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Optimal Monetary Policy with Heterogeneous Agents

Abstract

We analyze optimal monetary policy under commitment in an economy with uninsurable idiosyncratic risk, long-term nominal bonds and costly inflation. Our model features two transmission channels of monetary policy: a Fisher channel, arising from the impact of inflation on the initial price of long-term bonds, and a liquidity channel. The Fisher channel gives the central bank a reason to inflate for redistributive purposes, because debtors have a higher marginal utility than creditors. This inflationary motive fades over time as bonds mature and the central bank pursues a deflationary path to raise bond prices and thus relax borrowing limits. The result is optimal inflation front-loading. Numerically, we find that optimal policy achieves first-order consumption and welfare redistribution vis-à-vis a zero inflation policy.

JEL-Codes: E500, E620, F340.

Keywords: optimal monetary policy, incomplete markets, Gâteau derivative, nominal debt, inflation, redistributive effects, continuous time.

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1 Introduction

In recent years the redistributive effects of monetary policy have come to the forefront both of policy discussions and academic research.[1] On the research side, aided by the development of new computational techniques, a burgeoning literature has analyzed the transmission of monetary policy in macroeconomic models featuring rich household heterogeneity.

By and large, this literature has focused on positive questions, such as what are the different redistributive channels of monetary policy and how they shape its aggregate and redistributive effects (e.g. Auclert, 2019; Kaplan, Moll and Violante, 2018). However, progress on the normative front has been scarcer. This has been particularly true in the context of incomplete-markets models with uninsurable idiosyncratic risk in the Bewley-Huggett-Aiyagari tradition (Bewley, 1983; Huggett, 1993; Aiyagari, 1994). Such models have become very popular for analyzing policy in environments with heterogeneous agents, thanks to their rigorous microfoundations and realistic description of household heterogeneity. But solving for fully optimal policy in those models is a hard endeavor, because the policy-maker faces an infinite-dimensional, endogenously-evolving wealth and income distribution. Our paper aims at making progress in this direction.

Our starting point is Huggett's (1993) classic model of uninsurable idiosyncratic risk. As in the latter, households trade non-contingent claims, subject to an exogenous borrowing limit, in order to smooth consumption in the face of idiosyncratic income shocks. We depart from Huggett's real framework with one-period claims by considering nominal bonds with an arbitrarily long maturity. This opens two transmission channels of monetary policy. The first is a variant of the classic Fisher *channel* (Fisher, 1933): unexpected inflation reduces the initial (time-0) market price of the long-term nominal bond, and this revaluation of nominal claims implies a redistribution of resources from creditors to debtors. The second is a *liquidity channel*: both expected and unexpected *lower* inflation raises asset prices, and this enhances households' liquidity by relaxing their borrowing limit in market value terms.[2] We also assume that inflation is costly, which can be rationalized on the basis of nominal rigidities. Finally, we depart from the standard closed-economy setup by considering a small open economy, with the bonds being also held (and priced) by risk-neutral foreign investors. Therefore, the real interest rate in this economy is exogenous and monetary policy transmission does not rely on standard New Keynesian channels.

In this context, we analyze the Ramsey optimal monetary policy. To address the aforementioned difficulty of solving for optimal policy in models of this kind we employ a variational approach, which extends the concept of classical derivatives to infinite-dimensional spaces. This approach in particular allows us to obtain analytical first-order conditions for the Ramsey problem. As it turns out, we are able to provide a tight analytical characterization of optimal

[1]For discussions in policy-making circles, see e.g. Yellen (2016) or Constâncio (2017). For academic analyses of this issue, see our literature review below.

[2]This liquidity channel is reminiscent of the one in Aiyagari and McGrattan (1998). These authors show how, in a similar environment, the fiscal authority's provision of government debt effectively loosens households' borrowing constraint.

monetary policy. We derive an equation for optimal inflation in which the central bank trades off the disutility costs of inflation with its benefits. The latter are captured by the covariance between the *marginal utility of consumption* and the *net nominal position* (NNP), i.e. the market value of each household's net position in the long-term bond.[3]

This term sheds light on the crucial role played by the Fisher channel in shaping the path of optimal inflation. In particular, we highlight a 'redistributive inflationary bias' that is distinct from the classical inflationary bias in the New Keynesian literature. Since debtors have lower consumption and (under standard concave preferences) higher marginal utility of consumption than creditors, the central bank has an incentive to inflate so as to lower the initial (time-0) price of the long-term nominal bond, thus redistributing resources from creditors to debtors. The central bank however commits to gradually undoing the initial inflation. Indeed, as bonds mature progressively, the impact of future inflation on the initial bond price fades with the planning horizon. In the long-run, the Fisher channel does not operate and inflation is determined only by the trade-off between the liquidity channel –which calls for negative inflation in order to increase bond prices and thus loosen the borrowing limit in market-value terms– and the disutility cost of non-zero inflation. In sum, the Ramsey optimal policy is characterized by optimal inflation 'front-loading'. In this regard, we find another important result: provided households are patient enough –in the sense that their discount rate equals the real return required by international investors– the liquidity channel disappears and optimal long-run inflation is exactly zero. This result is reminiscent of the optimality of zero long-run inflation in standard (representative-agent) New Keynesian models, but it arises from entirely different reasons.

After calibrating the model, we then solve numerically for the full transitional dynamics. Initial inflation is first-order in magnitude, reflecting the above mentioned redistributive motive, and thereafter falls gradually towards its long-run value. The latter is negative, but very close to zero, consistently with the fact that under our calibration households' and investors' discount rates are very similar. To summarize, the central bank front-loads inflation so as to transitorily redistribute nominal wealth from creditors to debtors, but commits to gradually undo such initial inflation. This echoes standard results in the Ramsey optimal capital taxation literature (see, for instance Chari and Kehoe, 1999, and references therein).[4]

Besides the aggregate implications, we also analyze the *redistributive* effects of optimal monetary policy. We show that, relative to a zero-inflation regime, the optimal policy redistributes consumption from creditors to debtors. These effects find an echo in the welfare analysis. Com-

[3]In his positive analysis of the redistributive channels of monetary policy, Auclert (2019) shows that the impact of a monetary policy shock on aggregate consumption through the Fisher channel depends on the covariance between the NNP and the marginal propensity to consume (MPC). In our normative analysis, we show instead that it is the covariance of the former object with the *marginal utility of consumption* that determines optimal inflation.

