the case that $\hat{p}_Y < \hat{\mathcal{P}}$, because $\hat{p}_Y - \hat{\mathcal{P}} = \hat{p}_Y - (\theta_X \hat{p}_X + \theta_Y \hat{p}_Y) = \theta_X (\hat{p}_Y - \hat{p}_X) < 0$. As a result, effects can be more pronounced with $\mathcal{P}$ than with p_Y as anchoring price.

In Appendix C.1, we report additional results that hold constant all Allen substitution elasticities but allow factor intensities, as captured by $\gamma_K - \gamma_L$, to vary. Again we follow the model parameterizations investigated in Fullerton and Heutel (2007a). Our results confirm the importance of the choice of anchoring price for predictions on pollution outcomes and tax incidence. Notably, the variation in predictions induced by such choice is not dwarfed by that arising from the choice of factor intensities, and even exceeds it in the case of incidence on the uses side.

In Appendix C.2, we report results for a synthetic model that borrows the parameter values from Table 1, but assumes that pollution taxation is higher in the baseline, perhaps reflecting more advanced stages of environmental policy. Specifically, we assume that environmental taxes represent a cost share of 50% instead of 25% in industry Y. We keep the same relative cost shares for labor and capital, i.e., $\theta_{YL} = 0.3$ and $\theta_{YK} = 0.2$. For a given choice of Allen cross-price elasticities, the standard deviation of predicted pollution impacts across anchoring prices rises above 30% of the impact estimate obtained with the CPI. A notable insight from these parameterizations is that anchoring the pollution tax to p_X no longer produces comparative statics that are close to those obtained using the CPI, even if the largest discrepancies are still found with the use of p_Y. The reasons are that the magnitude of the change in the relative prices of the two consumption goods is larger (Table C.4 vs. Table 3), and the expenditure share of good Y, which is inversely related to the cost shares θ_{YL} and θ_{YK} for given values of the factor intensities,[27] is relatively large ($\frac{1}{3}$ vs. $\frac{1}{4}$), implying a substantial weight on p_Y in the CPI.

Finally, in Appendix C.3 we report results for a calibration of the model relevant for a US carbon tax and adapted from the recent study by Fullerton and Ta (2019). This calibration also leads to large discrepancies across anchoring indices. Specifically, because the dirty sector is relatively small in terms of its use of labor and capital, and its expenditure share on the polluting input is substantial, using p_Y as the anchoring index gives results very different from those obtained from the alternative indices.

[27]Specifically, $\theta_Y = \frac{\gamma_L \gamma_K}{\gamma_L \gamma_K + \gamma_K \theta_{YL} + \gamma_L \theta_{YK}}$.

6 Conclusion

This paper argues that numeraire choice can matter for assessing the general equilibrium effects of environmental policy on pollution levels and relative prices, both analytically and numerically. The reason is that pollution is a non-market good, making a pollution tax a nominal tax with no direct *ad valorem* equivalent. Anchoring the pollution tax to the equilibrium prices of other goods, as we have done here, renders the choice of numeraire innocuous, but comparative statics then depend on the choice of anchoring price, a choice that reflects different varieties of environmental policy.

We have shown that if one anchors the pollution tax to inflation, then pollution always decreases with an increase in the tax rate, the resulting equilibrium is always Marshallian stable, and tax incidence results that have previously been derived for specific model parameterizations in fact hold quite generally. Further, because indexing the tax on inflation eliminates the need to focus on special cases, we have derived new and general incidence results on the uses side that were not fully discussed in prior studies. These results highlight an interesting possibility, namely that a tax on a polluting input may raise the price of the good produced using that input relative to that of the "clean" good. This counterintuitive result can arise if the factor used more intensively by the clean industry is a better substitute for pollution than the other factor, and the price of the former rises relative to that of the latter as a result of the pollution tax.

Finally, our analysis implies that analytical and numerical models aimed at capturing the general equilibrium effects of environmental taxes that are actually indexed on inflation should use the relevant inflation index as numeraire, or explicitly anchor these taxes to inflation, so that comparative static results have the correct sign, and the correct magnitude. The fact that even in the pared-down model we use, the pollution and incidence outcomes of nominal taxes can vary widely with the choice of numeraire suggests that the same would be true in larger, more detailed models of the economy with a large set of candidate numeraire goods.

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Appendices

A Proofs and derivations

A.1 Useful equalities and inequalities implied by theory

Constant returns to scale in sectors X and Y imply that $\theta_{XL} + \theta_{XK} = 1$ and $\theta_{YL} + \theta_{YK} + \theta_{YZ} = 1$. Homogeneity of degree one of the conditional input demands implies that $\theta_{YL}e_{mL} + \theta_{YK}e_{mK} + \theta_{YZ}e_{mZ} = 0$, for $m \in \{L, K, Z\}$.

Concavity of the indirect cost function and Shephard's lemma imply that the Jacobian matrix of the conditional input demands is symmetric and negative semidefinite. These properties translate to the matrix of Allen substitution elasticities, defined as

$$\mathbf{E} \equiv \begin{pmatrix} e_{LL} & e_{LK} & e_{LZ} \\ e_{KL} & e_{KK} & e_{KZ} \\ e_{ZL} & e_{ZK} & e_{ZZ} \end{pmatrix}.$$

In particular, $e_{LL}, e_{KK}, e_{ZZ} \leq 0$, $e_{LL}e_{KK} - e_{LK}^2 \geq 0$, and similarly for other input pairs. In addition, $\mathbf{v}'\mathbf{E}\mathbf{v} \leq 0$ for any 3×1 vector $\mathbf{v}$. This property also implies that $\mathbf{v}'\mathbf{E}_{LK}\mathbf{v} \leq 0$ for any 2×1 vector $\mathbf{v}$, where

$$\mathbf{E}_{LK} \equiv \begin{pmatrix} e_{LL} & e_{LK} \\ e_{KL} & e_{KK} \end{pmatrix}.$$

Note that the Allen substitution matrix is singular, and that any triplet of values of substitution elasticities determine the remaining ones through homogeneity. Relatedly, the fact that $\mathbf{E}_{LK}$ be negative semidefinite suffices to ensure that the entire matrix $\mathbf{E}$ be negative semidefinite.

Simple algebra implies that $\frac{\theta_{XL}\gamma_L}{\theta_{YL}} = \frac{\theta_{XK}\gamma_K}{\theta_{YK}} = \frac{p_Y Y}{p_X X}$ and we will denote this ratio as δ. Further note that $\delta(\theta_{YL}\gamma_K + \theta_{YK}\gamma_L) = \gamma_L\gamma_K$.

A.2 Equilibrium displacement model

We denote with a "hat" the relative differential change in a variable, for example, $\hat{p}_Y \equiv \frac{dp_Y}{p_Y}$.

Resource availability constraints imply that $L_X + L_Y = L$, therefore $dL_X + dL_Y = 0$, and thus $\frac{dL_X}{L_X} + \frac{L_Y}{L_X}\frac{dL_Y}{L_Y} = \hat{L}_X + \gamma_L\hat{L}_Y = 0$. Similarly, $\hat{K}_X + \gamma_K\hat{K}_Y = 0$. These are referred to as Equations (A-1) and (A-2) below.

Denote by $X(L_X, K_X)$ the production function in industry X, and further denote X_L and X_K its derivatives with respect to labor and capital, respectively. Totally differentiating the equality $X = X(L_X, K_X)$, we obtain $dX = X_L dL_X + X_K dX_K$, which implies $\hat{X} = \frac{X_L}{X} dL_X + \frac{X_K}{X} dK_X$ and thus $\hat{X} = \frac{L_X X_L}{X} \hat{L}_X + \frac{K_X X_K}{X} \hat{K}_X$. But profit maximization implies that $p_X X_L = w$ and $p_X X_K = r$, so $\frac{L_X X_L}{X} = \frac{wL_X}{p_X X} = \theta_{XL}$ and $\frac{K_X X_K}{X} = \frac{rK_X}{p_X X} = \theta_{XK}$. Therefore, we have $\hat{X} = \theta_{XL} \hat{L}_X + \theta_{XK} \hat{K}_X$ (Equation (A-3)). A similar relationship can be derived in industry Y with respect to the three inputs L_Y, K_Y, and Z. That is, $\hat{Y} = \theta_{YL} \hat{L}_Y + \theta_{YK} \hat{K}_Y + \theta_{YZ} \hat{Z}$ (Equation (A-4)).

Constant returns to scale in industry X imply zero profit, that is, $p_X X = wL_X + rK_X$. Totally differentiating this condition, we get

$$p_X dX + X dp_X = w dL_X + L_X dw + r dK_X + K_X dr.$$

Dividing through by $p_X X$, we have $\hat{X} + \hat{p}_X = \frac{wL_X}{p_X X} \hat{L}_X + \frac{wL_X}{p_X X} \hat{w} + \frac{rK_X}{p_X X} \hat{K}_X + \frac{rK_X}{p_X X} \hat{r} = \theta_{XL} \hat{L}_X + \theta_{XK} \hat{K}_X + \theta_{XL} \hat{w} + \theta_{XK} \hat{r}$. Using $\hat{X} = \theta_{XL} \hat{L}_X + \theta_{XK} \hat{K}_X$, we get $\hat{p}_X = \theta_{XL} \hat{w} + \theta_{XK} \hat{r}$ (Equation (A-5)). Applying the same reasoning in industry Y, we obtain $\hat{p}_Y = \theta_{YL} \hat{w} + \theta_{YK} \hat{r} + \theta_{YZ} \hat{p}_Z$ (Equation (A-6)).

Profit maximization in industry X also implies that $\frac{X_L}{X_K} = \frac{w}{r}$, an equality that we can write as $rX_L - wX_K = 0$. Totally differentiating and using double subscripts to denote second-order derivatives, we get

$$r \left(X_{LL} dL_X + X_{LK} dK_X \right) + X_L dr - w \left(X_{LK} dL_X + X_{KK} dK_X \right) - X_K dw = 0,$$

where we have used $X_{KL} = X_{LK}$. Because is X is 1-homogeneous, both X_L and X_K are 0-homogeneous, and from Euler's theorem we have $L_X X_{LL} + K_X X_{LK} = 0$ and $L_X X_{LK} + K_X X_{KK} = 0$. Therefore, we can eliminate the two own-price second-order partials in the previous expression by using the cross-partial, leading to

$$r \left(-K_X X_{LK} \hat{L}_X + K_X X_{LK} \hat{K}_X \right) + X_L dr - w \left(L_X X_{LK} \hat{L}_X - L_X X_{LK} \hat{K}_X \right) - X_K dw = 0.$$

Rearranging, we get $X_{LK} \left(wL_X + rK_X \right) \left(\hat{K}_X - \hat{L}_X \right) = wX_K \hat{w} - rX_L \hat{r}$. Dividing through by $p_X X$ and using $w = p_X X_L$ and $r = p_X X_K$, we get $X_{LK} \left(\hat{K}_X - \hat{L}_X \right) = \frac{X_L X_K}{X} \left(\hat{w} - \hat{r} \right)$, or, remembering that $\sigma_X = \frac{X_L X_K}{X X_{LK}}$ since $X(L_X, K_X)$ is homogeneous of degree one, $\hat{K}_X - \hat{L}_X = \sigma_X \left(\hat{w} - \hat{r} \right)$ (Equation (A-7)).

