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Gravity Models and the Law of Large Numbers

Abstract

Modern quantitative theories of international trade rely on the probabilistic representation of technology and the assumption of the Law of Large Numbers (LLN), which ensures that when the number of traded goods goes to infinity, trade flows can be expressed via a deterministic gravity equation that is log-linear in exporter-specific, importer-specific and bilateral trade cost components. This paper shows that when the number of traded goods is finite, the gravity equation has a structural stochastic component not related to the fundamental gravity forces. It provides a novel explanation of the differences in the goodness of fit of gravity models across different sectors observed in the data. It also suggests that when the LLN does not hold, the welfare gains from trade have a considerable stochastic component and should be characterized via distributions rather than point estimates. We use sectoral trade data and Monte Carlo simulations to develop a procedure with minimal data requirements that allows estimation of intervals for the welfare gains from trade.

JEL-Codes: F100, F600, F140, F170.

Keywords: trade gravity, Law of Large Numbers, gains from trade.

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1 Introduction

The gravity model is, perhaps, the most widely used empirical tool in international economics. The reasons behind its success include a parsimonious specification, minimal data requirements, close isomorphic connection to several leading general equilibrium models of international trade, and good empirical fit. Despite these factors, surprisingly little is known about how much we can trust the predictions of gravity models and what factors influence the accuracy of their predictions. This work aims at filling this gap.

In this paper, we show that the goodness of fit of the gravity model is governed by how closely the true data generating process is approximated by the Law of Large Numbers (LLN). When the number of internationally traded varieties goes to infinity, the LLN holds and the gravity equation converges to a deterministic function that is log-linear in the exporter- and importer-specific economic fundamentals and bilateral trade barriers.[1]We show, however, that when the number of traded varieties is finite and the LLN is violated, the gravity equation has a stochastic component that arises structurally from theory and is not related to mismeasurement of fundamental factors that affect trade flows.[2] We show that when the number of total varieties is small, the variance of the stochastic component is large which leads to relatively poor fit of gravity models to data and, more importantly, has implications for the reliability of counterfactual predictions. Using these insights, we develop a straightforward procedure that characterized counterfactual predictions as distributions rather than point estimates.

First, we use a discrete version of the gravity model to illustrate that deviations from the LLN create a purely stochastic component that is not related to the fundamental forces behind the gravity model of trade. We demonstrate that the size of the stochastic component is decreasing in the number of traded varieties and disappears under the LLN. We take these predictions to the data on bilateral trade for 215 countries and 68 narrow sectors as well as 10 broad sectors. At both sectoral levels, we show that the number of traded varieties is negatively related to the residual sum of squares and positively related to a measure of the goodness of fit of the gravity model.[3] To the best of our knowledge, this provides one of the first structural explanations of why the gravity model is successful in fitting trade data in some sectors and unsuccessful in others.

Second, we construct a series of Monte Carlo simulations that confirm our theoretical predictions and the empirical patterns found in the data. Next, we develop a procedure based on the simulations that allows us to characterize the distribution of counterfactual predictions of the gravity model. The intuition behind the procedure relies on the fact that due to the presence of

[1]This holds in most general equilibrium models of international trade including Eaton and Kortum (2002) and Melitz (2003). The same holds for most migration and spatial equilibrium models, e.g., see Anderson (2011).

[2]See Egger and Nigai (2015) and Agnosteva et al. (2019) for discussion of mismeasurement of trade costs in gravity models.

[3]We provide details on how we define and measure *varieties* in Section 3.

the structural stochastic term, observed trade shares consist of deterministic and random components, where the distribution of the latter is governed by the number of traded varieties. From this perspective, observed trade shares should be viewed as draws from a known distribution. Hence, methods that rely on using observed trade shares for counterfactual predictions, often called *hat algebra*, calculate a single counterfactual outcome out of a distribution of possible outcomes.[4] Our simulation procedure relies on making multiple theory-consistent draws of trade shares from a multinomial distribution with gravity fundamentals as *event probabilities* and the number of traded varieties as the *number of trials*. The procedure yields a full distribution of counterfactual results.

Finally, we use our proposed simulation procedure to build a simple guide for practitioners on how to put theory-consistent bounds on counterfactual predictions of gravity models with minimal data requirements. In fact, together with the data on the number of traded varieties across different sectors that this paper provides, estimating the gravity model with a full set of origin and destination fixed effects as well as customary proxies for trade costs is sufficient to inform the simulation procedure. We apply the proposed procedure to the external data for 41 countries and 17 sectors from the World Input-Output Database against the backdrop of a neoclassical trade model with input-output linkages as in Caliendo and Parro (2015). Using the proposed procedure, we calculate the distribution of the counterfactual predictions of the welfare gains from a 10% reduction in bilateral trade costs in all manufacturing sectors. We find a large degree of heterogeneity in the precision of the counterfactual welfare predictions across countries. For example, while the medians of 100 simulations of the welfare gains from trade for Ireland and Portugal are both near 8%, the distribution of the predictions is substantially wider for the latter. The 5^{th} and 95^{th} percentiles of the distributions of the welfare gains from trade for the two countries are $\{6.4\%, 10.5\%\}$ and $\{6.4\%, 16.6\%\}$, respectively.

This paper is related to several strands of the literature. First, we relate to the literature that tries to explain the origin of the stochastic error term in the gravity equation. Much of this literature focuses on mismeasurement of the fundamental forces behind trade gravity. Egger and Nigai (2015) and Agnosteva, Anderson and Yotov (2019) emphasize how using imperfect proxies for bilateral trade costs, e.g., bilateral distance, leads to an unobserved residual trade costs. Bergstrand, Larch and Yotov (2015) look at the measurement and identification of the effects of trade agreements, borders and distances on trade flows. We also relate to Anderson and Yotov (2010) and Anderson and Yotov (2012) who test how accurately the multilateral resistance terms are captured by the exporter- and importer-specific fixed effects in the gravity framework. All these works point to mismeasurement of fundamental drivers of trade as the main source of the deviation between theory and data. We, however, argue that even when all fundamentals can be measured with precision, violation of the LLN leads to a structural stochastic component of

[4]For illustrations of the hat algebra approach, see Dekle, Eaton and Kortum (2007) and Costinot and Rodriguez-Clare (2014)

the gravity equation.