[4]Given our open economy assumption, the above redistributive motive also has a cross-border dimension. Under our calibration, the domestic economy is a net debtor *vis-à-vis* the rest of the World, so the central bank also has an incentive to redistribute from foreign investors to domestic debtors. However, our results show that, even if the country is assumed to start with a zero net nominal position (thus effectively shutting down cross-border redistribution at time zero), the purely (and more interesting) domestic redistributive motive is enough to justify relatively high (first-order) inflation rates in the first few years of the optimal plan.

pared to a zero inflation policy, optimal inflationary policy achieves welfare gains for debtors and losses for creditors which, when translated into consumption equivalents, are first-order in magnitude.

We also compute the optimal monetary policy response to an aggregate shock, such as a fall in aggregate income. In the analysis of aggregate shocks we focus on the optimal commitment plan 'from a timeless perspective,' as discussed in Woodford (2003). We find that inflation rises slightly on impact, as the central bank tries to partially counteract the negative effect of the shock on household consumption. However, the inflation reaction is an order of magnitude smaller than that of the shock itself. Intuitively, the value of sticking to past commitments to keep inflation near zero weighs more in the central bank's decision than the value of using inflation transitorily so as to stabilize consumption in response to an unforeseen event.

Overall, our findings shed some light on current policy and academic debates regarding the appropriate conduct of monetary policy once household heterogeneity is taken into account. In particular, our results suggest that, while some inflation may be justified in the short-run so as to redistribute resources to households with higher marginal utility, a central bank with the ability to commit should *not* sustain such an inflationary stance, but should instead promise to undo it over time. Finally, our results are *not* meant to suggest that monetary policy is the best tool to address redistributive issues, as there are probably more direct policy instruments. What our results indicate is that, in the context of economies with uninsurable idiosyncratic risk, the optimal design of monetary policy will to some extent reflect redistributive motives, the more so the less other policies (e.g. fiscal policy) are able to achieve optimal redistributive outcomes.[5]

Related literature. Our first main contribution relates to the normative insights on monetary policy. A recent literature addresses, from a *positive* perspective, the redistributive channels of monetary policy transmission in the context of general equilibrium models with incomplete markets and household heterogeneity. In terms of modeling, our paper is closest to Auclert (2019), Kaplan, Moll and Violante (2018), Gornemann, Kuester and Nakajima (2016), Hagedorn et al. (2019), McKay, Nakamura and Steinsson (2016) or Luetticke (2020), who also employ different versions of the incomplete markets model with uninsurable idiosyncratic risk.[6] Other contributions, such as Doepke and Schneider (2006b), Meh, Ríos-Rull and Terajima (2010), Sheedy (2014), Werning, (2015), Ravn and Sterk (2017), Challe et al. (2017), Sterk and Tenreyro (2018), Debortoli and Galí (2018) or Bilbiie (2019) analyze monetary policy in environments where heterogeneity is kept finite-dimensional. We contribute to this literature by analyzing *optimal* monetary policy, in an economy with uninsurable idiosyncratic risk.

As explained before, our analysis focuses on the Fisher channel. The latter channel is a long-standing topic that has experienced a revival in recent years. Doepke and Schneider (2006a) document net nominal asset positions across US sectors and household groups and estimate

[5] Bhandari et al. (2020) and Le Grand et al. (2020) analyze optimal monetary and fiscal policies in economies with heterogeneous agents.

[6] For work studying the effects of different aggregate shocks in related environments, see e.g. Guerrieri and Lorenzoni (2017), Den Haan, Rendahl and Riegler (2018), Auclert, Rognlie and Straub (2019) and Bayer et al. (2019).

empirically the redistributive effects of different inflation scenarios; Adam and Zhu (2014) perform a similar analysis for Euro Area countries. Pugsley and Rubinton (2019) quantify the distribution of welfare gains and losses of the US "Volcker" disinflation. Auclert (2019) identifies the redistributive channels of monetary policy, including the Fisher channel, in a general framework of dynamic consumer optimization and analyzes the importance of these channels for the aggregate consumption effects of monetary policy shocks using a sufficient statistics approach. We show how, in a model with uninsurable idiosyncratic risk featuring long-term nominal debt and costly inflation, the Fisher channel is crucial to shape optimal monetary policy under commitment. We uncover two novel normative insights. First, dispersion in net nominal positions gives rise to a 'redistributive inflationary bias': the central bank has an incentive to redistribute from creditors to debtors, who have a higher marginal utility of consumption. Second, inflation is front-loaded as the central bank commits to gradually reduce inflation in the future. We argue that these results would carry over to more fully-fledged incomplete-markets models that incorporate the above channels.

Our second main contribution is methodological. We contribute to the emergent literature on optimal policy problems in general equilibrium models with incomplete markets and uninsurable idiosyncratic risk. To the best of our knowledge, our paper is the first to solve for a fully dynamic optimal policy problem in a general equilibrium model with uninsurable idiosyncratic risk in which the cross-sectional net wealth distribution (an infinite-dimensional, endogenously evolving object) is a state in the planner's optimization problem. Even if the model considered here can be seen as a proof of concept, our methodology has been successfully applied to more complex settings, such as Bigio and Sannikov (2019), who analyze optimal monetary policy in an closed-economy incomplete-markets model where credit is intermediated by banks operating in an over-the-counter market.[7]

Different papers have analyzed Ramsey problems in incomplete-market models in which the policy-maker does not need to keep track of the wealth distribution; see Gottardi, Kajii, and Nakajima (2011), Bilbiie and Ragot (2017), Challe (2020) and Acharya et al. (2019). This is due either to particular assumptions that facilitate aggregation or to the fact that the equilibrium net wealth distribution is degenerate at zero. Le Grand et al. (2020) analyze optimal Ramsey monetary and fiscal policies in a model economy in which heterogeneity is kept finite-dimensional. In contrast to these papers, we introduce a methodology for computing the fully dynamic, nonlinear optimal policy under commitment in an incomplete-market setting where the policy-maker needs to keep track of the entire wealth distribution.

Dyrda and Pedroni (2018) and Itskhoki and Moll (2019) employ numerical optimization to study optimal dynamic Ramsey taxation in an Aiyagari economy and in a model with heterogeneous entrepreneurs, respectively. Dyrda and Pedroni (2018) assume that the paths for the optimal taxes follow splines with nodes set at a few exogenously selected periods whereas

[7]Although this paper focuses on optimal monetary policy, the techniques developed here lend themselves naturally to the analysis of other policy problems, e.g. optimal fiscal policy, in this class of models. Recent work analyzing fiscal policy issues in incomplete-markets, heterogeneous-agent models includes Heathcote (2005), Oh and Reis (2012), Kaplan and Violante (2014) and McKay and Reis (2016).