Equations describing the relationship between changes in input prices and input quantities in industry Y also need to be derived. Since there are three, rather than two, inputs, we use the Allen elasticities of substitution to make this connection. The Allen elasticity of substitution between inputs i and j is defined as $e_{ij} = \frac{a_{ij}}{\theta_{Yj}}$, where a_{ij} is the output-conditional elasticity of demand for input i with respect to the price of input j. Let us denote by $L_Y(w, r, p_Z, Y)$, $K_Y(w, r, p_Z, Y)$, and $Z(w, r, p_Z, Y)$ the conditional input demand functions in industry Y. Totally differentiating the equality $L_Y = L_Y(w, r, p_Z, Y)$, we get, using obvious notation,

$$dL_Y = L_{Y_w} dw + L_{Y_r} dr + L_{Y_{p_Z}} dp_Z + \frac{L_Y}{Y} dY$$

and, dividing through by L_Y, $\hat{L}_Y = \frac{wL_{Y_w}}{L_Y}\hat{w} + \frac{rL_{Y_r}}{L_Y}\hat{r} + \frac{p_Z L_{Y_{p_Z}}}{L_Y}\hat{p}_Z + \hat{Y}$, where we have used the fact that L_Y is proportional to Y due to constant returns to scale. But $\frac{wL_{Y_w}}{L_Y} = a_{LL}$ by definition, and similarly $\frac{rL_{Y_r}}{L_Y} = a_{LK}$ and $\frac{p_Z L_{Y_{p_Z}}}{L_Y} = a_{LZ}$. Thus, we have $\hat{L}_Y = a_{LL}\hat{w} + a_{LK}\hat{r} + a_{LZ}\hat{p}_Z + \hat{Y}$, or, using $a_{ij} = \theta_{Yj} e_{ij}$,

$$\hat{L}_Y = \theta_{YL} e_{LL} \hat{w} + \theta_{YK} e_{LK} \hat{r} + \theta_{YZ} e_{LZ} \hat{p}_Z + \hat{Y}.$$

Applying the same reasoning to the conditional demands for K_Y and Z, we obtain

$$\hat{K}_Y = \theta_{YL} e_{LK} \hat{w} + \theta_{YK} e_{KK} \hat{r} + \theta_{YZ} e_{KZ} \hat{p}_Z + \hat{Y} \quad \text{and} \quad \hat{Z} = \theta_{YL} e_{LZ} \hat{w} + \theta_{YK} e_{KZ} \hat{r} + \theta_{YZ} e_{ZZ} \hat{p}_Z + \hat{Y}.$$

These three relationships are not independent of the equality $\hat{Y} = \theta_{YL}\hat{L}_Y + \theta_{YK}\hat{K}_Y + \theta_{YZ}\hat{Z}$ derived previously. (For instance, taken together, they imply that equality given that $\theta_{YL} e_{mL} + \theta_{YK} e_{mK} + \theta_{YZ} e_{mZ} = 0$, for $m \in \{L, K, Z\}$.) Therefore, one may drop one of them, or, equivalently, subtract it from the other two. Following Fullerton and Heutel (2007a), we subtract the third one from each of the first two, leading to Equations (A-8) and (A-9) below.

A derivation similar to that leading to Equation (A-7), starting from the optimality condition $\frac{U_X}{U_Y} = \frac{p_X}{p_Y}$, leads to $\hat{X} - \hat{Y} = \sigma_u(\hat{p}_Y - \hat{p}_X)$, since, like $X(L_X, K_X)$, $U(X, Y)$ is homogenous of degree one (Equation (A-10) below).

We thus have 10 equations for 11 unknowns. The last equation comes from the choice of numeraire. Once a price index satisfying one-homogeneity, to which the pollution tax can be anchored through the relation $\hat{p}_Z = \hat{\mathcal{P}} + \hat{\tau}_Z$, has been defined, the choice of numeraire becomes innocuous. That is, identical results on goods provision, resource allocation, and *relative* prices will be obtained whether it is assumed that $\hat{p}_X = 0$, $\hat{w} = 0$, etc. Here we choose to set $\hat{\mathcal{P}} = 0$ (and thus $\hat{p}_Z = \hat{\tau}_Z$), but our results would be identical if we chose any other price normalization.

Using the normalization $\hat{\mathcal{P}} = 0$, the full system of equations describing the change in equilibrium values can be written as:

$$\begin{aligned}
\hat{L}_X + \gamma_L \hat{L}_Y &= 0 && \text{(A-1)}\\
\hat{K}_X + \gamma_K \hat{K}_Y &= 0 && \text{(A-2)}\\
\hat{X} - \theta_{XL}\hat{L}_X - \theta_{XK}\hat{K}_X &= 0 && \text{(A-3)}\\
\hat{Y} - \theta_{YL}\hat{L}_Y - \theta_{YK}\hat{K}_Y - \theta_{YZ}\hat{Z} &= 0 && \text{(A-4)}\\
\hat{p}_X - \theta_{XL}\hat{w} - \theta_{XK}\hat{r} &= 0 && \text{(A-5)}\\
\hat{p}_Y - \theta_{YL}\hat{w} - \theta_{YK}\hat{r} &= \theta_{YZ}\hat{\tau}_Z && \text{(A-6)}\\
\hat{L}_X - \hat{K}_X + \sigma_X\hat{w} - \sigma_X\hat{r} &= 0 && \text{(A-7)}\\
\hat{L}_Y - \hat{Z} - \theta_{YL}(e_{LL} - e_{LZ})\hat{w} - \theta_{YK}(e_{LK} - e_{KZ})\hat{r} &= \theta_{YZ}(e_{LZ} - e_{ZZ})\hat{\tau}_Z && \text{(A-8)}\\
\hat{K}_Y - \hat{Z} - \theta_{YL}(e_{LK} - e_{LZ})\hat{w} - \theta_{YK}(e_{KK} - e_{KZ})\hat{r} &= \theta_{YZ}(e_{KZ} - e_{ZZ})\hat{\tau}_Z && \text{(A-9)}\\
\hat{X} - \hat{Y} + \sigma_u\hat{p}_X - \sigma_u\hat{p}_Y &= 0 && \text{(A-10)}\\
\beta\hat{p}_Y + (1-\beta)\alpha\hat{w} + (1-\beta)(1-\alpha)\hat{r} &= 0 && \text{(A-11)}
\end{aligned}$$

which is a linear system in which $\hat{\tau}_Z$ is taken as the exogenous variable. We are particularly interested in deriving the effect $\frac{\hat{Z}}{\hat{\tau}_z}$ (in order to ensure that Condition 1 holds) and the effects $\frac{\hat{w}}{\hat{\tau}_z}$ and $\frac{\hat{r}}{\hat{\tau}_z}$ (to derive the incidence of the tax increase).

A.3 Derivation of the pollution effect $\frac{\hat{Z}}{\hat{\tau}_Z}$

Condition 1 mandates that the comparative static $\frac{\hat{Z}}{\hat{\tau}_z}$ be defined for all parameter values. Given that the system describing equilibrium displacement is linear, this implies that the determinant of the system matrix be nonzero, which, by continuity, implies that it must be of a determinate sign. We will show that this requirement is met by imposing a simple functional restriction on the weights of the price index.

Although we could compute the determinant of the system's matrix directly, following Fullerton and Heutel (2007a) it is easier to proceed by substitution to reduce the dimensionality of the problem. Equations (A-1) and (A-2) imply that $\hat{L}_X = -\gamma_L\hat{L}_Y$ and $\hat{K}_X = -\gamma_K\hat{K}_Y$. Using (A-3), we then have $\hat{X} = -\theta_{XL}\gamma_L\hat{L}_Y - \theta_{XK}\gamma_K\hat{K}_Y$ which, together with (A-4), implies that $\hat{X} - \hat{Y} = -(\theta_{XL}\gamma_L + \theta_{YL})\hat{L}_Y - (\gamma_K\theta_{XK} + \theta_{YK})\hat{K}_Y - \theta_{YZ}\hat{Z}$. Using (A-10) and defining $\delta \equiv \frac{\theta_{XL}\gamma_L}{\theta_{YL}} = \frac{\theta_{XK}\gamma_K}{\theta_{YK}}$, this in turn implies that

$$\theta_{YL}(1+\delta)\hat{L}_Y + \theta_{YK}(1+\delta)\hat{K}_Y + \theta_{YZ}\hat{Z} = \sigma_u(\hat{p}_X - \hat{p}_Y). \tag{A-12}$$

Using (A-5) and (A-6), we have

$$\hat{p}_X - \hat{p}_Y = (\theta_{XL} - \theta_{YL})\hat{w} + (\theta_{XK} - \theta_{YK})\hat{r} - \theta_{YZ}\hat{\tau}_Z$$

which together with (A-12) implies

$$\theta_{YL}(1+\delta)\hat{L}_Y + \theta_{YK}(1+\delta)\hat{K}_Y + \theta_{YZ}\hat{Z} + \sigma_u(\theta_{YL} - \theta_{XL})\hat{w} + \sigma_u(\theta_{YK} - \theta_{XK})\hat{r} = -\sigma_u\theta_{YZ}\hat{\tau}_Z. \tag{A-13}$$

One can then use (A-8) and (A-9) to eliminate $\hat{L}_Y$ and $\hat{K}_Y$ in Equation (A-13). To alleviate notation, define

$$\begin{aligned}
\bar{\theta}^1_{YL} &\equiv \theta_{YL}(e_{LL} - e_{LZ}) \\
\bar{\theta}^2_{YL} &\equiv \theta_{YL}(e_{LK} - e_{LZ}) \\
\bar{\theta}^1_{YK} &\equiv \theta_{YK}(e_{LK} - e_{KZ}) \\
\bar{\theta}^2_{YK} &\equiv \theta_{YK}(e_{KK} - e_{KZ}) \\
\bar{\theta}^1_{YZ} &\equiv \theta_{YZ}(e_{LZ} - e_{ZZ}) \\
\bar{\theta}^2_{YZ} &\equiv \theta_{YZ}(e_{KZ} - e_{ZZ}).
\end{aligned}$$

After rearrangement, we obtain:

$$\begin{aligned}
&\left[\sigma_u(\theta_{YL} - \theta_{XL}) + (1+\delta)(\theta_{YL}\bar{\theta}^1_{YL} + \theta_{YK}\bar{\theta}^2_{YL})\right]\hat{w} + \left[\sigma_u(\theta_{YK} - \theta_{XK}) + (1+\delta)(\theta_{YL}\bar{\theta}^1_{YK} + \theta_{YK}\bar{\theta}^2_{YK})\right]\hat{r} \\
&\quad + \left[\theta_{YZ} + (1+\delta)(\theta_{YL} + \theta_{YK})\right]\hat{Z} = -\left[\sigma_u\theta_{YZ} + (1+\delta)(\theta_{YL}\bar{\theta}^1_{YZ} + \theta_{YK}\bar{\theta}^2_{YZ})\right]\hat{\tau}_Z.
\end{aligned} \tag{A-14}$$

Equation (A-7) implies that $-\gamma_L\hat{L}_Y + \gamma_K\hat{K}_Y + \sigma_X\hat{w} - \sigma_X\hat{r} = 0$, which after eliminating $\hat{L}_Y$ and $\hat{K}_Y$ becomes

$$\left[\sigma_X - \gamma_L\bar{\theta}^1_{YL} + \gamma_K\bar{\theta}^2_{YL}\right]\hat{w} + \left[-\sigma_X - \gamma_L\bar{\theta}^1_{YK} + \gamma_K\bar{\theta}^2_{YK}\right]\hat{r} + \left[\gamma_K - \gamma_L\right]\hat{Z} = \left[\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ}\right]\hat{\tau}_Z. \tag{A-15}$$

Finally, (A-6) and (A-11) together imply that

$$\left[\beta\theta_{YL} + (1-\beta)\alpha\right]\hat{w} + \left[\beta\theta_{YK} + (1-\beta)(1-\alpha)\right]\hat{r} = -\beta\theta_{YZ}\hat{\tau}_Z. \tag{A-16}$$

Equations (A-14)-(A-16) constitute a linear system in the three unknowns $\hat{w}$, $\hat{r}$, and $\hat{Z}$. For the comparative static $\frac{\hat{Z}}{\hat{\tau}_Z}$ to be defined for all parameter values, the determinant of this system should have a constant sign. We can write this determinant as

$$\Delta \equiv \begin{vmatrix} \sigma_u(\theta_{YL}-\theta_{XL})+(1+\delta)(\theta_{YL}\bar{\theta}^1_{YL}+\theta_{YK}\bar{\theta}^2_{YL}) & \sigma_u(\theta_{YK}-\theta_{XK})+(1+\delta)(\theta_{YL}\bar{\theta}^1_{YK}+\theta_{YK}\bar{\theta}^2_{YK}) & \theta_{YZ}+(1+\delta)(\theta_{YL}+\theta_{YK}) \\ \sigma_X-\gamma_L\bar{\theta}^1_{YL}+\gamma_K\bar{\theta}^2_{YL} & -\sigma_X-\gamma_L\bar{\theta}^1_{YK}+\gamma_K\bar{\theta}^2_{YK} & \gamma_K-\gamma_L \\ \beta\theta_{YL}+(1-\beta)\alpha & \beta\theta_{YK}+(1-\beta)(1-\alpha) & 0 \end{vmatrix}$$

and, developing along the third column, we get

$$\begin{aligned}\Delta \;=\; & [\theta_{YZ}+(1+\delta)(\theta_{YL}+\theta_{YK})]\left[\left(\sigma_X-\gamma_L\bar{\theta}^1_{YL}+\gamma_K\bar{\theta}^2_{YL}\right)B_K+\left(\sigma_X+\gamma_L\bar{\theta}^1_{YK}-\gamma_K\bar{\theta}^2_{YK}\right)B_L\right] \\ & +(\gamma_L-\gamma_K)\left[\left(\sigma_u(\theta_{YL}-\theta_{XL})+(1+\delta)(\theta_{YL}\bar{\theta}^1_{YL}+\theta_{YK}\bar{\theta}^2_{YL})\right)B_K\right. \\ & \left.-\left(\sigma_u(\theta_{YK}-\theta_{XK})+(1+\delta)(\theta_{YL}\bar{\theta}^1_{YK}+\theta_{YK}\bar{\theta}^2_{YK})\right)B_L\right]\end{aligned}$$

where we have defined $B_L \equiv \beta\theta_{YL}+(1-\beta)\alpha$ and $B_K \equiv \beta\theta_{YK}+(1-\beta)(1-\alpha)$.