This work is also related to papers that examine the determinants of the welfare gains from trade. Arkolakis, Costinot and Rodriguez-Clare (2012) show that in many quantiative trade models the welfare gains from trade (relative to autarky) can be calculated using the trade elasticity parameter and the share of intra-trade. Ossa (2015) notes that the confidence intervals around the estimates of the elasticity of trade parameter lead to confidence intervals around the welfare gains from trade. We, on the other hand, prove that the share of intra-trade also has a stochastic component whenever the LLN does not hold, which emphasizes a different source of uncertainty in the predictions of the gains from trade.

We also relate to the literature on discrete gravity. Our benchmark data generating process is in line with Eaton, Kortum and Sotelo (2012). Our empirical strategy of estimating the gravity equation and formulating a practitioner's guide relies on the results derived in Sotelo (2019), who shows that Multinomial Pseudo Maximum Likelihood (MPML) estimator is equivalent to Poisson Pseudo Maximum Likelihood (PPML) when the dependant variable is specified in trade shares and estimation includes a full set of fixed effects. From that perspective, we also relate to Santos Silva and Tenreyro (2006) who pioneered PPML in gravity models and to Henderson and Millimet (2008) who advocate for estimating the gravity equation in levels. This work also relies on the results obtained in Fally (2015) who shows that PPML estimation respects general equilibrium constraints imposed in quantitative trade theory. Finally, our results are relevant for a plethora of empirical works that use gravity-type models for counterfactual analysis. For an overview of this literature refer to Anderson (2011), Costinot and Rodriguez-Clare (2014), and Head and Mayer (2014).

The rest of the paper is organized as follows. In the next section, we demonstrate how deviations from the LLN give rise to a stochastic error term in the gravity equation. We demonstrate empirically that the size of this residual term is negatively related to the number of traded varieties using data on 215 countries and multiple sectors in Section 3. In Section 4, we conduct a series of Monte Carlo experiments that confirm the patterns observed in the data and show how the number of traded varieties is related to the moments of the distribution of the welfare gains from trade. Section 5 provides a guide to practitioners for calculating the distribution of the welfare gains from trade that does not require any additional data apart from those provided in the paper and those typically used in gravity estimation. We provide an example of how to use this guide against the backdrop of a multi-country multi-sector neoclassical trade model. We discuss sensitivity of our results and possible extensions in Section 6. Section 7 provides a brief conclusion.

2 Discrete Gravity and the LLN

In this section, we present a version of the gravity equation for describing bilateral trade flows that is based on discrete numbers of goods where each good is produced with a unique technology. Most canonical models that provide micro-foundations for the gravity equation, e.g., Eaton and Kortum (2002) and Chaney (2008), rely on the assumption of the existence of a measure of goods produced and traded across countries which combined with the LLN lead to log-additive versions of trade gravity. To emphasize how departures from the LLN affect the gravity equation, we focus on a discrete data generating process. Our results, however, also apply to continuous versions of the gravity model.

Consider a world with I countries and S sectors. Let N^s denote the total number of different goods (varieties) in sector s that can be produced and potentially traded across all countries. This number can also be interpreted as the number of available technologies such that index z is used to denote both. Each country $i \in I$ can produce each good z. The number of goods that i can produce at prices below level $\bar{z}$ is distributed according to the Poisson distribution as follows:

$$N_i^s \sim \text{Poisson}\left(e^{\alpha \ln(F_i^s) - \beta \ln(C_i^s) - \beta \ln(T_{ii}^s)}\ \bar{z}\right), \tag{1}$$

where F_i^s captures country i productivity fundamentals in sector s, C_i^s reflects production costs, and T_{ii}^s denotes internal trade costs. Parameters α and β measure the relative importance of the three factors in determining the distribution of N_i^s. This specification is in line with Eaton, Kortum and Sotelo (2012) that specified micro-foundations for the gravity equation based on the discrete number of technologies available for production.

Country i may also export goods in each sector s to other countries. However, in order to ship to importer j it must pay trade cost, T_{ij}, such that the number of goods below $\bar{z}$ that i offers to j is distributed as follows:

$$N_{ij}^s \sim \text{Poisson}\left(e^{\alpha \ln(F_i^s) - \beta \ln(C_i^s) - \beta \ln(T_{ij}^s)}\ \bar{z}\right) \tag{2}$$

Given that the total number of goods in sector s consumed in j is $N^s = \sum_k N_{kj}^s$ and by properties of the Poisson distributions, the conditional distribution of N_{ij}^s is as follows:

$$N_{ij}^s \sim \text{Multinomial}\left(N^s, \frac{e^{\alpha \ln(F_i^s) - \beta \ln(C_i^s) - \beta \ln(T_{ij}^s)}}{\sum_k e^{\alpha \ln(F_k^s) - \beta \ln(C_k^s) - \beta \ln(T_{kj}^s)}}\right). \tag{3}$$

Let us define the ratio of realized N_{ij}^s relative to the total number of draws, N^s, from the

Multinomial distribution as:

$$\pi_{ij}^s = \frac{N_{ij}^s}{N^s} \tag{4}$$

The LLN states that as N^s goes to infinity, realization, π_{ij}, converges in probability to the event probability parameters of the Multinomial distribution such that:

$$\pi_{ij}^s \xrightarrow{p} e^{\alpha \ln(F_i^s) - \beta \ln(C_i^s) - \beta \ln(T_{ij}^s) - \ln(M_j^s)} \text{ , where } M_j^s = \sum_k F_k^{s\alpha}(C_k^s T_{kj}^s)^{-\beta}. \tag{5}$$