Itskhoki and Moll (2019) restrict the time paths of the tax policy to be an exponential function of time.[8] Both papers perform a numerical search of the optimal node values or coefficients. In our paper we do not impose *ex ante* any parametric form for the optimal policy, instead we derive analytically the first order conditions using infinite-dimensional calculus and compute the optimal policy. The computation of the optimal policy thus amounts to solving a standard heterogenous-agent model over an expanded set of variables that includes the Lagrange multipliers. Our approach improves considerably both the accuracy and the efficiency of the numerical computations.[9] Since this paper was first circulated, two papers have analyzed optimal policies in models with non-trivial heterogeneity. Bhandari et al. (2020) analyze numerically optimal fiscal and monetary policy in a heterogeneous agents New Keynesian environment with aggregate uncertainty and show how an across-person insurance motive prevails over conventional price stabilization motives. Their approach is based on perturbation theory and hence cannot address environments with exogenous, occasionally binding borrowing limits such as those used in models *à la* Aiyagari-Bewley-Huggett, which are precisely the focus of our paper. Açikgöz et al. (2018) analyze optimal fiscal policy with commitment in an Aiyagari economy. Similar to us, they solve a Lagrangian problem that includes the first order conditions of the households as constraints. However, instead of working with the complete income-wealth distribution as a state variable by means of infinite dimensional calculus, they analyze the problem of individual agents employing standard calculus and then aggregate, which imposes some extra constraints on the distributions that can be analyzed. Moreover, none of the above papers focuses on the implications of long-term nominal debt for the optimal design of monetary policy, which is at the core of our analysis.

An earlier literature has analyzed optimal monetary policy with heterogeneous agents in the context of monetary models. See, for instance, Bhattacharya et al. (2005), Costa and Werning (2008) or more recently Rocheteau et al. (2018). In contrast to these papers, the only role of money in our economy is as a unit of account, and redistribution operates through the presence of long-term nominal bonds, which opens the door to Fisher redistribution.

The use of infinite-dimensional calculus in problems with non-degenerate distributions is employed in Davila et al. (2012), Lucas and Moll (2014) and Nuño and Moll (2018) to find the first-best and the constrained-efficient allocation in heterogeneous-agents models. In these papers a social planner directly decides on individual policies in order to control a distribution of agents subject to idiosyncratic shocks. Here, by contrast, we show how these techniques may be extended to game-theoretical settings involving several agents who are moreover forward-looking. This requires the policy-maker to internalize how her promised future decisions affect private agents' expectations; the problem is then augmented by introducing costates that reflect

[8]Similarly, Lippi et al. (2015) perform a numerical search over a grid of values in order to analyze optimal *Markovian* monetary policy.

[9]As an illustration, Dyrda and Pedroni (2018) compute the optimal paths of three taxes employing 15 nodes in total. As reported in their paper, the numerical optimization took 120 hours using 576 cores at a Supercomputing Institute. In our case, we consider 9600 nodes (800 years at monthly frequency) and the code runs in less than 5 minutes in a home PC. Notwithstanding, the two models cannot be directly compared, as the open-economy nature of our model facilitates the computation.

the value of deviating from the promises made at time zero. This relates to the literature on *mean-field games* in mathematics. In particular, the paper is related to Bensoussan, Chau and Yam (2016), who analyze a model of a major player and a distribution of atomistic agents.

2 Model

We extend the basic Huggett framework to an open-economy setting with nominal, non-contingent, long-term debt and nominal rigidities. Time is continuous: $t \geq 0$. The domestic economy is composed of a measure-one continuum of households. There is a single, freely traded consumption good, the World price of which is normalized to 1. The domestic price (equivalently, the nominal exchange rate) at time t is denoted by P_t and evolves according to

$$dP_t = \pi_t P_t dt, \tag{1}$$

where π_t is the domestic inflation rate (equivalently, the rate of nominal exchange rate depreciation).

2.1 Households

2.1.1 Income and net assets

Household $k \in [0, 1]$ is endowed at time t with y_{kt} units of the good, where y_{kt} follows a two-state Poisson process: $y_{kt} \in \{y_1, y_2\}$, with $y_1 < y_2$. The process jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2.

Households trade nominal, non-contingent, long-term bonds (denominated in domestic currency) with one another and with foreign investors. Following standard practice in the literature, we model long-term debt in a tractable way by assuming that bonds pay exponentially decaying coupons (see, for instance, Leland and Toft, 1996). In particular, a bond issued at time t promises a stream of nominal payments $\left\{\delta e^{-\delta(s-t)}\right\}_{s\in(t,\infty)}$, totaling 1 unit of domestic currency over the (infinite) life of the bond. Thus, from the point of view of time t, a bond issued at $\tilde{t} < t$ is equivalent to $e^{-\delta\left(t-\tilde{t}\right)}$ newly issued bonds. This implies that a household's entire bond portfolio can be summarized by the current total nominal coupon payment, which we denote by δA_{kt}. One can then interpret δ as the 'amortization rate' and A_{kt} as the nominal face value of the bond portfolio. The latter evolves according to

$$dA_{kt} = \left(A_{kt}^{new} - \delta A_{kt}\right) dt,$$

where A_{kt}^{new} represents the face value of the flow of new bonds purchased at time t. For households with a negative net position, $(-)\, A_{kt}$ represents the face value of outstanding net liabilities ('debt' for short). Our formulation also implies that at each t one needs only consider the price of one bond cohort, e.g. newly issued bonds. Let Q_t denote the nominal market price of bonds issued at time t. The budget constraint of household k is then

$$Q_t A_{kt}^{new} = P_t \left(y_{kt} - c_{kt}\right) + \delta A_{kt},$$

where c_{kt} is the household's consumption. Combining the last two equations, we obtain the following dynamics for the nominal face value of net wealth,

$$dA_{kt} = \left(\frac{\delta A_{kt} + P_t \left(y_{kt} - c_{kt}\right)}{Q_t} - \delta A_{kt}\right) dt. \tag{2}$$

We define the *real* face value of net wealth as $a_{kt} \equiv A_{kt}/P_t$. Its dynamics are obtained by combining equations (1) and (2),

$$da_{kt} = \left[\frac{\delta a_{kt} + y_{kt} - c_{kt}}{Q_t} - \left(\delta + \pi_t\right) a_{kt}\right] dt, \tag{3}$$

where $\frac{\delta a_{kt} + y_{kt} - c_{kt}}{Q_t} = A_{kt}^{new}/P_t \equiv a_{kt}^{new}$ is the real face value of new bonds acquired at t. We assume that each household faces the following exogenous borrowing limit,

$$a_{kt} \geq \phi, \tag{4}$$

where $\phi \leq 0$. Therefore, the real face value of debt $(-)\, a_{kt}$ cannot exceed the level $(-)\, \phi \geq 0$. In Section 5 we analyze the alternative case, considered in Auclert (2019), in which the borrowing limit is defined in terms of the real *market* value of net wealth, given by $Q_t a_{kt}$ in our model.