Separating the terms in σ_X and σ_u, we can write $\Delta = C_1\sigma_X + C_2\sigma_u + C_3$, where

$$\begin{aligned} C_1 &\equiv (B_L+B_K)\,(1+\delta(\theta_{YL}+\theta_{YK})) \\ C_2 &\equiv (\gamma_L-\gamma_K)\,[B_K(\theta_{YL}-\theta_{XL})-B_L(\theta_{YK}-\theta_{XK})] \end{aligned}$$

and

$$\begin{aligned} C_3 \;\equiv\; & B_L\left[(1+(1+\delta)(\theta_{YL}+\theta_{YK}))\left(\gamma_L\bar{\theta}^1_{YK}-\gamma_K\bar{\theta}^2_{YK}\right)-(1+\delta)(\gamma_L-\gamma_K)\left(\theta_{YL}\bar{\theta}^1_{YK}+\theta_{YK}\bar{\theta}^2_{YK}\right)\right] \\ & +B_K\left[(1+(1+\delta)(\theta_{YL}+\theta_{YK}))\left(-\gamma_L\bar{\theta}^1_{YL}+\gamma_K\bar{\theta}^2_{YL}\right)+(1+\delta)(\gamma_L-\gamma_K)\left(\theta_{YL}\bar{\theta}^1_{YL}+\theta_{YK}\bar{\theta}^2_{YL}\right)\right]. \end{aligned}$$

Using the definitions of the terms $\bar{\theta}^1_{YL}$, $\bar{\theta}^2_{YL}$, $\bar{\theta}^1_{YK}$, $\bar{\theta}^2_{YK}$, and using the properties $\theta_{YZ}e_{KZ} = -\theta_{YL}e_{LK} - \theta_{YK}e_{KK}$ and $\theta_{YZ}e_{LZ} = -\theta_{YL}e_{LL} - \theta_{YK}e_{LK}$, we obtain

$$C_3 = -e_{LL}B_K\theta_{YL}\gamma_L(1+\gamma_K) - e_{KK}B_L\theta_{YK}\gamma_K(1+\gamma_L) + e_{LK}\left[B_K\theta_{YL}\gamma_K(1+\gamma_L)+B_L\theta_{YK}\gamma_L(1+\gamma_K)\right].$$

It is clear that $C_1 > 0$, therefore for the sign of Δ to be invariant to the choice of parameter values, which is required for it to always be nonzero, we need both $C_2 \geq 0$ and $C_3 \geq 0$. Let us start by discussing the sign of C_3. The first two terms in C_3 are clearly non-negative given that $e_{LL} \leq 0$ and $e_{KK} \leq 0$. The symmetry and negative semidefiniteness of the submatrix $\mathbf{E}_{LK}$ implies that $-e_{LL}v_1^2 - e_{KK}v_2^2 + 2e_{LK}v_1v_2 \geq 0$ for all vectors $\mathbf{v} = (v_1, v_2)$. If we can write C_3 in this form for some well-chosen vector $\mathbf{v}$, we can then conclude that $C_3 \geq 0$ for all parameter values. A sufficient condition is that $B_K\theta_{YL}\gamma_K(1+\gamma_L) = B_L\theta_{YK}\gamma_L(1+\gamma_K)$, that is, using the fact that $\gamma_L\theta_{XL}\theta_{YK} = \gamma_K\theta_{XK}\theta_{YL}$,

$$B_K\theta_{XL}(1+\gamma_L) = B_L\theta_{XK}(1+\gamma_K) \tag{A-17}$$

which implies that

$$C_3 = B_K\theta_{YL}\gamma_K(1+\gamma_L)\left(-e_{LL}\frac{\gamma_L(1+\gamma_K)}{\gamma_K(1+\gamma_L)} - e_{KK}\frac{\gamma_K(1+\gamma_L)}{\gamma_L(1+\gamma_K)} + 2e_{LK}\right) = -\mathbf{v}'\mathbf{E}_{LK}\mathbf{v}$$

for $\mathbf{v} \equiv \sqrt{B_K\theta_{YL}\gamma_K(1+\gamma_L)}\begin{pmatrix} \sqrt{\frac{\gamma_L(1+\gamma_K)}{\gamma_K(1+\gamma_L)}} \\ -\sqrt{\frac{\gamma_K(1+\gamma_L)}{\gamma_L(1+\gamma_K)}} \end{pmatrix}$.

The restriction in (A-17) also turns out to be necessary to guarantee that $C_3 \geq 0$ for all parameter values. To see why, first note that B_L and B_K depend on the cost shares θ_{YK} and θ_{YL} and the index weights α and β. Condition 2 further implies that B_L and B_K may not depend on the substitution elasticities e_{LL}, e_{KK}, or e_{LK} since α and β are themselves restricted to be independent of substitution elasticities. This means that values of α and β that satisfy Conditions 1 and 2 must ensure that $C_3 \geq 0$ for all possible values of (e_{LL}, e_{KK}, e_{LK}) satisfying the restrictions from production theory, namely: for all e_{LL}, e_{KK}, and e_{LK} such that $e_{LL} \leq 0$, $e_{KK} \leq 0$, and $e_{LL}e_{KK} - e_{LK}^2 \geq 0$:

$$-e_{LL}B_K\theta_{YL}\gamma_L(1+\gamma_K) - e_{KK}B_L\theta_{YK}\gamma_K(1+\gamma_L) + e_{LK}\left[B_K\theta_{YL}\gamma_K(1+\gamma_L) + B_L\theta_{YK}\gamma_L(1+\gamma_K)\right] \geq 0$$

that is,

$$B_K\theta_{YL}\left[\gamma_K(1+\gamma_L)e_{LK} - \gamma_L(1+\gamma_K)e_{LL}\right] \geq B_L\theta_{YK}\left[\gamma_K(1+\gamma_L)e_{KK} - \gamma_L(1+\gamma_K)e_{LK}\right].$$

Note that if $e_{LK} \geq 0$, this condition places no restriction on (B_L, B_K) and thus no restriction on (α, β). If $e_{LK} < 0$ however, it constrains the set of acceptable values of (α, β). Consider the subset of substitution elasticities

$$(e_{LL}, e_{KK}, e_{LK}) = \left(\frac{F\gamma_K(1+\gamma_L)}{\gamma_L(1+\gamma_K)}e_{LK}, \frac{\gamma_L(1+\gamma_K)}{F\gamma_K(1+\gamma_L)}e_{LK}, e_{LK}\right)$$

where $e_{LK} < 0$ and $F > 0$. These elasticities satisfy the theory restrictions. In addition, $\gamma_K(1+\gamma_L)e_{LK} - \gamma_L(1+\gamma_K)e_{LL} > 0 \Leftrightarrow F > 1$. Thus, we have that acceptable values of α and β must satisfy

$$B_K \geq B_L\frac{\theta_{YK}\left[\gamma_K(1+\gamma_L)e_{KK} - \gamma_L(1+\gamma_K)e_{LK}\right]}{\theta_{YL}\left[\gamma_K(1+\gamma_L)e_{LK} - \gamma_L(1+\gamma_K)e_{LL}\right]} = B_L\frac{\theta_{YK}\gamma_L(1+\gamma_K)}{\theta_{YL}\gamma_K(1+\gamma_L)}\left(\frac{\frac{1}{F}-1}{1-F}\right) = B_L\frac{\theta_{YK}\gamma_L(1+\gamma_K)}{\theta_{YL}\gamma_K(1+\gamma_L)}\left(\frac{1}{F}\right)$$

for all $F > 1$ while also satisfying

$$B_K \leq B_L \frac{\theta_{YK}\left[\gamma_K(1+\gamma_L)e_{KK} - \gamma_L(1+\gamma_K)e_{LK}\right]}{\theta_{YL}\left[\gamma_K(1+\gamma_L)e_{LK} - \gamma_L(1+\gamma_K)e_{LL}\right]} = B_L \frac{\theta_{YK}\gamma_L(1+\gamma_K)}{\theta_{YL}\gamma_K(1+\gamma_L)}\left(\frac{1}{F}\right)$$

for all $F < 1$. Taking limits as $F \to 1, F > 1$, and as $F \to 1, F < 1$ yields

$$B_L \frac{\theta_{YK}\gamma_L(1+\gamma_K)}{\theta_{YL}\gamma_K(1+\gamma_L)} \leq B_K \leq B_L \frac{\theta_{YK}\gamma_L(1+\gamma_K)}{\theta_{YL}\gamma_K(1+\gamma_L)}$$

which implies (A-17).

Finally, it is easy to check that (A-17) implies that

$$B_K(\theta_{YL} - \theta_{XL}) - B_L(\theta_{YK} - \theta_{XK}) = \frac{B_L(\gamma_L - \gamma_K)\theta_{YK}(1+\delta)}{\gamma_K(1+\gamma_L)} \tag{A-18}$$

so that $C_2 \geq 0$ as well.

In order to ensure that Condition 3 is satisfied, we use Cramer's rule to derive the effect $\frac{\hat{Z}}{\hat{\tau}_Z}$. Having shown that the system determinant is positive, we only need to ensure that the following determinant

$$\Delta_Z \equiv \begin{vmatrix} \sigma_u(\theta_{YL} - \theta_{XL}) + (1+\delta)(\theta_{YL}\bar{\theta}^1_{YL} + \theta_{YK}\bar{\theta}^2_{YL}) & \sigma_u(\theta_{YK} - \theta_{XK}) + (1+\delta)(\theta_{YL}\bar{\theta}^1_{YK} + \theta_{YK}\bar{\theta}^2_{YK}) & -\sigma_u\theta_{YZ} - (1+\delta)(\theta_{YL}\bar{\theta}^1_{YZ} + \theta_{YK}\bar{\theta}^2_{YZ}) \\ \sigma_X - \gamma_L\bar{\theta}^1_{YL} + \gamma_K\bar{\theta}^2_{YL} & -\sigma_X - \gamma_L\bar{\theta}^1_{YK} + \gamma_K\bar{\theta}^2_{YK} & \gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ} \\ B_L & B_K & -B_Z \end{vmatrix}$$

is non-positive, where we have defined $B_Z \equiv \beta\theta_{YZ}$. (Note that $B_L + B_K + B_Z = 1$.) This determinant can be written as $\Delta_Z = D_1\sigma_X + D_2\sigma_u + D_3\sigma_X\sigma_u + D_4$, with