However, when N^s is finite, π_{ij}^s deviates from $e^{\alpha \ln(F_i^s) - \beta \ln(C_i^s) - \beta \ln(T_{ij}^s) - \ln(M_j^s)}$ by the stochastic term ϵ_{ij}^s. Hence, when the LLN does not hold, the gravity equation can be specified as follows:

$$\pi_{ij}^s = e^{\alpha \ln(F_i^s) - \beta \ln(C_i^s) - \beta \ln(T_{ij}^s) - \ln(M_j^s)} + \epsilon_{ij}^s. \tag{6}$$

Equation 6 suggests two insights. First, observed π_{ij}^s should be viewed as a draw from a Multinomial distribution with parameters defined by the trade gravity forces and N^s. Second, the absolute value and variance of ϵ_{ij}^s is finite and decreasing in N^s. Note that this specification provides an exact foundation for estimating the gravity equation in levels as in Santos Silva and Tenreyro (2006). In the next Section, we show using real data that the size of ϵ_{ij}^s is governed by differences in N^s across sectors which explains why trade data from certain sectors are more suitable for gravity-type estimation.

3 Data and Estimation

We start with the 10-digit Harmonized System ($HS10$) to define a variety belonging to N^s. This classification is the most detailed encompassing system that is generally used to denote products in export and import data in the US. In our context, one could also interpret each $HS10$ as a production technology . We use the list of all recorded $HS10$ products during 1989 - 2006 from Feenstra, Romalis and Schott (2002). We assign each $HS10$ to sector s and count how many products fall within each category. We define sectors according to the second revision of the Standard International Trade Classification (SITC2) using two aggregation levels. First, we assign 11,398 $HS10$ products to 68 narrow sectors defined according to the SITC2 2-digit classification. Examples of such narrow sectors include {*Vegetables and Fruits*}, {*Paper and Paper Manufactures*}, and {*Road Vehicles*}. We report the number of varieties within each narrowly defined sector in the Appendix. Second, we also assign $HS10$ products to 10 broad sectors according to the SITC2 1-digit classification. We report these ten broad sectors along with the corresponding number of narrow sectors and $HS10$ products within each broad sector in Table 1.

Table 1: SECTOR CLASSIFICATION AND NUMBER OF $HS10$ PRODUCTS

SITC2 (1-digit sector)	# SITC2 (2-digit sector)	# $HS10$
Food and Live Animals	10	1260
Beverages and Tobacco	2	118
Crude Materials, Inedible, Except Fuels	9	796
Mineral Fuels, Lubric. and Related Mtrls	4	106
Animal and Vegetable Oils,Fats and Waxes	3	77
Chemicals and Related Products, N.E.S.	9	1518
Manufactured Goods Classif. By Material	9	2916
Machinery and Transport Equipment	9	3065
Miscellaneous Manufactured Articles	8	1456
Goods not Classif. Elsewhere	5	86
Total	68	11398
Average	7	1140
Standard Deviation	3	1135

Table 1 suggests that on average each broad sector consists of 7 narrow sectors and subsumes over a thousand different $HS10$ products. Broad sectors, however, are heterogeneous as indicated by the standard deviation of 1,135. The same holds for narrow sectors. Hence, at both levels of aggregation the data seem to offer variation across sectors, which we use to assess how N^s affects the size and variance of the stochastic error term, ϵ_{ij}^s, in Equation (7).

We next turn to estimating the gravity equation specified in 6 separately for each sector s. As is customary in the empirical gravity literature, we collect is-specific terms such that $\ln(X_i^s) = \alpha \ln(F_i^s) - \beta \ln(C_i^s)$ and the gravity equations is as follows:

$$\pi_{ij}^s = e^{\ln(X_i^s) - \beta \ln(T_{ij}^s) - \ln(M_j^s)} + \epsilon_{ij}^s. \tag{7}$$

It is customary to capture is- and js-specific terms via exporter-sector and importer-sector fixed effects and specify bilateral frictions as a function of observable bilateral variables related to geography as well as institutional and cultural similarities. For empirical purposes, we adopt three common proxies for bilateral trade frictions and parameterize $\ln(T_{ij}^s)$ as follows:

$$\beta \ln(T_{ij}^s) = \gamma^s \ln(\text{distance}_{ij}) + \mu^s \text{language}_{ij} + \eta^s \text{contiguity}_{ij}, \tag{8}$$

where language_{ij} and contiguity_{ij} are indicator functions that take the value of one whenever i and j share a common language and border, respectively. The data on distance_{ij}, language_{ij}, and contiguity_{ij} are from CEPII. Given parameterization of trade costs, we can now estimate Equation (7).

We estimate the gravity model specified in Equations (7) and (8) using Multinomial Pseudo

Maximum Likelihood (MPML), which in our context is identical to employing Poisson Pseudo Maximum Likelihood (PPML) customarily used in the gravity literature. Sotelo (2019) shows that the two estimators are identical as long as the dependent variable is specified in trade shares and a full set of importer-specific fixed effects are included. Fally (2015) shows that estimating Equation (6) using PPML is warranted from the structural perspective. Our focus, however, is different as we highlight the fact that ϵ_{ij}^s arises structurally when LLN does not hold and that the size of ϵ_{ij}^s depends on how many products exist in a sector.