2.1.2 Preferences

Household have preferences over paths for consumption c_{kt} and domestic inflation π_t discounted at rate $\rho > 0$,

$$\mathbb{E}_0 \left\{ \int_0^\infty e^{-\rho t} \left[u(c_{kt}) - x\left(\pi_t\right)\right] dt \right\}.$$

The consumption utility function u is bounded and continuous, with $u' > 0$, $u'' < 0$ for $c > 0$. The inflation disutility function x satisfies $x' > 0$ for $\pi > 0$, $x' < 0$ for $\pi < 0$, $x'' > 0$ for all π, and $x(0) = x'(0) = 0$. Utility costs of inflation can be microfounded on the basis e.g. of costly price adjustment. In Appendix A we show that a quadratic specification of the form $x(\pi) = \frac{\psi}{2}\pi^2, \psi > 0$, can be derived in a model version where firms are modeled explicitly and set prices subject to quadratic price adjustment costs *à la* Rotemberg (1982).

From now onward we drop subscripts k for ease of exposition. The household chooses consumption at each point in time in order to maximize its welfare. The *value function* of the household at time t can be expressed as

$$v_t(a, y) = \max_{\{c_s\}_{s \in [t, \infty)}} \mathbb{E}_t \left\{ \int_t^\infty e^{-\rho(s-t)} \left[u(c_s) - x\left(\pi_s\right)\right] ds \right\},$$

subject to the law of motion of net wealth (3) and the borrowing limit (4). We use the shorthand notation $v_{it}(a) \equiv v_t(a, y_i)$ for the value function when household income is low ($i = 1$) and high ($i = 2$). The *Hamilton-Jacobi-Bellman* (HJB) equation corresponding to the problem above is

$$\rho v_{it}(a) = \frac{\partial v_{it}}{\partial t} + \max_c \left\{ u(c) - x(\pi_t) + \mathbf{s}_{it}(a, c) \frac{\partial v_{it}}{\partial a} \right\} + \lambda_i \left[v_{jt}(a) - v_{it}(a) \right], \quad (5)$$

for $i, j = 1, 2$, and $j \neq i$, where $\mathbf{s}_{it}(a, c)$ is the *drift* function, given by

$$\mathbf{s}_{it}(a, c) \equiv \frac{\delta a + y_i - c}{Q_t} - (\delta + \pi_t) a, \quad (6)$$

$i = 1, 2$. The first order condition for consumption is

$$u'(c_{it}(a)) = \frac{1}{Q_t} \frac{\partial v_{it}(a)}{\partial a}, \quad (7)$$

where $c_{it}(a) \equiv c_t(a, y_i)$. Therefore, household consumption increases with nominal bond prices and falls with the slope of the value function. Intuitively, a higher bond price (equivalently, a lower yield) gives the household an incentive to save less and consume more. A steeper value function, on the contrary, makes it more attractive to save so as to increase net bond holdings.

2.2 Foreign investors

Households trade bonds with competitive risk-neutral foreign investors that can invest elsewhere at the risk-free real rate $\bar{r}$. As explained before, bonds are amortized at rate δ. Foreign investors also discount future nominal payoffs with the accumulated domestic inflation (i.e. exchange rate depreciation) between the time of the bond purchase and the time such payoffs accrue. Therefore, the nominal price of the bond at time t is given by

$$Q_t = \int_t^\infty \delta e^{-(\bar{r}+\delta)(s-t) - \int_t^s \pi_u du} ds. \quad (8)$$

Taking the derivative with respect to time, we obtain

$$Q_t (\bar{r} + \delta + \pi_t) = \delta + \dot{Q}_t, \quad (9)$$

where $\dot{Q}_t \equiv dQ_t/dt$. The ordinary differential equation (9) provides the risk-neutral pricing of the nominal bond. The boundary condition is $\lim_{T\to\infty} e^{-(\bar{r}+\delta)T - \int_0^T \pi_u du} Q_T = 0$. The steady state bond price is $Q_\infty = \frac{\delta}{\bar{r}+\delta+\pi_\infty}$, where π_∞ is the inflation level in the steady state.

2.3 Central Bank

There is a central bank that chooses monetary policy. We assume that there are no monetary frictions so that the only role of money is that of a unit of account. The central bank can trade a short-term (instantaneous) nominal claim with foreign investors. The central bank sets the instantaneous nominal interest rate R_t of that facility. A no-arbitrage condition implies

$R_t = \bar{r} + \pi_t$. This is the Fisher equation with a constant real interest rate $\bar{r}$. Therefore the monetary authority effectively chooses the inflation rate π_t. In Section 3, we will study in detail the optimal inflation policy of the central bank.

2.4 Equilibrium

The state of the economy at time t is the joint density of net wealth and income, $f_t(a,y) \equiv \{f_t(a,y_i)\}_{i=1}^2 \equiv \{f_{it}(a)\}_{i=1}^2$. Let $\mathbf{s}_{it}(a, c_{it}(a)) \equiv s_{it}(a)$ be the drift of individual real net wealth evaluated at the optimal consumption policy. The density satisfies the normalization $\sum_{i=1}^2 \int_\phi^\infty f_{it}(a)\, da = 1$. The dynamics of the density are given by the *Kolmogorov Forward* (KF) equation,

$$\frac{\partial f_{it}(a)}{\partial t} = -\frac{\partial}{\partial a}\left[s_{it}(a) f_{it}(a)\right] - \lambda_i f_{it}(a) + \lambda_j f_{jt}(a),\ i,j = 1,2, j \neq i. \tag{10}$$

Given a central bank interest rate policy $\{R_t\}_{t\geq 0}$, we define an equilibrium in this economy as paths for prices $\{w_t, Q_t, \pi_t\}_{t\geq 0}$, the consumption policy function $\{c_t(\cdot)\}_{t\geq 0}$, the household value function $\{v_t(\cdot)\}_{t\geq 0}$ and the income-wealth density $\{f_t(\cdot)\}_{t\geq 0}$ such that, at every time t, (i) households and firms maximize their corresponding objective functions taking as given equilibrium prices, (ii) all markets clear. There are three markets in the economy: the bond market, the labor market, and the goods market. Notice that, given the inflation path, bond prices can be computed independently of the rest of the economy.