$$\begin{aligned}
D_1 &\equiv (1+\delta)\left[B_Z(\theta_{YL}\bar{\theta}^1_{YK} + \theta_{YK}\bar{\theta}^2_{YK} + \theta_{YL}\bar{\theta}^1_{YL} + \theta_{YK}\bar{\theta}^2_{YL}) - (B_L + B_K)(\theta_{YL}\bar{\theta}^1_{YZ} + \theta_{YK}\bar{\theta}^2_{YZ})\right] \\
D_2 &\equiv (\theta_{YL} - \theta_{XL})\left[B_Z(\gamma_L\bar{\theta}^1_{YK} - \gamma_K\bar{\theta}^2_{YK}) - B_K(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ})\right] \\
&\quad -(\theta_{YK} - \theta_{XK})\left[B_Z(\gamma_L\bar{\theta}^1_{YL} - \gamma_K\bar{\theta}^2_{YL}) - B_L(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ})\right] \\
&\quad +\theta_{YZ}\left[B_K(\gamma_L\bar{\theta}^1_{YL} - \gamma_K\bar{\theta}^2_{YL}) - B_L(\gamma_L\bar{\theta}^1_{YK} - \gamma_K\bar{\theta}^2_{YK})\right] \\
D_3 &\equiv (\theta_{YL} - \theta_{XL} + \theta_{YK} - \theta_{XK})B_Z - \theta_{YZ}(B_L + B_K) \\
D_4 &\equiv (1+\delta)\left[B_L\left((\theta_{YL}\bar{\theta}^1_{YK} + \theta_{YK}\bar{\theta}^2_{YK})(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ}) - (\gamma_L\bar{\theta}^1_{YK} - \gamma_K\bar{\theta}^2_{YK})(\theta_{YL}\bar{\theta}^1_{YZ} + \theta_{YK}\bar{\theta}^2_{YZ})\right)\right. \\
&\quad -B_K\left((\theta_{YL}\bar{\theta}^1_{YL} + \theta_{YK}\bar{\theta}^2_{YL})(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ}) - (\gamma_L\bar{\theta}^1_{YL} - \gamma_K\bar{\theta}^2_{YL})(\theta_{YL}\bar{\theta}^1_{YZ} + \theta_{YK}\bar{\theta}^2_{YZ})\right) \\
&\quad \left.+B_Z\left((\theta_{YL}\bar{\theta}^1_{YL} + \theta_{YK}\bar{\theta}^2_{YL})(\gamma_L\bar{\theta}^1_{YK} - \gamma_K\bar{\theta}^2_{YK}) - (\gamma_L\bar{\theta}^1_{YL} - \gamma_K\bar{\theta}^2_{YL})(\theta_{YL}\bar{\theta}^1_{YK} + \theta_{YK}\bar{\theta}^2_{YK})\right)\right].
\end{aligned}$$

It turns out that all the D_i parameters are non-positive, so that $\Delta_Z \leq 0$. To start with, note that $D_3 = -\theta_{YZ}(B_Z + B_L + B_K) = -\theta_{YZ} < 0$. Let us then show that $D_1 \leq 0$. First, note that

$$\begin{aligned}
\theta_{YL}\bar{\theta}^1_{YZ} + \theta_{YK}\bar{\theta}^2_{YZ} &= \theta_{YZ}\left[\theta_{YL}(e_{LZ} - e_{ZZ}) + \theta_{YK}(e_{KZ} - e_{ZZ})\right] \\
&= \theta_{YZ}\left[\theta_{YL}e_{ZL} + \theta_{YK}e_{ZK} - (\theta_{YL} + \theta_{YK})e_{ZZ}\right] \\
&= \theta_{YZ}\left[-\theta_{YZ}e_{ZZ} - (\theta_{YL} + \theta_{YK})e_{ZZ}\right] \\
&= -\theta_{YZ}e_{ZZ} \geq 0
\end{aligned}$$

Second, note that

$$\begin{aligned}
\theta_{YL}\bar{\theta}^1_{YK} + \theta_{YK}\bar{\theta}^2_{YK} + \theta_{YL}\bar{\theta}^1_{YL} + \theta_{YK}\bar{\theta}^2_{YL} &= \theta_{YL}\theta_{YK}(2e_{LK} - e_{KZ} - e_{LZ}) + (\theta_{YL})^2(e_{LL} - e_{LZ}) + (\theta_{YK})^2(e_{KK} - e_{KZ}) \\
&= e_{LL}(\theta_{YL})^2 + e_{KK}(\theta_{YK})^2 + 2e_{LK}\theta_{YL}\theta_{YK} - (\theta_{YL} + \theta_{YK})(\theta_{YL}e_{LZ} + \theta_{YK}e_{KZ}) \\
&= -(\theta_{YL} + \theta_{YK} + \theta_{YZ})(\theta_{YL}e_{LZ} + \theta_{YK}e_{KZ}) \\
&= \theta_{YZ}e_{ZZ} \leq 0
\end{aligned}$$

Therefore, $D_1 = (1+\delta)\theta_{YZ}e_{ZZ} \leq 0$. Let us now show that $D_4 \leq 0$. We can rewrite

$$D_4 = (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\left[B_L(\bar{\theta}^2_{YK}\bar{\theta}^1_{YZ} - \bar{\theta}^1_{YK}\bar{\theta}^2_{YZ}) + B_K(\bar{\theta}^1_{YL}\bar{\theta}^2_{YZ} - \bar{\theta}^2_{YL}\bar{\theta}^1_{YZ}) + B_Z(\bar{\theta}^2_{YL}\bar{\theta}^1_{YK} - \bar{\theta}^1_{YL}\bar{\theta}^2_{YK})\right].$$

We will show that the coefficients on B_L, B_K, and B_Z in the square bracket are all non-positive. Consider for instance the coefficient $\bar{\theta}^1_{YL}\bar{\theta}^2_{YZ} - \bar{\theta}^2_{YL}\bar{\theta}^1_{YZ}$ on B_K. We have:

$$\begin{aligned}
\frac{\bar{\theta}^1_{YL}\bar{\theta}^2_{YZ} - \bar{\theta}^2_{YL}\bar{\theta}^1_{YZ}}{\theta_{YL}\theta_{YZ}} &= (e_{LL} - e_{LZ})(e_{KZ} - e_{ZZ}) - (e_{LK} - e_{LZ})(e_{LZ} - e_{ZZ}) \\
&= -(e_{LL}e_{ZZ} - e^2_{LZ}) + e_{LL}e_{KZ} - e_{LZ}e_{KZ} - e_{LK}e_{LZ} + e_{LK}e_{ZZ} \\
&= -(e_{LL}e_{ZZ} - e^2_{LZ}) + e_{LL}e_{KZ} + e_{LK}e_{ZZ} - \frac{e_{LZ}}{\theta_{YK}}(\theta_{YK}e_{ZK} + \theta_{YK}e_{LK}) \\
&= -(e_{LL}e_{ZZ} - e^2_{LZ}) + e_{LL}e_{KZ} + e_{LK}e_{ZZ} + \frac{e_{LZ}}{\theta_{YK}}(\theta_{YL}e_{ZL} + \theta_{YZ}e_{ZZ} + \theta_{YL}e_{LL} + \theta_{YZ}e_{LZ}) \\
&= -(e_{LL}e_{ZZ} - e^2_{LZ}) + \frac{1}{\theta_{YK}}\left[e_{LL}(\theta_{YK}e_{KZ} + \theta_{YL}e_{LZ}) + e_{ZZ}(\theta_{YK}e_{LK} + \theta_{YZ}e_{LZ}) + e^2_{LZ}(\theta_{YL} + \theta_{YZ})\right] \\
&= -\frac{(e_{LL}e_{ZZ} - e^2_{LZ})}{\theta_{YK}} \leq 0
\end{aligned}$$

where we have used the fact that $e_{LL}e_{ZZ} - e^2_{LZ} \geq 0$. Similarly, the coefficient on B_Z has the same sign as

$$\begin{aligned}
\frac{\bar{\theta}^2_{YL}\bar{\theta}^1_{YK} - \bar{\theta}^1_{YL}\bar{\theta}^2_{YK}}{\theta_{YL}\theta_{YK}} &= (e_{LK} - e_{LZ})(e_{LK} - e_{KZ}) - (e_{LL} - e_{LZ})(e_{KK} - e_{KZ}) \\
&= -(e_{LL}e_{KK} - e^2_{LK}) + e_{LL}e_{KZ} - e_{LK}e_{KZ} - e_{LK}e_{LZ} + e_{KK}e_{LZ}.
\end{aligned}$$

A reasoning similar to that used to sign the coefficient on B_K can be used by swapping the indices K and Z to determine the sign of this expression, and similarly for the coefficient on B_L. Therefore, $D_4 \leq 0$. It can further be shown that

$$D_4 = -(\gamma_L \theta_{YK} + \gamma_K \theta_{YL} + \gamma_L \gamma_K) \left(\theta_{YL} e_{LK} e_{LZ} + \theta_{YK} e_{LK} e_{KZ} + \theta_{YZ} e_{LZ} e_{KZ} \right)$$

so that D_4 is independent of α and β. It remains to be shown that $D_2 \leq 0$. Rewrite

$$\begin{aligned} D_2 &= B_L \left[\theta_{YZ}(\gamma_K \bar{\theta}^2_{YK} - \gamma_L \bar{\theta}^1_{YK}) + (\theta_{YK} - \theta_{XK})(\gamma_L \bar{\theta}^1_{YZ} - \gamma_K \bar{\theta}^2_{YZ}) \right] \\ &\quad + B_K \left[\theta_{YZ}(\gamma_L \bar{\theta}^1_{YL} - \gamma_K \bar{\theta}^2_{YL}) + (\theta_{YL} - \theta_{XL})(\gamma_K \bar{\theta}^2_{YZ} - \gamma_L \bar{\theta}^1_{YZ}) \right] \\ &\quad + B_Z \left[(\theta_{YL} - \theta_{XL})(\gamma_L \bar{\theta}^1_{YK} - \gamma_K \bar{\theta}^2_{YK}) + (\theta_{YK} - \theta_{XK})(\gamma_K \bar{\theta}^2_{YL} - \gamma_L \bar{\theta}^1_{YL}) \right]. \end{aligned}$$

Using the definitions of the $\bar{\theta}^i_{Yj}$ parameters and the relationships $\theta_{YL} e_{mL} + \theta_{YK} e_{mK} + \theta_{YZ} e_{mZ} = 0$, one can show that the three terms in square brackets are equal to each other, so that the value of D_2 is independent of (α, β) due to $B_L + B_K + B_Z = 1$. Using the first of these bracketed terms, we thus have