We employ data on international trade flows for 215 countries from the COMTRADE database for 2006. We follow the usual practice of using a C.I.F. measure of imports complemented with F.O.B. observations of exports whenever C.I.F. data are missing. This gives us $215 \cdot (215 - 1)$ country pairs in each sector s.[5] First, we estimate the gravity model in (7) for 68 narrow sectors and record two statistics:

$$RSS^s = \sum_{i,j} \left(\pi_{ij}^s - \overline{\pi}_{ij}^s\right)^2 \text{ and } PR^s = 1 - \frac{var(\pi_{ij}^s - \overline{\pi}_{ij}^s)}{var(\pi_{ij}^s)} \text{ for } i \neq j, \tag{9}$$

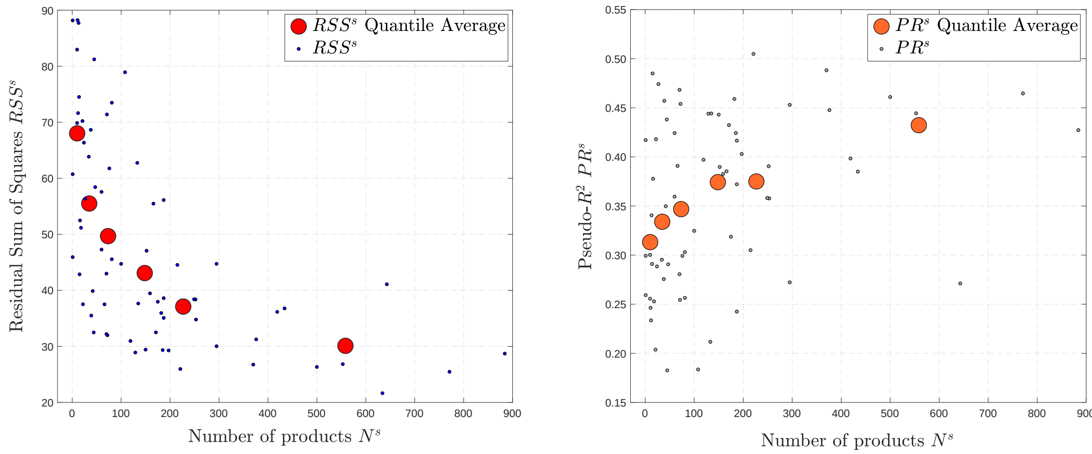
where $\overline{\pi}_{ij}^s$ are fitted values that capture fundamental gravity forces governing trade shares. The difference between the observed trade shares and the fitted values, then, captures the stochastic term, ϵ_{ij}^s.

The first statistics in Equation (9) is the residual sum of squares, RSS^s, which captures the absolute size of the error terms in sector s. The second measure is one minus the share of variance of the error term in the total variance in that sector, which is interpreted as pseudo-R^2.

We present the results obtained from estimating the gravity equation on 68 narrow sectors in Figure 1. In the left panel, we report RSS^s for each of the 68 sectors as well as average results for 6 quantiles defined according to N^s. The results show that the predictions stated in Section 2 hold in the data. The size of the stochastic residual term in the gravity equation is negatively related to the number of varieties in each sector. When N^s is low, realization π_{ij}^s significantly deviates from the gravity fundamentals captured in $\overline{\pi}_{ij}^s$ such that the RSS^s is high. As N^s increases, the size of the stochastic residual term decreases. In the right panel of Figure 1, we demonstrate the relationship between N^s and the Pseudo-R^2 measure based on the share of variance of ϵ_{ij}^s in total variance of π_{ij}^s. The results are also consistent with the predictions in Section 2 – as the number of varieties increases the share of variance of ϵ_{ij}^s declines and the goodness of fit of the gravity equation increases.

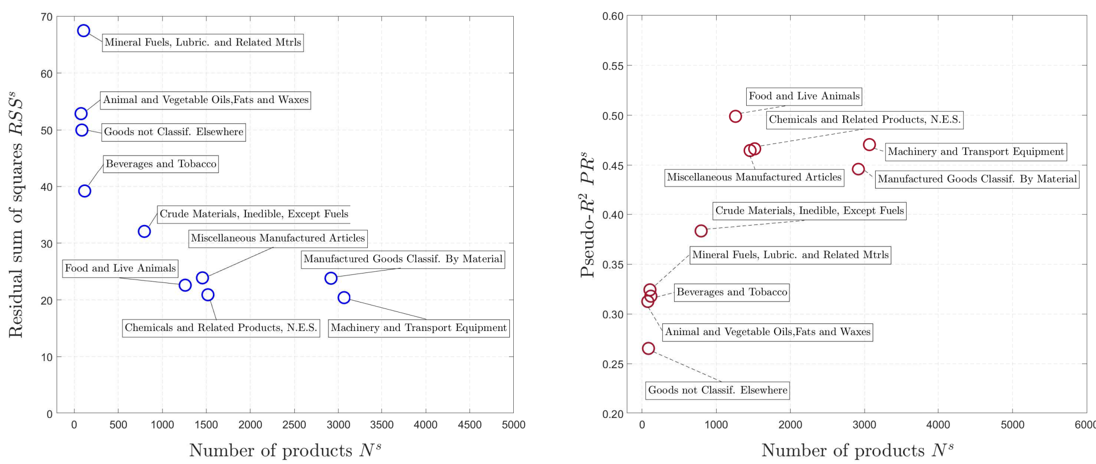
[5] As production data is generally not available for this sample of countries at sectoral levels, we compute trade shares using total sectoral imports as denominators.

Figure 1: Residual Sum of Squares & Pseudo-R^2 (SITC2 2-DIGIT)



Next, we report the results obtained from the data on 10 broad sectors in Figure 2. In the left panel, we demonstrate the relationship between N^s and RSS^s. Again, the results are consistent with the prediction that the number of varieties within a sector is negatively related to the size of the stochastic error term. The same applies to the predicted positive relationship between N^s and Pseudo-R^2 illustrated in the right panel of Figure 2.

Figure 2: Residual Sum of Squares & Pseudo-R^2 (SITC2 1-DIGIT)



At both levels of aggregation, the number of $HS10$ products within each sector seems to be an important determinant of the goodness of fit of the gravity model. Hence, violation of the LLN assumption customarily utilized in quantitative trade models offers one of the first explanations of why certain sectors, such as manufactured goods, are relatively more suitable for gravity estimation. As we will see, this result will also have important implications for comparative statics results that rely on the gravity framework.