We henceforth use the notation

$$\mathbb{E}_{f_t(a,y)}\left[g_t(a,y)\right] \equiv \sum_{i=1}^{2} \int_\phi^\infty g_t(a,y_i)\, f_t(a,y_i)\, da$$

to denote the cross-household average at time t of any function g_t of individual net wealth and income levels, or equivalently the aggregate value of such a function (given that the household population is normalized to 1). We can define some aggregate variables of interest. The aggregate real face value of net wealth in the economy is $\bar{a}_t \equiv \mathbb{E}_{f_t(a,y)}[a]$. Aggregate consumption is $\bar{c}_t \equiv \mathbb{E}_{f_t(a,y)}[c_t(a,y)]$, and aggregate income is $\bar{y}_t \equiv \mathbb{E}_{f_t(a,y)}[y]$. These quantities are linked by the current account identity,

$$\frac{d\bar{a}_t}{dt} = \frac{\delta \bar{a}_t + \bar{y}_t - \bar{c}_t}{Q_t} - (\delta + \pi_t)\,\bar{a}_t \equiv \bar{a}_t^{new} - (\delta + \pi_t)\,\bar{a}_t, \tag{11}$$

For future reference, we may also define the real face value of gross household debt, $\bar{b}_t \equiv \sum_{i=1}^2 \int_\phi^0 (-a)\, f_{it}(a)\, da$.

2.5 Transmission channels of monetary policy

Before starting the formal analysis of optimal monetary policy, it is worthwhile to briefly discuss the channels through which inflation operates in this economy. The first one affects all households symmetrically, namely the fact that non-zero inflation reduces household welfare as shown in equation (5). As argued before, such welfare costs from inflation can be microfounded in a model version with price-setting firms and costly price adjustment. In this sense, this is a standard channel that is present in the (representative-agent) New Keynesian literature. Our simple model abstract from other New-Keynesian transmission channels. In particular, the fixed labor supply precludes changes in aggregate output, despite the presence of nominal rigidities. This allows us to simplify the analysis and focus on the already hard question of how to optimally design monetary policy in the presence of uninsurable idiosyncratic risk.

Monetary policy has two additional transmission channels. Both are readily visible if we first rewrite the model in terms of the real market value of net wealth. Denoting the latter as $a_t^m \equiv Q_t a_t$, its dynamics are given by

$$\begin{aligned} \dot{a}_t^m &= \dot{Q}_t a_t + Q_t \dot{a}_t = \bar{r} a_t^m + y_i - c_t, && (12) \\ a_0^m &= Q_0 a_0, && (13) \end{aligned}$$

where we have used equations (3) and (9) in the second equality of (12). Also, the borrowing limit (4) can be expressed in terms of the market value of net wealth as

$$a_t^m \geq Q_t \phi. \quad (14)$$

Notice first that monetary policy can*not* affect the real return on a_t^m, which is given by the World real interest rate $\bar{r}$ as shown in equation (12). This is a consequence of the fact that bonds are priced by competitive foreign investors. However, as shown by equation (13) the central bank can affect the *initial* market value of net wealth through changes in the time-0 bond price Q_0 −the initial asset position in face value, a_0, is predetermined as of time 0. That is, *unanticipated* inflation may redistribute net wealth in market value terms from creditors (those with $a_0^m > 0$) to debtors (those with $a_0^m < 0$). This is a variant of the classic Fisher redistributive channel of monetary policy. Notice that it is the entire path of *future* inflation $\{\pi_t\}_{t \geq 0}$ that affects the time-0 price of the long-term bond, as shown in the bond pricing condition (8), which we rewrite here for $t = 0$,

$$Q_0 = \int_0^\infty \delta e^{-(\bar{r}+\delta)t - \int_0^t \pi_s ds} dt. \quad (15)$$

The second transmission channel operates by altering households' ability to borrow. As shown by equation (14), at *any* time t the central bank can tighten or relax the borrowing limit in market value terms by affecting contemporaneous bond prices Q_t. In particular, deflationary policies relax the borrowing limit –expressed in market value terms– by increasing bond prices, and vice versa for inflationary policies. We denote this channel as the *liquidity channel* of monetary policy.

3 Optimal monetary policy

3.1 Central bank problem

We now turn to the design of the optimal monetary policy. Following standard practice, we assume that the central bank is utilitarian, i.e. it gives the same Pareto weight to each household. We consider the case in which the central bank can credibly commit to a future inflation path (the Ramsey problem).

The central bank is assumed to be benevolent and hence maximizes economy-wide aggregate welfare, defined as

$$W_0 \equiv \mathbb{E}_{f_0(a,y)}\left[v_0\left(a,y\right)\right]. \tag{16}$$

It will turn out to be useful to express the above welfare criterion as follows. Lemma 1 in Appendix B shows how the welfare criterion (16) can alternatively be expressed as

$$W_0 = \int_0^{\infty} e^{-\rho t}\mathbb{E}_{f_t(a,y)}\left[u\left(c_t\left(a,y\right)\right) - x\left(\pi_t\right)\right]dt.$$

The central bank credibly commits at time zero to an inflation path $\{\pi_t\}_{t\geq 0}$. The value functional of the central bank is given by

$$W\left[f_0\left(\cdot\right)\right] = \max_{\{\pi_t, Q_t, v_t(\cdot), c_t(\cdot), f_t(\cdot)\}_{t\geq 0}} W_0, \tag{17}$$

subject to the law of motion of the distribution (10), the bond pricing equation (9), and households' HJB equation (5) and optimal consumption choice (7).

Notice that the optimal value W and the optimal policies are not ordinary functions, but *functionals*, as they map the infinite-dimensional initial distribution $f_0\left(\cdot\right)$ into $\mathbb{R}$. In particular, the optimal inflation path depends on the initial distribution $f_0\left(a,y\right)$ and on time: $\pi_t \equiv \pi\left[f_0\left(\cdot\right), t\right]$. The central bank maximizes welfare taking into account not only the state dynamics (10), but *also* households' HJB equation (5) and investors' bond pricing condition (9), both of which are forward-looking. That is, the central bank understands how it can steer households' and foreign investors' expectations by committing to an inflation path.

Given an initial distribution $f_0(a,y)$, a *Ramsey allocation* is composed of a sequence of inflation rates $\{\pi_t\}_{t\geq 0}$, a household value function $\{v_t\left(\cdot\right)\}_{t\geq 0}$, a consumption policy $\{c_t\left(\cdot\right)\}_{t\geq 0}$, a bond price function $\{Q_t\}_{t\geq 0}$ and a distribution $\{f_t\left(\cdot\right)\}_{t\geq 0}$ such that they solve the central bank problem (17).