$$
\begin{aligned}
D_2 &= \theta_{YZ}\theta_{YK}\left(\gamma_K(e_{KK}-e_{KZ})-\gamma_L(e_{LK}-e_{KZ})\right)+(\theta_{YK}-\theta_{XK})\theta_{YZ}\left(\gamma_L(e_{LZ}-e_{ZZ})-\gamma_K(e_{KZ}-e_{ZZ})\right)\\
&= e_{KK}\theta_{YZ}\theta_{YK}\gamma_K+e_{ZZ}(\theta_{YK}-\theta_{XK})\theta_{YZ}(\gamma_K-\gamma_L)+e_{KZ}\left(\theta_{YZ}\theta_{YK}(\gamma_L-\gamma_K)-(\theta_{YK}-\theta_{XK})\theta_{YZ}\gamma_K\right)\\
&\quad -e_{LK}\theta_{YZ}\theta_{YK}\gamma_L+e_{LZ}(\theta_{YK}-\theta_{XK})\theta_{YZ}\gamma_L\\
&= e_{KK}\theta_{YZ}\theta_{YK}\left(\gamma_K+\frac{\theta_{YK}\gamma_L}{\theta_{YL}}\right)+e_{ZZ}\theta_{YZ}(\theta_{YK}-\theta_{XK})\left(\gamma_K-\gamma_L-\frac{\theta_{YZ}\gamma_L}{\theta_{YL}}\right)\\
&\quad +e_{KZ}\left(\theta_{YZ}\theta_{YK}(\gamma_L-\gamma_K)-(\theta_{YK}-\theta_{XK})\theta_{YZ}\gamma_K+\theta_{YZ}\theta_{YK}\frac{\theta_{YZ}\gamma_L}{\theta_{YL}}-(\theta_{YK}-\theta_{XK})\theta_{YZ}\frac{\theta_{YK}\gamma_L}{\theta_{YL}}\right)\\
&= e_{KK}\frac{\theta_{YZ}\theta_{YK}\gamma_L}{\delta\theta_{YL}}\left(\frac{\delta\gamma_K\theta_{YL}}{\gamma_L}+\theta_{YK}\delta\right)+e_{ZZ}\frac{\theta_{YZ}\theta_{YK}}{\gamma_K\theta_{YL}}(\gamma_K-\delta)\left((\gamma_K-\gamma_L)\theta_{YL}-\gamma_L\theta_{YZ}\right)\\
&\quad +e_{KZ}\theta_{YZ}\theta_{YK}\left(\gamma_L-\gamma_K-\gamma_K\left(1-\frac{\delta}{\gamma_K}\right)+\frac{\theta_{YZ}\gamma_L}{\theta_{YL}}-\left(1-\frac{\delta}{\gamma_K}\right)\frac{\theta_{YK}\gamma_L}{\theta_{YL}}\right)\\
&= e_{KK}\frac{\theta_{YZ}\theta_{YK}\gamma_L}{\delta\theta_{YL}}(\gamma_K\theta_{XL}+\gamma_K\theta_{XK})+e_{ZZ}\frac{\theta_{YZ}\theta_{YK}}{\gamma_K\theta_{YL}}(\gamma_K-\delta)(\gamma_K\theta_{YL}+\gamma_L\theta_{YK}-\gamma_L)\\
&\quad +e_{KZ}\theta_{YZ}\theta_{YK}\left(\gamma_L-2\gamma_K+\delta+\frac{\theta_{YZ}}{\theta_{YL}}\gamma_L-(\gamma_K-\delta)\frac{\theta_{YK}\gamma_L}{\theta_{YL}\gamma_K}\right)\\
&= e_{KK}\frac{\theta_{YZ}\theta_{YK}\gamma_L\gamma_K}{\delta\theta_{YL}}+e_{ZZ}\frac{\theta_{YZ}\theta_{YK}\gamma_L}{\gamma_K\theta_{YL}\delta}(\gamma_K-\delta)\left(\frac{\delta\gamma_K\theta_{YL}}{\gamma_L}+\delta\theta_{YK}-\delta\right)\\
&\quad +e_{KZ}\theta_{YZ}\theta_{YK}\left(\gamma_L-2\gamma_K+\delta+\frac{\gamma_L}{\theta_{YL}}(1-\theta_{YK}-\theta_{YL})-\frac{\theta_{YK}\gamma_L}{\theta_{YL}}+\delta\frac{\theta_{YK}\gamma_L}{\theta_{YL}\gamma_K}\right)\\
&= e_{KK}\frac{\theta_{YZ}\theta_{YK}\gamma_K}{\theta_{XL}}+e_{ZZ}\frac{\theta_{YZ}\theta_{YK}}{\gamma_K\theta_{XL}}(\gamma_K-\delta)^2+e_{KZ}\theta_{YZ}\theta_{YK}\left(-2\gamma_K+\delta+\frac{\delta}{\theta_{XL}}-\frac{2\gamma_K\theta_{XK}}{\theta_{XL}}+\frac{\delta\theta_{XK}}{\theta_{XL}}\right)\\
&= e_{KK}\frac{\theta_{YZ}\theta_{YK}\gamma_K}{\theta_{XL}}+e_{ZZ}\frac{\theta_{YZ}\theta_{YK}}{\gamma_K\theta_{XL}}(\gamma_K-\delta)^2-2e_{KZ}\frac{\theta_{YZ}\theta_{YK}}{\theta_{XL}}(\gamma_K-\delta)\\
&= \frac{\theta_{YZ}\theta_{YK}}{\theta_{XL}}\left(e_{KK}\gamma_K+e_{ZZ}\frac{(\gamma_K-\delta)^2}{\gamma_K}-2e_{KZ}(\gamma_K-\delta)\right)
\end{aligned}
$$

which is non-positive from the negative semidefiniteness of the submatrix $\mathbf{E}_{KZ}$.

A.4 Tax incidence: sources side

We apply Cramer's rule to derive the effect $\frac{\hat{w}}{\hat{\tau}_Z}$. The effect $\frac{\hat{r}}{\hat{\tau}_Z}$ can be obtained using symmetry. We have that $\frac{\hat{w}}{\hat{\tau}_Z}=\frac{\Delta_w}{\Delta}$, where

$$
\Delta_w \equiv \begin{vmatrix} -\sigma_u\theta_{YZ}-(1+\delta)(\theta_{YL}\bar{\theta}^1_{YZ}+\theta_{YK}\bar{\theta}^2_{YZ}) & \sigma_u(\theta_{YK}-\theta_{XK})+(1+\delta)(\theta_{YL}\bar{\theta}^1_{YK}+\theta_{YK}\bar{\theta}^2_{YK}) & \theta_{YZ}+(1+\delta)(\theta_{YL}+\theta_{YK}) \\ \gamma_L\bar{\theta}^1_{YZ}-\gamma_K\bar{\theta}^2_{YZ} & -\sigma_X-\gamma_L\bar{\theta}^1_{YK}+\gamma_K\bar{\theta}^2_{YK} & \gamma_K-\gamma_L \\ -B_Z & B_K & 0 \end{vmatrix}.
$$

This determinant can be written as $\Delta_w = F_1\sigma_X + F_2\sigma_u + F_3$, with

$$
\begin{aligned}
F_1 &= -\left[\theta_{YZ} + (1+\delta)(\theta_{YL} + \theta_{YK})\right] B_Z \\
F_2 &= (\gamma_K - \gamma_L)\left[\theta_{YZ}B_K - (\theta_{YK} - \theta_{XK})B_Z\right] \\
F_3 &= (1+\delta)(\gamma_K - \gamma_L)\left[(\theta_{YL}\bar{\theta}^1_{YZ} + \theta_{YK}\bar{\theta}^2_{YZ})B_K - (\theta_{YL}\bar{\theta}^1_{YK} + \theta_{YK}\bar{\theta}^2_{YK})B_Z\right] \\
&\quad + \left[\theta_{YZ} + (1+\delta)(\theta_{YL} + \theta_{YK})\right]\left[(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ})B_K - (\gamma_L\bar{\theta}^1_{YK} - \gamma_K\bar{\theta}^2_{YK})B_Z\right].
\end{aligned}
$$

Simplifying, we obtain:

$$
\begin{aligned}
F_3 &= \theta_{YZ}\left[(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ})B_K - (\gamma_L\bar{\theta}^1_{YK} - \gamma_K\bar{\theta}^2_{YK})B_Z\right] \\
&\quad +(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\left[(\bar{\theta}^1_{YZ} - \bar{\theta}^2_{YZ})B_K - (\bar{\theta}^1_{YK} - \bar{\theta}^2_{YK})B_Z\right] \\
&= \theta_{YZ}\left[(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ})B_K - (\gamma_L\bar{\theta}^1_{YK} - \gamma_K\bar{\theta}^2_{YK})B_Z\right] \\
&\quad +(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\left[\theta_{YZ}(e_{LZ} - e_{KZ})B_K - \theta_{YK}(e_{LK} - e_{KK})B_Z\right].
\end{aligned}
$$

Using symmetry, we then have that $\frac{\hat{r}}{\hat{\tau}_Z} = \frac{G_1\sigma_X + G_2\sigma_u + G_3}{\Delta}$ with

$$
\begin{aligned}
G_1 &= F_1 \\
G_2 &= (\gamma_K - \gamma_L)\left[-\theta_{YZ}B_L + (\theta_{YL} - \theta_{XL})B_Z\right]
\end{aligned}
$$

and

$$
\begin{aligned}
G_3 &= \theta_{YZ}\left[-(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ})B_L - (\gamma_K\bar{\theta}^2_{YL} - \gamma_L\bar{\theta}^1_{YL})B_Z\right] \\
&\quad +(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\left[-\theta_{YZ}(e_{LZ} - e_{KZ})B_L - \theta_{YL}(e_{LK} - e_{LL})B_Z\right].
\end{aligned}
$$

We have $\frac{\hat{w}-\hat{r}}{\hat{\tau}_Z} = \frac{\sigma_X(F_1-G_1)+\sigma_u(F_2-G_2)+F_3-G_3}{\Delta}$. It is easy to see that $F_2 - G_2 = (\gamma_K - \gamma_L)\theta_{YZ}$. In addition, we have

$$
\begin{aligned}
F_3 - G_3 &= \theta_{YZ}\left[(B_L + B_K)(\gamma_L\bar{\theta}^1_{YZ} - \gamma_K\bar{\theta}^2_{YZ}) + B_Z\left(\gamma_K\bar{\theta}^2_{YL} - \gamma_L\bar{\theta}^1_{YL} - \gamma_L\bar{\theta}^1_{YK} + \gamma_K\bar{\theta}^2_{YK}\right)\right] \\
&\quad +(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\left[\theta_{YZ}(e_{LZ} - e_{KZ})(B_L + B_K) + \left(\theta_{YL}(e_{LK} - e_{LL}) - \theta_{YK}(e_{LK} - e_{KK})\right)B_Z\right] \\
&= e_{ZZ}(\theta_{YZ})^2(B_L + B_K)(\gamma_K - \gamma_L) \\
&\quad +e_{LZ}\left[(\theta_{YZ})^2(B_L + B_K)\gamma_L + \theta_{YZ}\theta_{YL}B_Z(\gamma_L - \gamma_K) + (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YZ}(B_L + B_K)\right] \\
&\quad +e_{KZ}\left[-(\theta_{YZ})^2(B_L + B_K)\gamma_K + \theta_{YZ}\theta_{YK}B_Z(\gamma_L - \gamma_K) - (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YZ}(B_L + B_K)\right] \\
&\quad +e_{LL}\left[-\theta_{YZ}\theta_{YL}B_Z\gamma_L - (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YL}B_Z\right] \\
&\quad +e_{LK}\left[\theta_{YZ}B_Z\left(\gamma_K\theta_{YL} - \gamma_L\theta_{YK}\right) + (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})(\theta_{YL} - \theta_{YK})B_Z\right] \\
&\quad +e_{KK}\left[\theta_{YZ}\theta_{YK}B_Z\gamma_K + (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YK}B_Z\right].
\end{aligned}
$$

Using $\theta_{YL}e_{LL} = -\theta_{YK}e_{LK} - \theta_{YZ}e_{LZ}$ and $\theta_{YK}e_{KK} = -\theta_{YL}e_{LK} - \theta_{YZ}e_{KZ}$ and $B_L + B_K = 1 - B_Z$, we get

$$\begin{aligned}
F_3 - G_3 \quad = \quad & e_{ZZ}(\theta_{YZ})^2(1 - B_Z)(\gamma_K - \gamma_L) \\
& + e_{LZ}\left[(\theta_{YZ})^2\gamma_L + \theta_{YZ}\theta_{YL}B_Z(\gamma_L - \gamma_K) + (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YZ}(1 - B_Z)\right] \\
& + e_{KZ}\left[-(\theta_{YZ})^2\gamma_K + \theta_{YZ}\theta_{YK}B_Z(\gamma_L - \gamma_K) - (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YZ}(1 - B_Z)\right] \\
& - e_{LL}(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YL}B_Z \\
& + e_{LK}(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})(\theta_{YL} - \theta_{YK})B_Z \\
& + e_{KK}(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YK}B_Z.
\end{aligned}$$

Now using $\theta_{YZ}e_{ZZ} + \theta_{YL}e_{LZ} + \theta_{YK}e_{KZ} = 0$, we get

$$\begin{aligned}
F_3 - G_3 \quad = \quad & e_{ZZ}(\theta_{YZ})^2(\gamma_K - \gamma_L) \\
& + e_{LZ}\left[(\theta_{YZ})^2\gamma_L + (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YZ}(1 - B_Z)\right] \\
& + e_{KZ}\left[-(\theta_{YZ})^2\gamma_K - (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YZ}(1 - B_Z)\right] \\
& - e_{LL}(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YL}B_Z \\
& + e_{LK}(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})(\theta_{YL} - \theta_{YK})B_Z \\
& + e_{KK}(1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YK}B_Z \\
= \quad & e_{ZZ}(\theta_{YZ})^2(\gamma_K - \gamma_L) \\
& + e_{LZ}\left[(\theta_{YZ})^2\gamma_L + (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YZ}\right] \\
& + e_{KZ}\left[-(\theta_{YZ})^2\gamma_K - (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\theta_{YZ}\right] \\
= \quad & e_{LZ}\theta_{YZ}\left[-\theta_{YL}(\gamma_K - \gamma_L) + \theta_{YZ}\gamma_L + (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\right] \\
& + e_{KZ}\theta_{YZ}\left[-\theta_{YK}(\gamma_K - \gamma_L) - \theta_{YZ}\gamma_K - (1+\delta)(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\right] \\
= \quad & e_{LZ}\theta_{YZ}\left[\gamma_L + \delta(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\right] - e_{KZ}\theta_{YZ}\left[\gamma_K + \delta(\gamma_L\theta_{YK} + \gamma_K\theta_{YL})\right] \\
= \quad & e_{LZ}\theta_{YZ}\gamma_L(1 + \gamma_K) - e_{KZ}\theta_{YZ}\gamma_K(1 + \gamma_L).
\end{aligned}$$

Therefore, we have

$$\frac{\hat{w} - \hat{r}}{\hat{\tau}_Z} = \frac{\theta_{YZ}\left[\sigma_u(\gamma_K - \gamma_L) + \gamma_L(1 + \gamma_K)e_{LZ} - \gamma_K(1 + \gamma_L)e_{KZ}\right]}{\Delta}.$$

Note that variations in the choices for α and β affect incidence on the sources side only through the denominator Δ.