4 Monte Carlo Simulations

In this section, we design a Monte Carlo simulation consistent with a quantitative general equilibrium model of trade with two goals. First, we aim to confirm the empirical results in Section 3. Second, we use the simulation design to develop a procedure that allows characterizing the distribution of the welfare gains from trade in comparative statics exercises.

For simplicity let us consider a single-sector version and drop superscript s. As we will see in the next section, this can be done without loss of generality. We set parameters related to productivity levels to unity such that $\alpha = 1$ and $F_i = 1$. Parameter β is the trade elasticity parameter, which we set to a customary value of 4. We sample the remaining economic primitives from the following distributions:

$$T_{ij} \sim \text{Normal}(3, 0.5); \quad L_i \sim \text{Normal}(100, 20), \tag{10}$$

where L_i reflects the population size of country i and though it does not directly enter the gravity equation in trade shares, it in part determines endogenous variable C_i, which in this setting we interpret as the wage in country i. Note that our results do not qualitatively depend on the choice of the distribution family of T_{ij} and L_i nor on the moments of the distributions.

Next, given the fundamentals, we solve for the endogenous variables, C_i and M_i using the general equilibrium constraints. The trade balance condition, which states that total exports must equal total imports, implicitly determines the values of C_i and M_i as follows:

$$L_i C_i = \sum_j e^{\alpha \ln(F_i) - \beta \ln(C_i) - \beta \ln(T_{ij}) - \ln(M_j)} (L_j C_j), \text{ where } M_j = \sum_k {F_k}^{\alpha} (C_k T_{kj})^{-\beta}. \tag{11}$$

We conduct each simulation in the following steps:

[1] Choose I and N and draw a vector of L_i and a matrix of T_{ij} as in Equation (10).

[2] Given fundamentals $\{\alpha, \beta, F_i, L_i, T_{ij}\}$, we solve the system in Equation (14)

[3] Specify $\overline{\pi}_{ij} = e^{\alpha \ln(F_i) - \beta \ln(C_i) - \beta \ln(T_{ij}) - \ln(M_j)}$. Given $\overline{\pi}_{ij}$ and N, we draw N_{ij} from the Multinomial(N, $\overline{\pi}_{ij}$) and calculate π_{ij}.

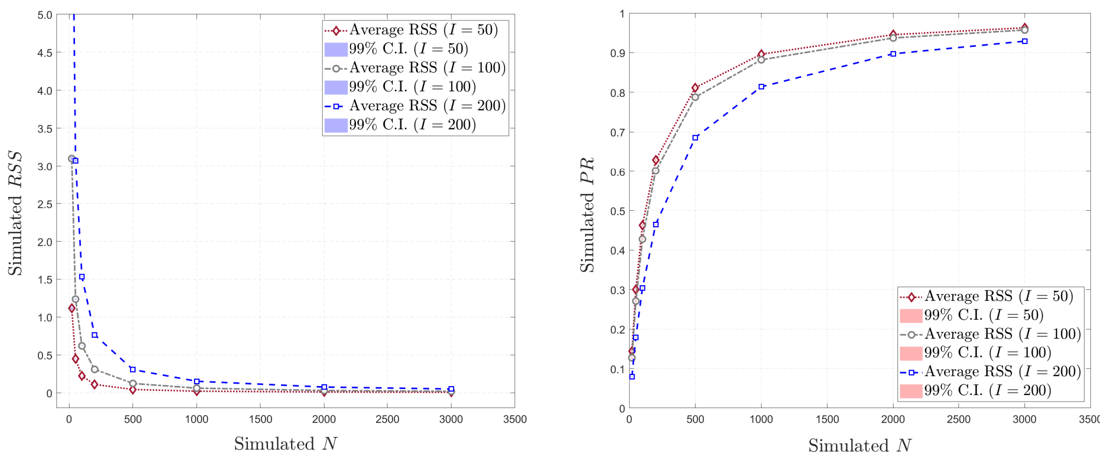
For each simulation, we calculate RSS and PR as follows:

$$RSS = \sum_{i,j} (\pi_{ij} - \overline{\pi}_{ij})^2 \text{ and } PR = 1 - \frac{var(\pi_{ij} - \overline{\pi}_{ij})}{var(\pi_{ij})} \text{ for } i \neq j. \tag{12}$$

We conduct 100 simulations for each value of $N = \{100, 200, 500, 1000, 3000, 5000\}$ and $I =$

$\{50, 100, 200\}$. We calculate average RSS and PR across 100 simulations for each $\{N, I\}$ pair and present the results in Figure 3.

Figure 3: SIMULATED RESIDUAL SUM OF SQUARES & PSEUDO-R^2



In the left panel of Figure 3, we present the residual sum of squares for different values of N and I. Consistent with out findings in Section 3, RSS is decreasing in N. On the other hand, as the number of countries in the sample increases, note that RSS also tends to increase. In the right panel of Figure 3, we see that the results obtained using simulated data suggests that the goodness of fit of the gravity model is increasing in N and decreasing in I. Hence, we conclude that deviations from the LLN as measured by N alone is sufficient to reproduce the patterns seen in the data in Section 3.

When the LLN does not hold, the realized trade shares, π_{ij}, depend on the gravity forces in $\overline{\pi}_{ij}$ and the stochastic error term:

$$\pi_{ij} = \overline{\pi}_{ij} + \epsilon_{ij}. \tag{13}$$

Meanwhile, the trade balance condition along with the other structural constraints must still hold. This means that wages, prices, and ultimately welfare also have a random component that is not driven by economic fundamentals. Let us use W_i and P_i to denote realized wages and prices. They are determined according to the trade balance condition as follows:

$$L_i W_i = \sum_j \pi_{ij}(L_j W_j) \quad \text{and} \quad P_j = \left(\sum_k {F_k}^{\alpha} (W_k T_{kj})^{-\beta} \right)^{-\frac{1}{\beta}}. \tag{14}$$

We can then measure real income as W_i/P_i.