The Ramsey problem is an optimal control problem in a suitable function space. In order to solve this problem, we should construct a Lagrangian in such a space. In Appendix B, we show that the Lagrangian $\mathcal{L}[\pi, Q, f, v, c] \equiv \mathcal{L}_0$ is given by

$$\begin{aligned}
\mathcal{L}_0 \;\equiv\; & \int_0^\infty e^{-\rho t} \sum_{i=1}^{2} \int_\phi^\infty \{ [u(c_{it}(a)) - x(\pi_t)] f_{it}(a) \\
& + \zeta_{it}(a) \left[-\frac{\partial f_{it}(a)}{\partial t} - \frac{\partial}{\partial a} [s_{it}(a) f_{it}(a)] - \lambda_i f_{it}(a) + \lambda_j f_{jt}(a) \right] \\
& + \theta_{it}(a) \left[\frac{\partial v_{it}}{\partial t} + u(c_{it}(a)) - x(\pi_t) + s_{it}(a) \frac{\partial v_{it}}{\partial a} + \lambda_i [v_{jt}(a) - v_{it}(a)] - \rho v_{it}(a) \right] \\
& + \eta_{it}(a) \left[u'(c_{it}(a)) - \frac{1}{Q_t} \frac{\partial v_{it}}{\partial a} \right] \} da dt \\
& + \int_0^\infty e^{-\rho t} \mu_t \left[Q_t (\bar{r} + \pi_t + \delta) - \delta - \dot{Q}_t \right] dt,
\end{aligned} \tag{18}$$

where $j = 1, 2, j \neq i$ and ζ, θ, η and μ are Lagrange multipliers.

Notice that the optimal monetary policy problem is represented in terms of the face value of net wealth a. The reason is that this problem is mathematically much easier to handle than its corresponding representation in market value terms (a^m). The difficulty with the market-value formulation is that the central bank controls a wealth distribution with a (non-zero) domain $[Q_t\phi, \infty)$ which depends endogenously on the entire inflation path $\{\pi_s\}_{s \geq t}$ through bond prices. This complicates both the mathematical and computational solution. However, the two transmission channels discussed in section 2.5 continue to operate throughout as far as the dynamics of the net wealth distribution in market value terms is concerned.

3.2 Optimal inflation

In order to maximize the Lagragian (18) with respect to the functions π, Q, f, v, c we employ a variational approach. In particular, we compute the *Gâteaux derivatives*, which extend the concept of derivative from $\mathbb{R}$ to infinite-dimensional spaces (see Appendix B.1 for further details). As an example, the Gâteaux derivative with respect to the income-wealth density f is

$$\lim_{\alpha \to 0} \frac{\mathcal{L}_0 [f + \alpha h, \pi, Q, v, c] - \mathcal{L}_0 [f, \pi, Q, v, c]}{\alpha}$$

where h is an arbitrary function in the same function space as f. The first-order conditions require that the Gâteaux derivatives should be zero for *any* function h.

In Appendix B.2 we show that in equilibrium the Lagrange multiplier ζ associated with the Kolmogorov Forward equation (10), which represents the social value of an individual household, coincides with the private value v. In addition, the Lagrange multipliers θ and η associated with the households' HJB equation (5) and first-order condition (7), respectively, are both zero. That is, households' forward-looking optimizing behavior *does not* represent a source of time-inconsistency, as the monetary authority would choose at all times the same individual consumption and saving policies as the households themselves. Therefore, the only nontrivial Lagrange multiplier is μ_t, the one associated with the bond pricing equation (9). As shown in the appendix, the first order condition with respect to inflation is

$$x'(\pi_t) = \mathbb{E}_{f_t(a,y)}[-NNP_t(a)\,u'(c_t(a,y))] + \mu_t Q_t, \tag{19}$$

where

$$NNP_t(a) \equiv Q_t a_t \tag{20}$$

is the *net nominal position* (NNP), i.e. the real market value of a household's net position in the long-term bond. The first order condition with respect to bond prices gives us the law of motion of the costate μ_t,

$$\frac{d\mu_t}{dt} = (\rho - \bar{r} - \pi_t - \delta)\,\mu_t + \frac{1}{Q_t}\mathbb{E}_{f_t(a,y)}[URE_t(a,y)\,u'(c_t(a,y))], \tag{21}$$

with initial condition $\mu_0 = 0$, where

$$URE_t(a,y) \equiv \delta a_t + y_t - c_t(a,y) \tag{22}$$

is the *unhedged interest rate exposure* (URE), defined following Auclert (2019) as the difference between maturing assets (including income) and liabilities (including planned consumption), or equivalently the household's net saving requirement in a given period.

The following proposition characterizes the solution to this problem.

Proposition 1 (Optimal inflation) *In addition to equations (10), (9), (5) and (7), if a solution to the Ramsey problem (17) exists, the inflation path π_t must satisfy (19), where the costate μ_t follows (21).*

Equations (19) and (21) reflect the central bank's motives to inflate or disinflate under the Ramsey optimal commitment. Moreover, they encapsulate incentives to redistribute both among domestic households and between the latter and foreign investors (cross-border redistribution). To see both aspects more clearly, we use the identity $\text{cov}(x,y) = \mathbb{E}(xy) - \mathbb{E}(x)\mathbb{E}(y)$ to express (19) and (21) as:

$$\begin{aligned} x'(\pi_t) &= \overbrace{\text{cov}_{f_t(a,y)}[-NNP_t(a), MUC_t(a,y)]}^{\text{Domestic net nominal position motive}} \\ &\quad + \overbrace{\mathbb{E}_{f_t(a,y)}[-NNP_t(a)]\,\mathbb{E}_{f_t(a,y)}[MUC_t(a,y)]}^{\text{Cross-border net nominal position motive}} + \mu_t Q_t, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \mu_t &= \int_0^t e^{-\int_s^t(\bar{r}+\pi_z+\delta-\rho)dz}\frac{1}{Q_s}\{\overbrace{\text{cov}_{f_s(a,y)}[URE_s(a,y), MUC_s(a,y)]}^{\text{Domestic interest rate exposure motive}} \\ &\quad + \overbrace{\mathbb{E}_{f_s(a,y)}[URE_s(a,y)]\,\mathbb{E}_{f_s(a,y)}[MUC_s(a,y)]}^{\text{Cross-border interest rate exposure motive}}\}ds, \end{aligned} \tag{24}$$

where $MUC_t(a,y) \equiv u'(c_t(a,y))$ denotes the *marginal utility of consumption* (MUC) of a household with net wealth-income pair (a,y) and in equation (24) we have solved for μ_t forward. According to equation (23), marginal inflation disutility x' (which is increasing in inflation) equals the sum of three terms. The first term, $\text{cov}_{f_t(\cdot)}[-NNP_t(\cdot), MUC_t(\cdot)]$, represents the *domestic net nominal position motive.* It captures the fact that, to the extent that such covariance is *positive* such that households with a negative nominal net position (i.e. indebted households) have a higher marginal utility, then the central bank has an incentive to create inflation so as to redistribute resources from domestic creditors to domestic debtors. Indeed, Lemma 2 in the Appendix proves that, given strictly concave preferences, MUC falls with net wealth, $\partial u'/\partial a < 0$: debtors (those with $NNP < 0$) have a higher marginal utility of consumption than creditors ($NNP > 0$) and hence receive a higher effective weight in the central bank's inflation decision. Thus, even if the country has a zero net position *vis-à-vis* the rest of the World, as long as there is dispersion in net wealth the central bank has a reason to redistribute from creditors to debtors.