A.5 Tax incidence: uses side

Using the equalities $\hat{p}_Y = \theta_{YL}\hat{w} + \theta_{YK}\hat{r} + \theta_{YZ}\hat{\tau}_Z$ and $\hat{p}_X = \theta_{XL}\hat{w} + \theta_{XK}\hat{r}$ together with the expressions for $\frac{\hat{w}}{\hat{\tau}_Z}$ and $\frac{\hat{r}}{\hat{\tau}_Z}$, one can derive the following expression for the incidence of the pollution tax on the uses side:

$$\begin{aligned}\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} &= \frac{\theta_{YZ}}{\Delta}\left[\sigma_X\left(1 + \gamma_L\theta_{XL} + \gamma_K\theta_{XK}\right) + \left(\theta_{YL}\gamma_K(1+\gamma_L) + \theta_{YK}\gamma_L(1+\gamma_K)\right)e_{LK}\right.\\ &\quad \left. +\gamma_L(1+\gamma_K)(\theta_{XK} - \theta_{YK})e_{LZ} + \gamma_K(1+\gamma_L)(\theta_{XL} - \theta_{YL})e_{KZ}\right].\end{aligned}$$

Note that variations in the choices for α and β affect incidence on the uses side only through the denominator Δ. In addition, the price of Y could increase or decrease relative to X. But note that, as in Fullerton and Heutel (2007a), the ambiguity is resolved when $\gamma_L = \gamma_K \equiv \gamma$. Indeed, we can then write $\theta_{XL} - \theta_{YL} = \theta_{XL} - \theta_{YL}(\theta_{XL} + \theta_{XK}) = \theta_{XL}(1 - \theta_{YL} - \theta_{YK}) = \theta_{XL}\theta_{YZ}$, and similarly $\theta_{XK} - \theta_{YK} = \theta_{XK}\theta_{YZ}$, which yields

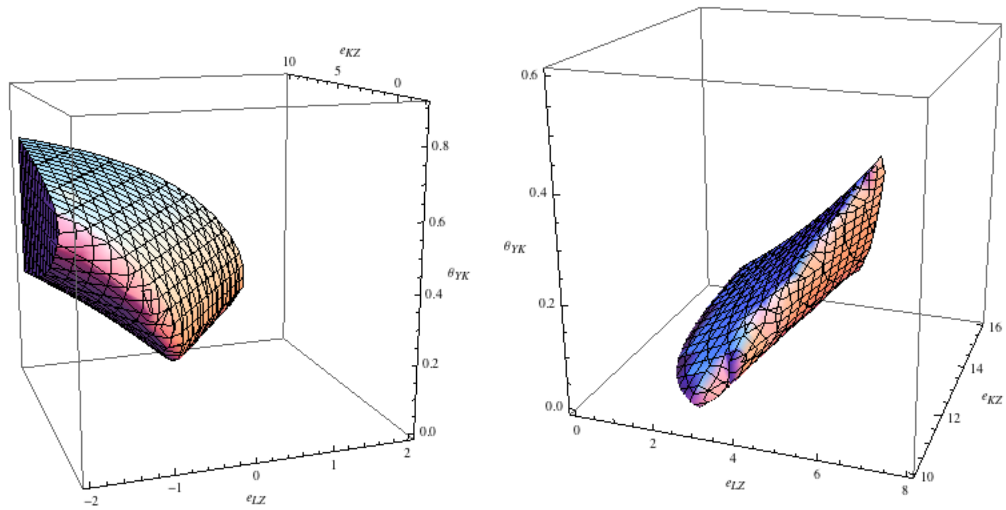
$$\begin{aligned}\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} &= \frac{\theta_{YZ}(1+\gamma)}{\Delta}\left[\sigma_X + \gamma\left((\theta_{YL} + \theta_{YK})e_{LK} + \theta_{XK}\theta_{YZ}e_{LZ} + \theta_{XL}\theta_{YZ}e_{KZ}\right)\right]\\ &= \frac{\theta_{YZ}(1+\gamma)}{\Delta}\left[\sigma_X + \gamma\left((\theta_{XK}\theta_{YL} + \theta_{XL}\theta_{YK})e_{LK} - \theta_{XK}\theta_{YL}e_{LL} - \theta_{XL}\theta_{YK}e_{KK}\right)\right]\end{aligned}$$

which is positive given the negative semidefiniteness of $\mathbf{E}_{LK}$ and the fact that $\theta_{XL}\theta_{YK} = \theta_{XK}\theta_{YL}$.

A.5.1 Depiction of cases where $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} \leq 0$

Figure A.1 provides depictions of the parameter region that supports $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} \leq 0$. In order to reduce the dimensionality of the problem, we assume $\sigma_X = 0$ (larger values of σ_X always make the occurrence of $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} \leq 0$ less likely, *ceteris paribus*), $\gamma_L = 1$, and $\gamma_K = 0.20$. (It is necessary that $\gamma_L \neq \gamma_K$ in order for $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z}$ to be negative, thus we assume that sector Y is labor-intensive relative to sector X.) We further assume that $\theta_{YZ} = 0.10$. Because $\sigma_X = 0$, the sign of the effect only depends on the relative magnitudes of the Allen cross-price elasticities of substitution, thus without loss of generality we set $|e_{LK}| = 1$ and consider the three-dimensional space spanned by the values of $(e_{LZ}, e_{KZ}, \theta_{YK})$. Note that θ_{XL} (and thus θ_{XK}) are determined by the values of γ_L, γ_K, θ_{YZ}, and θ_{YK}. We impose the restrictions pertaining to the negative semidefiniteness of the Slutsky matrix. Panel (a) of Figure A.1 considers the case where $e_{LK} > 0$. In that case, either e_{LZ} or e_{KZ} may be negative, although the relative magnitudes of the Allen elasticities are then constrained by the negative semidefiniteness of the Slutsky matrix. Panel (b) considers the case $e_{LK} < 0$.

Figure A.1: Depictions of cases where $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} < 0$



(a) $e_{LK} = 1$ (b) $e_{LK} = -1$

Note: We set $\sigma_X = 0$, $\gamma_L = 1$, $\gamma_K = 0.20$, $\theta_{YZ} = 0.10$, and $|e_{LK}| = 1$. The colored region represents the parameter space supporting $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} < 0$.

In that case both e_{LZ} or e_{KZ} must be positive (and sufficiently so to ensure that e_{LL} and e_{KK} are nonpositive). Overall, Figure A.1 confirms that $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} \leq 0$ occurs for a non-negligible set of values of the model parameters.

A.5.2 Proof of Proposition 4

Without loss of generality, assume that $\gamma_L > \gamma_K$. We will show that $\frac{\hat{w} - \hat{r}}{\hat{\tau}_Z} \geq 0 \Rightarrow \frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} \geq 0$, which is the contrapositive of the statement that $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} < 0 \Rightarrow \frac{\hat{w} - \hat{r}}{\hat{\tau}_Z} < 0$.

Suppose that $\frac{\hat{w} - \hat{r}}{\hat{\tau}_Z} \geq 0$. Given Proposition 2 and the fact that $\sigma_u \geq 0$, it must be that $\gamma_L(1 + \gamma_K)e_{LZ} \geq \gamma_K(1 + \gamma_L)e_{KZ}$. Note that we would have arrived at a similar conclusion if we had assumed that $e_{KZ} \leq e_{LZ}$, given that $\gamma_L > \gamma_K$; therefore, the fact that we will show that $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} \geq 0$ also serves to show that we must have $e_{KZ} > e_{LZ}$.

Given that $\sigma_X \geq 0$, in order to show that $\frac{\hat{p}_Y - \hat{p}_X}{\hat{\tau}_Z} \geq 0$ it suffices to show that

$$(\theta_{YL}\gamma_K(1 + \gamma_L) + \theta_{YK}\gamma_L(1 + \gamma_K))\, e_{LK} + \gamma_L(1 + \gamma_K)(\theta_{XK} - \theta_{YK})e_{LZ} + \gamma_K(1 + \gamma_L)(\theta_{XL} - \theta_{YL})e_{KZ} \geq 0.$$

Using the fact that $\theta_{XL} - \theta_{YL} = \theta_{YK} + \theta_{YZ} - \theta_{XK}$, we can rewrite the left-hand side of this inequality as

$$[\gamma_L(1+\gamma_K)e_{LZ}-\gamma_K(1+\gamma_L)e_{KZ}]\,(\theta_{XK}-\theta_{YK})+\gamma_L(1+\gamma_K)\theta_{YK}e_{LK}+\gamma_K(1+\gamma_L)\,(\theta_{YZ}e_{KZ}+\theta_{YL}e_{LK})\,. \tag{A-19}$$

Because $\theta_{XK}-\theta_{YK}>0$ (a consequence of $\gamma_L>\gamma_K$) and $\theta_{YZ}e_{KZ}+\theta_{YL}e_{LK}=-\theta_{YK}e_{KK}\geq 0$, whenever $e_{LK}\geq 0$ the expression in (A-19) is clearly nonnegative. Let us thus focus on the case where $e_{LK}<0$, in which case we must have $e_{LZ}\geq 0$ and $e_{KZ}\geq 0$. If $e_{LZ}=0$, then from $\gamma_L(1+\gamma_K)e_{LZ}\geq\gamma_K(1+\gamma_L)e_{KZ}$ we must also have $e_{KZ}=0$, and therefore also $\theta_{YK}e_{KK}+\theta_{YL}e_{LK}=0$. But this is impossible since $e_{KK}\leq 0$ and we have assumed $e_{LK}<0$. Therefore, we must have $e_{LZ}>0$, and also $e_{KZ}>0$.