It is customary to rely on π_{ij} in the counterfactual analysis. This method known as the *hat algebra* approach has become very popular because in many cases it does not require calibration of economic fundamentals such as productivity levels and trade costs but rather relies on observable data (see Dekle et al., 2007; Caliendo and Parro, 2015; Ossa, 2015). Let a' denote the counterfactual value of an arbitrary variable a such that the relative change is $\widehat{a} = a'/a$. Next, consider a counterfactual change in trade costs, $\widehat{T}_{ij}$. We can then specify the counterfactual equilibrium as follows:

$$Y_i \widehat{W}_i = \sum_j \pi'_{ij} Y_j \widehat{w}_j \quad \text{and} \quad \widehat{P}_j = \left(\sum_k \pi_{ij} (\widehat{W}_i \widehat{T}_{ij})^{-\beta} \right)^{-\frac{1}{\beta}}, \tag{15}$$

where π_{ij} and $Y_i = L_i W_i$ are observed in the data. Then, we can specify the gains from the reduction in trade costs as:

$$\text{Welfare Gains} = 100\% \cdot \left(\frac{\widehat{W}_i}{\widehat{P}_i} - 1 \right). \tag{16}$$

The welfare gains from trade are not deterministic as they depend on the realized trade shares, π_{ij}. As we have demonstrated, π_{ij} has a structural stochastic component whenever the LLN does not hold. Hence, to characterize the welfare gains from trade, one has to consider the whole distribution of π_{ij} rather than a single realization.

To illustrate how to derive the distribution of the welfare gains from a 10% reduction in all international trade costs, let us consider a sample of 50 countries with the gravity forces summarized in $\overline{\pi}_{ij}$ as before. Then we take the following steps:

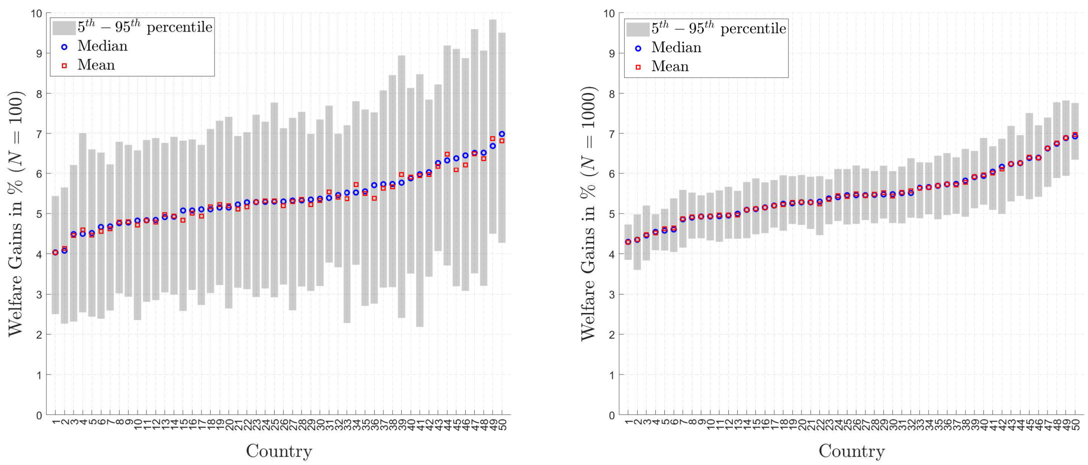
[1] Fix N and solve for $\overline{\pi}_{ij}$ given fundamentals as in Equation (14)

[2] Draw 100 samples of N_{ij} from the Multinomial(N, $\overline{\pi}_{ij}$) and calculate $\pi_{ij} = \dfrac{N_{ij}}{N}$

[3] Solve for the counterfactual equilibrium and calculate the welfare gains from trade for each realization, π_{ij}, as in Equations (15) and (16)

Without loss of generality let us index countries according to the median gains from trade in 100 samples that we draw. In Figure 4, we characterize the distribution of the welfare gains from trade by reporting the mean, median as well as the 5^{th} and 95^{th} percentiles. We report the results for $N = 100$ in the left panel of Figure 4, whereas the right panel illustrates the distribution of the welfare gains when $N = 1000$.

Figure 4: Welfare Gains from Trade



The results in Figure 4 suggest that the distribution of the welfare gains from trade is wide when N is small. For example, for a country where the median gains are at 5% the 5^{th} and 95^{th} percentiles are around 3% in 7%, respectively. However, as N increases and the realized trade shares are more closely linked to the gravity fundamentals, the distribution of the gains becomes narrower as illustrated in the right panel of Figure 4 where $N = 1000$. In the next section, we describe how to extend the procedure of calculating the distribution of the gains from trade to real-world data sets.

5 Guide for Practitioners

Implementing the simulation procedure laid out in the previous section to calculate the distribution of the welfare gains from trade under an arbitrary change in fundamentals such as a reduction of trade costs is straightforward as long as one knows $\overline{\pi}_{ij}$ and N. In this section, we provide a guide for practitioners with minimal data requirements on how to obtain the distribution of the welfare gains from trade based on real-world data.

We use external data from the World Input-Output Database (WIOD) for 2006 where we observe trade shares, π_{ij}^s, and total imports, Y_i^s. The database includes 40 countries plus the Rest of the World. We consider 16 manufacturing sectors and 1 sector which captures all service sectors. We match WIOD sectors to the SITC2 (2-digit) sectors to calculate the number of varieties N^s in each WIOD sector and report the results in Table 2.[6] As it is not possible for us to count the number of traded goods in the service sector we set it to 1,000, which is roughly the average number across all other sectors.

[6] We use two concordances to match SITC2 to WIOD sectors. First we use the concordance between SITC2 and ISIC2 available at `https://wits.worldbank.org/product_concordance.html`. Second, we use the concordance between ISIC2 and ISIC3 available at `https://unstats.un.org/unsd/classifications/econ/`.