The second term, $\mathbb{E}_{f_t(\cdot)}[-NNP_t(\cdot)]\,\mathbb{E}_{f_t(\cdot)}[MUC_t(\cdot)]$, captures the *cross-border net nominal position motive.* If the country is a net debtor, such that $\mathbb{E}_{f_t(\cdot)}(-NNP_t) > 0$, the central bank has a motive to redistribute wealth from foreign investors to domestic borrowers (who have an average marginal utility of $\mathbb{E}_{f_t(\cdot)}[MUC_t(\cdot)]$) as it only cares about the welfare of the latter.

The third term on the right-hand side of equation (23) captures the value to the central bank of promises about time-t inflation made to foreign investors (the agents effectively pricing the bond) at time 0. The costate μ_t is zero at the time of announcing the Ramsey plan ($t = 0$), because the central bank is not bound by previous commitments. From then on, it evolves according to equation (24). In the latter equation, the term $\text{cov}_{f_s(\cdot)}[URE_s(\cdot), MUC_s(\cdot)]$ represents the *domestic interest rate exposure motive.* Intuitively, the central bank understands that a commitment to higher inflation in the future lowers bond prices today, which hurts −in terms of face value of debt− those households that need to *sell* new bonds (i.e. those with $URE < 0$) and vice versa for those that purchase new bonds ($URE > 0$). If the former households have a higher marginal utility than the latter ones, such that the covariance between URE and MUC is negative, then μ_t should become more and more *negative* over time. Indeed this will be the case both in the theoretical long-run characterization below and in our numerical analysis of transitional dynamics in Section 4. From equation (23), this would give the central bank an incentive to *reduce* inflation over time, thus tempering the above-discussed net nominal position redistributive motive.

Finally, the term $\mathbb{E}_{f_s(\cdot)}[URE_s(\cdot)]\,\mathbb{E}_{f_s(\cdot)}[MUC_s(\cdot)]$ captures the *cross-border interest rate exposure motive.* Intuitively, to the extent that the domestic economy is a net issuer of new bonds to the rest of the World, such that $\mathbb{E}_{f_t(\cdot)}[URE_t] < 0$, then expectations of future inflation that reduce bond prices today are welfare-detrimental for domestic households in the aggregate, thus giving the central bank a further incentive to commit to lowering inflation in the future. Notice how the interest rate exposure motives are discounted at a rate $(\bar{r} + \pi_t + \delta - \rho)$, the difference between the instantaneous rate at which future nominal coupons are discounted (see equation 15) and household's discount factor ρ.

Equations (23) and (24) provide us with a simple formulation of optimal monetary policy. They parallel the analysis in Auclert (2019), who analyzes the response of aggregate consumption to a monetary policy shock in a general equilibrium framework with heterogeneous households. In Auclert (2019), the impact of such a shock through the Fisher and unhedged interest rate exposure channels depends on the covariances between the marginal propensity to consume (MPC) and the NNP and URE, respectively. Here, instead, it is the covariance of the latter objects with the marginal utility of consumption –as opposed to the MPC– that determines the central bank's optimal policy. Another important difference is that in Auclert (2019) the above objects appear contemporaneously. In (24), by contrast, the whole future path of the covariance between URE and MUC affects current inflation. This reflect the fact that the Ramsey problem is not *time-consistent*, due to the presence of the forward-looking bond pricing condition (8). If at some future point in time $t > 0$ the central bank decided to re-optimize given the state at that point, $f_t(\cdot)$, the new path for optimal inflation would not need to coincide with the original path, as the costate at that point would be $\mu_t = 0$ (corresponding to a new commitment formulated at time t).

3.3 Further analytical results

Even though equations (23) and (24) provide a tight analytical characterization of the forces behind the optimal inflation path, a closed-form solution for the latter is elusive in this framework. However, it is possible to obtain a number of analytical results on the optimal inflation policy. In doing so, we will also establish the mapping between equation (23) and the transmission channels discussed in section 2.5.

Initial inflation and the redistributive inflationary bias. From now on, we assume that the economy as a whole is *not* a net creditor. First we characterize initial inflation.

Proposition 2 (Redistributive inflationary bias at time 0) *Provided the aggregate nominal net position is non-positive, $\mathbb{E}_{f_0(\cdot)}[-NNP_0(a)]/Q_0 = -\bar{a}_0 \geq 0$, then optimal inflation at time-0 is strictly positive, $\pi_0 > 0$.*

The formal proof can be found in Appendix B.3, although the result follows quite directly from equation (23) evaluated at time 0 (at which $\mu_0 = 0$). As explained above, the strict concavity of preferences implies that indebted households ($NNP < 0$) have a higher marginal utility of consumption than lending ones ($NNP > 0$) and hence $\text{cov}_{f_0(.)}[-NNP_0, MUC_0]$ is strictly positive. Provided the economy as a whole is not a net creditor, i.e. $\mathbb{E}_{f_0(.)}[-NNP_0] \geq 0$, then the right-hand side of equation (23) is strictly positive. Since $x'(\pi) > 0$ only for $\pi > 0$, it follows that initial inflation must be strictly positive. It is important to stress that this result is independent of the open economy dimension: even if the economy as a whole is neither a creditor or a debtor ($\bar{a}_0 = 0$), the fact that u' is strictly decreasing in net wealth implies that, as long as there is dispersion in net wealth, the central bank will have a reason to inflate.