The negative semidefiniteness of $\mathbf{E}_{LK}$ implies that $e_{LL}e_{KK}-e_{LK}^2\geq 0$, and, using $\theta_{YL}e_{LL}=-\theta_{YK}e_{LK}-\theta_{YZ}e_{LZ}$ and $\theta_{YK}e_{KK}=-\theta_{YL}e_{LK}-\theta_{YZ}e_{KZ}$, that

$$\theta_{YK}e_{LK}e_{KZ}+\theta_{YL}e_{LK}e_{LZ}+\theta_{YZ}e_{LZ}e_{KZ}\geq 0. \tag{A-20}$$

Given that $e_{LZ}>0$, Equation (A-20) implies that $\theta_{YZ}e_{KZ}+\theta_{YL}e_{LK}\geq-\theta_{YK}\frac{e_{LK}e_{KZ}}{e_{LZ}}$. Therefore, the expression in (A-19) is larger than or equal to

$$[\gamma_L(1+\gamma_K)e_{LZ}-\gamma_K(1+\gamma_L)e_{KZ}]\,(\theta_{XK}-\theta_{YK})+\theta_{YK}\left[\gamma_L(1+\gamma_K)e_{LK}-\gamma_K(1+\gamma_L)\frac{e_{LK}e_{KZ}}{e_{LZ}}\right],$$

that is,

$$[\gamma_L(1+\gamma_K)e_{LZ}-\gamma_K(1+\gamma_L)e_{KZ}]\left(\theta_{XK}-\theta_{YK}+\frac{\theta_{YK}e_{LK}}{e_{LZ}}\right).$$

Given that $\gamma_L(1+\gamma_K)e_{LZ}\geq\gamma_K(1+\gamma_L)e_{KZ}$, it remains to be shown that $\theta_{XK}-\theta_{YK}+\frac{\theta_{YK}e_{LK}}{e_{LZ}}\geq 0$, that is, using $\theta_{XK}=\frac{\gamma_L\theta_{YK}}{\gamma_K\theta_{YL}+\gamma_L\theta_{YK}}$, that

$$\frac{\gamma_L}{\gamma_K\theta_{YL}+\gamma_L\theta_{YK}}-1+\frac{e_{LK}}{e_{LZ}}\geq 0. \tag{A-21}$$

The inequality in (A-20) implies that $\theta_{YZ}e_{LZ}\geq-\theta_{YK}e_{LK}-\theta_{YL}\frac{e_{LK}e_{LZ}}{e_{KZ}}$, and, using $\gamma_L(1+\gamma_K)e_{LZ}\geq\gamma_K(1+\gamma_L)e_{KZ}$ again, that

$$e_{LZ}\geq-\frac{\theta_{YK}}{\theta_{YZ}}e_{LK}-\frac{\theta_{YL}\gamma_K(1+\gamma_L)}{\theta_{YZ}\gamma_L(1+\gamma_K)}e_{LK}=\left[\frac{\theta_{YK}\gamma_L(1+\gamma_K)+\theta_{YL}\gamma_K(1+\gamma_L)}{\theta_{YZ}\gamma_L(1+\gamma_K)}\right](-e_{LK}),$$

leading to

$$\frac{e_{LK}}{e_{LZ}}\geq-\frac{\theta_{YZ}\gamma_L(1+\gamma_K)}{\theta_{YK}\gamma_L(1+\gamma_K)+\theta_{YL}\gamma_K(1+\gamma_L)}.$$

Plugging this lower bound into Equation (A-21) and simplifying yields a lower bound equal to

$$\frac{\theta_{YL}\left(\gamma_L-\gamma_K\right)\left(\theta_{YK}\gamma_L+\theta_{YL}\gamma_K+\gamma_L\gamma_K\right)}{\left(\gamma_K\theta_{YL}+\gamma_L\theta_{YK}\right)\left(\theta_{YK}\gamma_L(1+\gamma_K)+\theta_{YL}\gamma_K(1+\gamma_L)\right)},$$

which is positive given that $\gamma_L > \gamma_K$. Q.E.D.

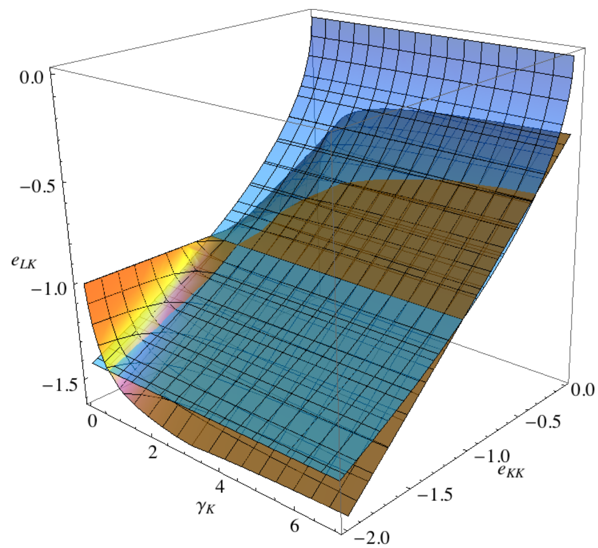
A.6 Example of pollution-enhancing tax in Fullerton and Heutel (2007a)

In the modeling approach of Fullerton and Heutel (2007a) ($\beta = 0$, $\alpha = \theta_{XL}$), an increase in the tax on the polluting input may lead to increased pollution. For instance, the following set of parameter values leads to the counterintuitive outcome: $\sigma_u = \sigma_X = 0$, $\theta_{YL} = \frac{1}{8}$, $\theta_{YK} = \frac{1}{2}$, $\gamma_L = 1$, $\gamma_K = \frac{12}{13}$, $e_{LL} = -1$, $e_{KK} = -1.1005$, $e_{ZZ} = -3$. The remaining parameters can be deduced using the relationships $\frac{\gamma_L\theta_{XL}}{\theta_{YL}} = \frac{\gamma_K\theta_{XK}}{\theta_{YK}}$, which determines $\theta_{XL} = 0.1875$, as well as the three equalities $\theta_{YL}e_{mL} + \theta_{YK}e_{mK} + \theta_{YZ}e_{mZ} = 0$, for $m \in \{L, K, Z\}$, which jointly determine the three cross-price Allen elasticities, $e_{LK} = -1.049$, $e_{LZ} = 1.732$, and $e_{KZ} = 1.817$. One can check that the resulting substitution matrix satisfies negative semidefiniteness. With these parameters, we obtain $\hat{Z} = 0.96\hat{\tau}_Z$, that is, a positive elasticity of pollution with respect to the pollution tax. Not surprisingly, in that case the value of the pollution tax actually decreases relative to the value of our preferred price indices. Indeed, denoting $\theta_X = \frac{p_X X}{p_X X + p_Y Y}$ and $\theta_Y = 1 - \theta_X$, we obtain $\hat{p}_Z - (\theta_X\hat{p}_X + \theta_Y\hat{p}_Y) = -0.8\hat{\tau}_Z$. Denoting $\theta_L = \frac{w\bar{L}}{w\bar{L}+r\bar{K}}$ and $\theta_K = 1 - \theta_L$, we obtain $\hat{p}_Z - (\theta_L\hat{w} + \theta_K\hat{r}) = -1.0\hat{\tau}_Z$.

Note that the assumption that $\sigma_u = \sigma_X = 0$ is not necessary for the sign reversal to occur. For instance, it happens for the following set of parameter values: $\sigma_u = 0.25$, $\sigma_X = 0.25$, $\theta_{YL} = 0.51$, $\theta_{YK} = 0.22$, $\gamma_L = 1$, $\gamma_K = 0.128$, $e_{LL} = -11$, $e_{KK} = -396$, and $e_{LZ} = -65.9989$.

Finally, note that the numerical example given in Fullerton and Heutel (2007a), footnote 14, to illustrate the counterintuitive outcome violates the negative semidefiniteness property of the substitution matrix in sector Y. This is easily seen by computing the determinant of the submatrix $\begin{pmatrix} e_{LL} & e_{LZ} \\ e_{LZ} & e_{ZZ} \end{pmatrix}$, which turns out to be negative rather than non-negative for their choice of substitution elasticities. Since their example assumes $e_{LK} > 0$, the value of the equilibrium system determinant (Δ) has the right (positive) sign. In their example, the sign reversal is due to the numerator (Δ_Z) having the wrong (positive) sign. We have shown in Appendix A.3 that this numerator is in fact non-positive for all admissible values of the Allen substitution elasticities.

Figure A.2: Violations of the law of input demand with p_Y as numeraire



Note: We set $\sigma_X = \sigma_u = 0$, $\gamma_L = 1$, $e_{LL} = -1$, and assume $e_{LK} \leq 0$. All points located below the blue surface violate the restrictions from theory (the negative semidefiniteness of the Slutsky matrix implies that $e_{LL}e_{KK} \geq e_{LK}^2$). Within the relevant subspace located above the blue surface, the parameter region located below the orange surface supports violations of the law of input demand.

A.7 Violation of the law of input demand with p_Y as numeraire

Figure A.2 depicts the parameter space supporting violations of the law of input demand when p_Y is used as the anchoring price index. As in the main text, we set $\sigma_X = \sigma_u = 0$, $\gamma_L = 1$, $e_{LL} = -1$, and consider $e_{LK} \leq 0$. The choice $e_{LL} = -1$ is a normalization that only affects the scale of e_{KK} and e_{LK}. As such, the values of e_{KK} and e_{LK} can be reinterpreted as those of $\frac{e_{KK}}{|e_{LL}|}$ and $\frac{e_{LK}}{|e_{LL}|}$, respectively. Similarly shaped violation regions exist for alternative values of γ_L.

B Model with one clean input

Consider an economy with one clean input, labor. We consider price indices of the form $\mathcal{P} = p_Y^{\beta} p_X^{1-\beta}$. ($\hat{p}_X = \hat{w}$ due to constant returns to scale in sector X.) The displaced equilibrium is given by the following set of equations:

$$\begin{aligned}
\hat{L}_X + \gamma_L \hat{L}_Y &= 0 \\
\hat{X} - \hat{L}_X &= 0 \\
\hat{Y} - \theta_{YL}\hat{L}_Y - \theta_{YZ}\hat{Z} &= 0 \\
\hat{p}_Y - \theta_{YL}\hat{p}_X &= \theta_{YZ}\hat{\tau}_Z \\
\hat{L}_Y - \hat{Z} + \sigma_Y \hat{p}_X &= \sigma_Y \hat{\tau}_Z \\
\hat{X} - \hat{Y} + \sigma_u \hat{p}_X - \sigma_u \hat{p}_Y &= 0 \\
\beta \hat{p}_Y + (1-\beta)\hat{p}_X &= 0.
\end{aligned}$$

where the notation is as before and $\theta_{YL} + \theta_{YZ} = 1$. It is easy to show that the determinant of the system has a constant sign, and that

$$\frac{\hat{Z}}{\hat{\tau}_Z} = -\frac{\sigma_u \theta_{YZ} + \sigma_Y(\gamma_L + \theta_{YL})}{(1+\gamma_L)(1-\beta\theta_{YZ})} < 0.$$

The result can also be derived directly from the model with two clean inputs, specialized to the case where $\sigma_X = 0$, $e_{LK} = 0$, and $\gamma_L = \gamma_K$. (To see why one must have $\gamma_L = \gamma_K$, consider Equations (A-1) and (A-2). Fixed proportions in industry X imply $\hat{L}_X = \hat{K}_X$. Fixed proportions between labor and capital in industry Y imply $\hat{L}_Y = \hat{K}_Y$. Hence $\gamma_L = \gamma_K$.) We thus have $\Delta = C_3$, and since $e_{LK} = 0$, $\Delta > 0$ for all acceptable parameter values.

C Additional simulations

C.1 Parameter values considered in Fullerton and Heutel (2007a)

Simulations reported in Tables C.1 and C.2 are performed using the parameter values: $\sigma_u = \sigma_X = e_{LK} = 1$, $e_{LZ} = 1$, $e_{KZ} = -0.5$, and $\theta_{YZ} = 0.25$. As in Fullerton and Heutel (2007a), the values of the remaining model parameters are determined by the values of $\gamma_K - \gamma_L$ given in the tables and the conditions $K_X + K_Y = 0.4$, $L_X + L_Y = 0.6$, and $L_Y + K_Y = 0.2$.