Table 2: WIOD Sectors and N^s

WIOD sector	N^s
Agriculture, Hunting, Forestry and Fishing	1,193
Mining and Quarrying	292
Food, Beverages and Tobacco	945
Textiles and Textile Products	749
Leather, Leather and Footwear	337
Wood and Products of Wood and Cork	335
Pulp, Paper, Paper , Printing and Publishing	269
Coke, Refined Petroleum and Nuclear Fuel	78
Chemicals and Chemical Products	1,528
Rubber and Plastics	505
Other Non-Metallic Mineral	1,059
Basic Metals and Fabricated Metal	690
Machinery, Nec	1,061
Electrical and Optical Equipment	1,835
Transport Equipment	1,735
Manufacturing, Nec; Recycling	2,651
Services	1,000

We combine the WIOD data with the data on (log) bilateral distance, common language and contiguity from CEPII and estimate the gravity equation for each sector s to obtain fitted values $\overline{\pi}^s_{ij}$. Given $\overline{\pi}^s_{ij}$ and N^s we draw 100 samples of N^s_{ij} and calculate 100 realizations π^s_{ij}. We then use the *hat algebra* approach against the backdrop of a neoclassical multi-sector model of international trade with input-output linkages as in Caliendo and Parro (2015). We consider a 10% reduction in bilateral trade costs in all manufacturing sectors and compute the welfare gains for each country for each draw of π^s_{ij}. We briefly sketch the model in changes below.

Given the counterfactual change in trade costs, $\widehat{T}^s_{ij}$, we first solve for the changes in the unit cost of production:

$$\widehat{C}^s_i = \widehat{W}_i^{\gamma^s_i} \left(\prod_{s'} (\widehat{P}^s_i)^{\nu_i^{s's}} \right)^{1-\gamma^s_i}, \tag{17}$$

where γ^s_i and $\nu_i^{s's}$ are the value added share and input-output share that producers in i in sector s source from s', respectively. Both parameters are calculated using data from the WIOD. Next, we characterize changes in prices:

$$\widehat{P}^s_j = \left(\sum_k \pi^s_{kj} (\widehat{C}^s_k)^{-\beta^s} (\widehat{T}^s_{kj})^{-\beta^s} \right)^{-\frac{1}{\beta^s}}, \tag{18}$$

where β^s is the sectoral trade elasticity parameter with the values from Caliendo and Parro

(2015). Next, we calculate the level of counterfactual expenditures:

$$Y_i^{s\prime} = \sum_{s'} \nu_i^{s's} \sum_k \pi_{ik}^{s}{}' Y_k^{s\prime} + \alpha_i^s (\widehat{W}_i (W_i L_i) + D_i), \tag{19}$$

where α_i^s is the Cobb-Douglas consumption share calculated from WIOD. D_i is the deficit constant observed in the data and $L_i W_i$ is total value added from WIOD.[7] We also specify the counterfactual trade shares as:

$$\pi_{ij}^{s}{}' = \pi_{ij}^s \left(\frac{\widehat{C}_k^s \widehat{T}_{kj}^s}{\widehat{P}_j^s} \right)^{-\beta^s} \tag{20}$$

Finally, we calculate counterfactual changes in wages from the trade balance condition:

$$\sum_s \sum_j \pi_{ij}^{s}{}' Y_j^{s\prime} + D_i = \sum_s \sum_j \pi_{ji}^{s}{}' Y_i^{s\prime} \tag{21}$$

Given the calculated values of $\widehat{W}_i$ and $\widehat{P}_i$ we calculate the counterfactual changes in welfare as in Equation (16).

We characterize the distribution of the welfare gains from trade for all countries in Figure 5 by reporting the mean as ○ and median as □. We use ▽ to denote the welfare gains from trade when using unconditional observed data. These are the gains that would be predicted by the existing *hat algebra* approach. We also report the interval between the 1^{st} and 99^{th} percentiles and between the 5^{th} and 95^{th} percentiles.

Figure 5: Distribution of the Welfare Gains from Trade

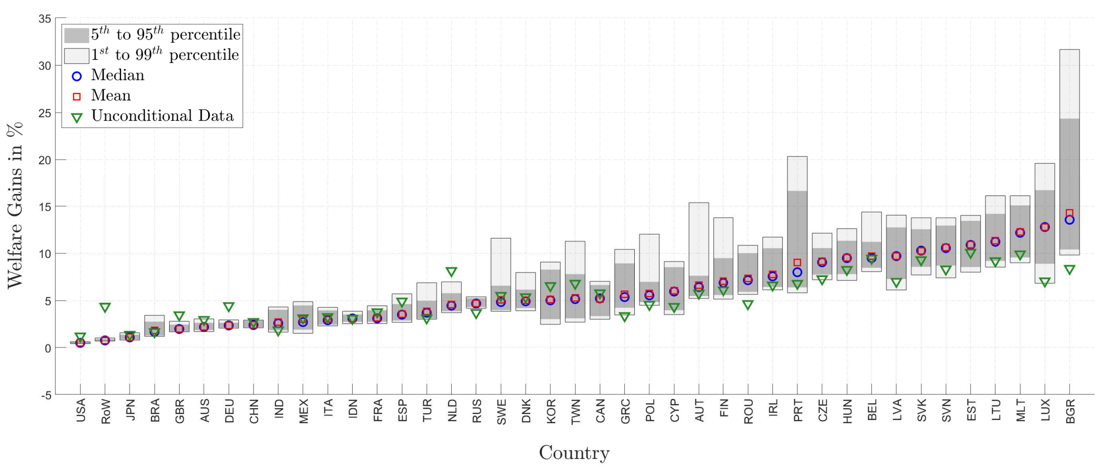


Figure 5 suggests that the distributions of the gains are generally wider for countries with higher median gains. There is, however, a substantial degree of heterogeneity. For example, while Ireland and Portugal have similar median gains of roughly 8%, the 5^{th} and 95^{th} percentiles of

[7] We keep D_i constant to the world GDP in the counterfactual experiments.

the gains for the two countries are $\{6.4\%, 10.5\%\}$ and $\{6.4\%, 16.6\%\}$, respectively. The results in Figure 5 also show that using unconditional data on π_{ij}^s to calculate the welfare gains from trade does not produce unbiased estimates of the average or median gains. While most of the estimates based on the unconditional data fall inside the intervals between the 1^{st} and 99^{th} percentiles, some estimates could be considered outliers. For example, the gains based on the unconditional data for Germany and Bulgaria are 4.4% and 8.4%, respectively. The former is above the 99^{th} percentile of the gains for Germany and the latter is below the 1^{st} percentile for Bulgaria. Hence, these two results based on the unconditional data are not theory consistent as they are not explained by the respective economic fundamentals but rather occur because the LLN does not hold.