C.2 Model with higher pollution tax

Tables C.3 and C.4 summarize results for a model with parameters identical to those in Table 1, except for the expenditure share on the dirty input in industry X, which is doubled

Table C.1: Pollution effect of a 10% increase in the nominal pollution tax (%)

$\gamma_K - \gamma_L$	θ_{YK}	θ_{XK}	$\hat{Z}$						
			p_X	p_Y	$\mathcal{P}$	w	r	$\mathcal{R}$	st. dev.
-0.25	0.1515	0.4495	-7.50	-10.10	-8.02	-7.64	-7.35	-7.52	0.95
-0.20	0.1818	0.4394	-7.00	-9.42	-7.48	-7.14	-6.82	-7.01	0.89
-0.15	0.2118	0.4294	-6.49	-8.72	-6.93	-6.65	-6.29	-6.50	0.82
-0.10	0.2416	0.4195	-5.98	-8.03	-6.39	-6.14	-5.77	-5.99	0.76
-0.05	0.2710	0.4097	-5.48	-7.33	-5.85	-5.64	-5.26	-5.48	0.69
0.00	0.3000	0.4000	-4.97	-6.63	-5.31	-5.14	-4.75	-4.97	0.62
0.05	0.3286	0.3905	-4.47	-5.94	-4.77	-4.63	-4.25	-4.47	0.55
0.10	0.3566	0.3811	-3.98	-5.26	-4.24	-4.13	-3.76	-3.97	0.49
0.15	0.3841	0.3720	-3.49	-4.59	-3.71	-3.63	-3.28	-3.48	0.42
0.20	0.4110	0.3630	-3.00	-3.93	-3.19	-3.13	-2.81	-2.99	0.36

Table C.2: Incidence effects of a 10% increase in the nominal pollution tax (%)

$\gamma_K - \gamma_L$	Sources side: $\hat{w} - \hat{r}$							Uses side: $\hat{p}_Y - \hat{p}_X$						
	p_X	p_Y	$\mathcal{P}$	w	r	$\mathcal{R}$	st. dev.	p_X	p_Y	$\mathcal{P}$	w	r	$\mathcal{R}$	st. dev.
-0.25	0.39	0.52	0.41	0.39	0.38	0.39	0.05	2.57	3.46	2.75	2.62	2.52	2.58	0.33
-0.20	0.47	0.63	0.50	0.48	0.45	0.47	0.06	2.57	3.46	2.75	2.62	2.50	2.57	0.33
-0.15	0.55	0.73	0.58	0.56	0.53	0.55	0.07	2.56	3.44	2.74	2.62	2.48	2.56	0.33
-0.10	0.63	0.84	0.67	0.64	0.60	0.63	0.08	2.55	3.42	2.72	2.61	2.46	2.55	0.32
-0.05	0.71	0.94	0.75	0.73	0.68	0.71	0.09	2.53	3.38	2.70	2.60	2.42	2.53	0.32
0.00	0.79	1.05	0.84	0.81	0.75	0.79	0.10	2.50	3.33	2.67	2.58	2.39	2.50	0.31
0.05	0.86	1.15	0.92	0.89	0.82	0.86	0.11	2.47	3.28	2.63	2.56	2.35	2.47	0.31
0.10	0.94	1.25	1.00	0.98	0.89	0.94	0.12	2.43	3.22	2.59	2.52	2.30	2.43	0.30
0.15	1.02	1.34	1.09	1.06	0.96	1.02	0.12	2.39	3.15	2.54	2.49	2.25	2.39	0.29
0.20	1.10	1.44	1.17	1.15	1.03	1.09	0.13	2.35	3.07	2.49	2.45	2.19	2.34	0.28

to $\theta_{YZ} = 0.50$. Expenditure shares on labor and capital are reduced to $\theta_{YL} = 0.30$ and $\theta_{YK} = 0.20$, preserving their ratio.

A notable consequence of this change is that the use of p_X as anchoring index yields predictions that do not closely approximate those obtained using the CPI.

C.3 Parameter values adapted from Fullerton and Ta (2019)

Fullerton and Ta (2019) calibrate a Cobb-Douglas model of a closed economy to U.S. data to analyze the effects of a carbon tax on prices, inputs, outputs, and welfare. Their model specification does not squarely fit the model discussed here because the dirty input (fossil fuels) is used in the production of both electricity (a good consumed by households) and a composite consumption good. The consumption good (X) is produced using electricity and fossil fuels. Because production of the consumption good uses mostly labor and

Table C.3: Pollution effect of a 10% increase in the nominal pollution tax (%)

e_{KZ}	e_{LZ}	$\hat{Z}$					
		p_X ($\mathcal{R}$)	p_Y	$\mathcal{P}$	w	r	st. dev.
0.0	0.0	-4.00	-8.00	-4.80	-4.00	-4.00	1.46
0.5	0.0	-5.16	-10.32	-6.19	-5.05	-5.33	1.88
1.0	0.0	-6.25	-12.50	-7.50	-6.00	-6.67	2.28
0.0	0.5	-5.76	-11.52	-6.91	-5.89	-5.58	2.11
0.5	0.5	-7.00	-14.00	-8.40	-7.00	-7.00	2.56
1.0	0.5	-8.16	-16.33	-9.80	-8.00	-8.42	2.97
0.0	1.0	-7.45	-14.89	-8.94	-7.78	-7.00	2.74
0.5	1.0	-8.76	-17.53	-10.52	-8.95	-8.50	3.21
1.0	1.0	-10.00	-20.00	-12.00	-10.00	-10.00	3.65

Table C.4: Incidence effect of a 10% increase in the nominal pollution tax (%)

e_{KZ}	e_{LZ}	Sources side: $\hat{w} - \hat{r}$						Uses side: $\hat{p}_Y - \hat{p}_X$					
		p_X ($\mathcal{R}$)	p_Y	$\mathcal{P}$	w	r	st. dev.	p_X ($\mathcal{R}$)	p_Y	$\mathcal{P}$	w	r	st. dev.
0.0	0.0	0.00	0.00	0.00	0.00	0.00	0.00	5.00	10.00	6.00	5.00	5.00	1.83
0.5	0.0	-0.54	-1.08	-0.65	-0.53	-0.56	0.20	5.00	10.00	6.00	4.89	5.17	1.82
1.0	0.0	-1.04	-2.08	-1.25	-1.00	-1.11	0.38	5.00	10.00	6.00	4.80	5.33	1.82
0.0	0.5	0.54	1.09	0.65	0.56	0.53	0.20	5.00	10.00	6.00	5.11	4.84	1.83
0.5	0.5	0.00	0.00	0.00	0.00	0.00	0.00	5.00	10.00	6.00	5.00	5.00	1.83
1.0	0.5	-0.51	-1.02	-0.61	-0.50	-0.53	0.19	5.00	10.00	6.00	4.90	5.16	1.82
0.0	1.0	1.06	2.13	1.28	1.11	1.00	0.39	5.00	10.00	6.00	5.22	4.70	1.84
0.5	1.0	0.52	1.03	0.62	0.53	0.50	0.19	5.00	10.00	6.00	5.11	4.85	1.83
1.0	1.0	0.00	0.00	0.00	0.00	0.00	0.00	5.00	10.00	6.00	5.00	5.00	1.83

Table C.5: Model parameters adapted from Fullerton and Ta (2019)

Parameter	Notation in Fullerton and Ta (2019), Table 1	Value
e_{LK}	N/A (Cobb-Douglas)	1
σ_u	N/A (Cobb-Douglas)	1
σ_X	N/A (Cobb-Douglas)	1
γ_L	$\frac{L_E^0}{L_X^0}$	0.012
γ_K	$\frac{K_E^0}{K_X^0}$	0.031
θ_{XL}	$\frac{\beta}{\alpha+\beta}$	0.455
θ_{YL}	ϵ	0.138
θ_{YK}	δ	0.415

Note: To ensure internal consistency, once we have set γ_L, γ_K, θ_{YL}, and θ_{YK} we recompute $\theta_{XL} = \frac{\gamma_K \theta_{YL}}{\gamma_K \theta_{YL} + \gamma_L \theta_{YK}}$ to ensure that the equality $\frac{\gamma_L \theta_{XL}}{\theta_{YL}} = \frac{\gamma_K \theta_{XK}}{\theta_{YK}}$ holds exactly.

capital (the cost shares on electricity and fossil fuels are very small), to fit their calibration data within our framework we assume that production of X rests solely on capital and labor. Thus sector Y (the dirty sector) is represented by the electricity sector, which uses a large share of fossil fuels in terms of input costs. Table C.5 summarizes the calibration data.

Tables C.6 and C.7 summarize results for a set of selected substitution elasticities. We set $e_{LK} = 1$ but allow the substitution elasticities e_{KZ} and e_{LZ} to vary. We exclude sets of substitution elasticities that violate the negative semidefiniteness of the Slutsky matrix. Note that the choices of p_X and $\mathcal{R}$ as anchoring indices lead to indistinguishable results with the level of precision used here, although they are not identical.

The results indicate that choosing p_Y as the anchoring index leads to predictions very different from those obtained with other indices. These discrepancies are driven by the large expenditure share on the polluting input in sector Y and the small shares of labor and capital used in that sector. To see why, note that differences in predictions are solely due to differences in Δ, the determinant of the equilibrium system. But $\Delta = C_1\sigma_X + C_2\sigma_u + C_3$, and the expressions for C_1, C_2, and C_3 in Section A.3 make it clear that whenever γ_L and γ_K are small relative to one, which is the case here, the term C_1 dominates and thus determines the magnitude of Δ. (Recall that $\sigma_X = \sigma_u = 1$ in the parameterization.) Since the magnitude of C_1 directly depends on $B_L + B_K = 1 - B_Z$, choices of anchoring indices leading to very different magnitudes of $B_Z \equiv \beta\theta_{YZ}$ will lead to very different predictions. When p_Y is chosen, $\beta = 1$, while for all other choices considered in Tables C.6 and C.7 B_Z is either close to or equal to zero. It is equal to zero whenever $\beta = 0$ (choice of p_X, w, r, or

Table C.6: Pollution effect of a 10% increase in the nominal pollution tax (%)

e_{KZ}	e_{LZ}	$\hat{Z}$						
		p_X	p_Y	$\mathcal{P}$	w	r	$\mathcal{R}$	st. dev.
1.0	-0.5	-7.88	-14.28	-8.02	-7.85	-7.91	-7.88	2.37
0.0	0.0	-4.36	-7.88	-4.44	-4.38	-4.35	-4.36	1.30
0.5	0.0	-6.48	-11.72	-6.60	-6.49	-6.48	-6.48	1.94
1.0	0.0	-8.59	-15.55	-8.74	-8.57	-8.61	-8.59	2.58
0.0	0.5	-5.06	-9.13	-5.15	-5.09	-5.03	-5.06	1.51
0.5	0.5	-7.18	-12.98	-7.31	-7.20	-7.17	-7.18	2.15
1.0	0.5	-9.30	-16.82	-9.46	-9.28	-9.31	-9.30	2.79
0.0	1.0	-5.75	-10.37	-5.85	-5.79	-5.72	-5.75	1.71
0.5	1.0	-7.88	-14.23	-8.02	-7.91	-7.86	-7.88	2.36
1.0	1.0	-10.00	-18.08	-10.18	-10.00	-10.00	-10.00	3.00

Table C.7: Incidence effect of a 10% increase in the nominal pollution tax (%)

e_{KZ}	e_{LZ}	Sources side: $\hat{w} - \hat{r}$							Uses side: $\hat{p}_Y - \hat{p}_X$						
		p_X	p_Y	$\mathcal{P}$	w	r	$\mathcal{R}$	st. dev.	p_X	p_Y	$\mathcal{P}$	w	r	$\mathcal{R}$	st. dev.
1.0	-0.5	-0.08	-0.15	-0.08	-0.08	-0.08	-0.08	0.02	4.48	8.11	4.56	4.46	4.50	4.48	1.35
0.0	0.0	0.08	0.14	0.08	0.08	0.08	0.08	0.02	4.46	8.05	4.54	4.48	4.44	4.46	1.33
0.5	0.0	0.01	0.02	0.01	0.01	0.01	0.01	0.00	4.47	8.08	4.55	4.47	4.47	4.47	1.34
1.0	0.0	-0.05	-0.10	-0.06	-0.05	-0.05	-0.05	0.02	4.48	8.10	4.56	4.46	4.49	4.48	1.35
0.0	0.5	0.11	0.19	0.11	0.11	0.11	0.11	0.03	4.46	8.04	4.54	4.48	4.44	4.46	1.33
0.5	0.5	0.04	0.07	0.04	0.04	0.04	0.04	0.01	4.47	8.07	4.54	4.48	4.46	4.47	1.34
1.0	0.5	-0.03	-0.05	-0.03	-0.03	-0.03	-0.03	0.01	4.47	8.09	4.55	4.47	4.48	4.47	1.34
0.0	1.0	0.13	0.24	0.14	0.14	0.13	0.13	0.04	4.45	8.03	4.53	4.49	4.43	4.45	1.33
0.5	1.0	0.07	0.12	0.07	0.07	0.07	0.07	0.02	4.46	8.06	4.54	4.48	4.45	4.46	1.33
1.0	1.0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	4.47	8.08	4.55	4.47	4.47	4.47	1.34

$\mathcal{R}$) and it is close to zero when $\mathcal{P}$ is chosen because θ_Y is small (0.039 for this calibration).