Overall, Figure 5 suggests that accounting for sectoral differences in N^s and considering the distribution of the welfare gains from trade rather than point estimates is important. On the one hand, for certain countries the distributions of the gains are narrow and centered around the median such that the counterfactual results based on a single realization of π_{ij} are informative. On the other hand, for a majority of countries that we consider, the distributions of the gains are sufficiently wide. In this case, the *hat algebra* approach based only on the observable data cannot provide the whole picture and using the simulation approach proposed in this paper is warranted.

6 Discussion and Extensions

So far, we have used a discrete version of the gravity equation to illustrate how deviations from the LLN affect the goodness of fit and what implications it has for comparative statics analysis. It should be clear, however, that the main mechanisms highlighted in this paper apply to models that lead to continuous gravity equations and that the proposed simulation procedure to derive the distribution of the welfare gains from trade remains valid.

For example, consider the gravity equation produced by a trade model based on perfect competition as in Eaton and Kortum (2002):

$$\pi_{ij} = e^{\alpha \ln(F_i) - \beta \ln(C_i) - \beta \ln(T_{ij}) - \ln(M_j)} + \epsilon_{ij}, \tag{22}$$

where $\alpha \ln(F_i)$ is now interpreted as the scale parameter of the Fréchet productivity distribution in exporter i and β – as the shape parameter of the Fréchet distribution common to all countries. In this case, we interpret N as the number of draws from the Fréchet distribution in each country such that under the LLN the stochastic term, ϵ_{ij}, converges to zero. Violations of the LLN would mean that the number of the productivity draws is finite and entail the same implication that we have considered in the discrete case.

A similar logic can applied to models based on monopolistic competition such as Krugman (1980) and Melitz (2003) if the productivity distribution in the latter is Pareto as in Chaney (2008). In this case, the structural gravity equation above would still hold though the interpretation of the exporter- and importer-specific terms would change. We would interpret N as the number of firms and the LLN would be violated whenever the number of firms is finite. As is customary in the literature, one would have to assume that each firm produces a single unique product.

In this paper, we have considered deviations from the LLN to be the only source of the stochastic term ϵ_{ij}. However, there are also other potential sources of the stochastic term. For example, if the gravity fundamentals are measured with error the total error term would consist of two components:

$$\epsilon_{ij} = \varepsilon_{ij} + \xi_{ij}, \tag{23}$$

where ε_{ij} is the term due to deviations from the LLN and ξ_{ij} is the measurement error associated with the gravity fundamentals. However, the measurement error, ξ_{ij}, has no structural interpretation and should not be related to N. As we have seen in Section 3, the total sum of squares of ϵ_{ij} is largely explained by N. This suggests that even if ξ_{ij} exists its quantitative role in explaining total error ϵ_{ij} is likely smaller than ε_{ij}. One could also potentially address such a measurement error by including more proxies for trade costs and employing panel data with bilateral fixed effects (see Egger and Nigai, 2015), which is not the case for the component due to violations of the LLN.

7 Conclusion

This paper offers a novel explanation of why the goodness of fit of the gravity model of international trade is heterogeneous across different sectors. We show that when the LLN does not hold and the number of traded varieties is finite, the gravity model has a structural stochastic component that is unrelated to the forces underlying the gravity model of trade. When the number of traded varieties is small, the share of the variance of this stochastic component is high relative to the total variance. We confirmed this prediction empirically by using trade data for 215 countries and multiple sectors.

We show that violations of the LLN observed in the data have important implications for calculating the welfare gains from trade within the gravity frameworks. In cases when the LLN does not hold, it is important to characterize the distribution of the welfare gains from trade. We have developed a procedure based on simulations that allows for obtaining such distributions with minimal data requirements. We have also demonstrated the advantages and simplicity of the procedure by applying it to the external data for 40 countries and 17 sectors sectors against the backdrop of a neoclassical model of trade with input-output linkages.

In this work, we have focused on characterizing and dealing with the uncertainty in the estimates of the welfare gains from trade stemming from violations of the LLN. It will be potentially fruitful to combine this source of uncertainty with the uncertainty in the estimates of the gravity forces such as the elasticity of trade. We leave this for future research.

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Appendix

Table 3: SITC2 (2-DIGIT) & N^s

SITC2 (2-digit)	N^s	SITC2 (2-digit)	N^s	SITC2 (2-digit)	N^s	SITC2 (2-digit)	N^s
00	27	26	81	57	10	76	185
01	152	27	100	58	129	77	771
02	39	28	108	59	150	78	171
03	295	29	187	61	187	79	133
04	135	32	11	62	119	81	22
05	376	33	81	63	159	82	44
06	34	34	13	64	295	83	15
07	70	35	1	65	884	84	419
08	60	41	14	66	253	85	60
09	72	42	47	67	434	87	221
11	42	43	16	68	215	88	175
12	76	51	643	69	370	89	500
21	71	52	252	71	187	93	18
22	38	53	70	72	553	94	12
23	21	54	197	73	249	95	45
24	166	55	66	74	634	96	1
25	24	56	1	75	182	97	10