To delve further into the forces behind initial optimal inflation, and in particular to establish the link between initial inflation and the transmission channels discussed in section 2.5, we evaluate equation (19) at time 0 and express it in terms of the real market value of each household's net bond position, $a_t^m \equiv Q_t a_t$ (see derivation in Lemma 3 in Appendix B):

$$\begin{aligned} \overbrace{x'(\pi_0)}^{\text{disutility of inflation}} &= -Q_0 \sum_{i=1}^{2} \int_{\phi Q_0}^{\infty} \overbrace{v_{i0}\left(\frac{a^m}{Q_0}\right)}^{\text{time-0 value function}} \frac{d}{dQ_0} \overbrace{\left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)\right]}^{\text{initial wealth dist. in market value}} da^m \quad (25) \\ &= \mathbb{E}_{f_0^m(a^m,y)}\left[-NNP_0\left(a^m\right) MUC_0\left(a^m, y\right)\right], \end{aligned}$$

where $f_0^m(a^m, y)$ denotes the time-0 distribution of net wealth in terms of market value (i.e., the distribution of NNPs), given by

$$f_0^m(a^m, y) = \frac{1}{Q_0} f_0\left(\frac{a^m}{Q_0}, y\right), \quad (26)$$

for $a^m \in [\phi Q_0, \infty)$. According to equation (25), the marginal disutility of inflation $x'(\pi_0)$ must equal its marginal benefit. The latter reflects the marginal increase in aggregate welfare due to the fall in the initial bond price Q_0, due in turn to the marginal increase in inflation. The impact of inflation on the initial bond price modifies the initial distribution of net wealth in terms of market value, in particular by redistributing wealth from creditors to debtors: the fall in Q_0 reduces the market value of creditors' assets and also reduces the market value of the debtors' liabilities. This captures the welfare effect of the Fisher channel. As discussed above, as long as the country is not a net creditor the central bank has an incentive to redistribute by creating surprise inflation.

The Fisher channel is not the only one affecting initial inflation. Equation (25) also shows two subtle ways in which the liquidity channel influences optimal time-0 inflation. First, notice that the lower limit of the net wealth distribution in market value terms f_0^m is given by ϕQ_0 and is therefore endogenous to policy through its impact on the initial bond price Q_0. Second, the complete path of bond prices affects the time-0 value function $v_{i0}(\cdot)$ by altering at each time t the borrowing limit expressed in terms of market value, ϕQ_t. Though this channel would call for a deflationary path in order to improve households' liquidity, Proposition 2 demonstrates that this motive takes second place compared to the Fisher redistributive motive. The overall effect is the bias towards positive initial inflation discussed above.

To the best of our knowledge, this *redistributive inflationary bias* is a novel result in the context of incomplete markets models with uninsurable idiosyncratic risk. It is also different from the classical inflationary bias of discretionary monetary policy originally emphasized by Kydland and Prescott (1977) and Barro and Gordon (1983). In those papers, and more generally in the New Keynesian literature, the source of the inflation bias is a persistent attempt by the monetary authority to raise output above its natural level. Here, by contrast, it arises from the aggregate welfare gains that can be achieved by redistributing wealth towards indebted households. Importantly, while the model analyzed here is deliberately simple with a view to illustrating our methodology, this redistributive motive to inflate would carry over to fully fledged models with uninsurable idiosyncratic risk that feature a Fisher channel.

Optimal long-run inflation. As bonds mature progressively, the impact of future inflation on the initial bond price decreases exponentially with time. This implies that inflation in the far away future plays no role through the Fisher channel. Long-run inflation will thus be determined by the trade-off between the liquidity channel, which calls for negative inflation rates in order to increase bond prices and thus loosen the households' borrowing limit –in market value terms–, and the disutility cost of non-zero inflation. Though the exact value of steady-state inflation cannot be derived analytically for the general case, the following proposition shows how, as long as households are patient enough, it is exactly zero.

Proposition 3 (Optimal long-run inflation) *In the limit as* $\rho \to \bar{r}$*, the optimal steady-state inflation rate tends to zero:* $\lim_{\rho \to \bar{r}} \pi_\infty = 0$*.*

The proof can be found in Appendix B.4. Proposition 3 provides a useful benchmark to understand the long-run properties of optimal policy when ρ is close to $\bar{r}$. This will indeed be the case in our numerical analysis.

Proposition 3 is reminiscent of a well-known result from the New Keynesian literature, namely that optimal long-run inflation in the standard New Keynesian framework is exactly zero (see e.g. Benigno and Woodford, 2005). In that framework, the optimality of zero long-run inflation arises from the fact that, at that level, the welfare gains from trying to exploit the short-run output-inflation trade-off (i.e. raising output towards its socially efficient level) exactly cancel out with the welfare losses from permanently worsening that trade-off (through higher inflation expectations). Key to that result is the fact that, in that model, price-setters and the (benevolent) central bank have the same (steady-state) discount factor. Here, the optimality of zero long-run inflation reflects instead the fact that, provided the discount rate of the investors pricing the bonds is arbitrarily close to that of the central bank, the liquidity channel becomes ineffective in the long run as households accumulate enough wealth to avoid the borrowing limit.

The limiting case of instantaneous bonds. The presence of long-term nominal bonds is key for understanding the front-loaded nature of optimal inflation in our model, i.e. the initial inflation followed by a gradual disinflationary path. In the next proposition we show how, in the instantaneous-debt limit, optimal inflation converges to zero right after time 0.

Proposition 4 (Instantaneous bonds) *In the limit as* $\delta \to \infty$*, the optimal inflation rate converges to zero:* $\lim_{\delta \to \infty} \pi_t = 0$*,* $t > 0$*.*

The proof can be found in Lemma 4. With instantaneous bonds, the Fisher channel loses its effectiveness and therefore there is no reason for using inflation for redistributive purposes.

Representative agent. To conclude the analytical results, we consider the case of a representative agent. This can be seen as a special instance of the model above in which $y_1 = y_2 = y$, the borrowing limit is not binding, $\phi = -\infty$, and the initial distribution is degenerate at position a_0. For simplicity, we also consider the case with log utility $u(c) = \log(c)$. The solution to this problem can be derived analytically, as shown in Lemma 5. Consumption evolves according to

$$c_t = \rho(\overbrace{Q_0 a_0}^{NNP_0} + y/\bar{r})e^{-(\rho-\bar{r})t},$$

which clearly illustrates how the initial market value of debt $a_0^m = Q_0 a_0$ determines the whole path of of consumption. Notice how in this case the central bank can only influence consumption through the Fisher channel: if the country is a debtor, such that $a_0 < 0$, the central bank may redistribute from foreign investors to the representative household by reducing bond prices through surprise inflation. The liquidity channel is not active given the fact that the borrowing limit is not binding. In this case, the optimal inflation path is pinned down by

$$x'(\pi_t) = \underbrace{\overbrace{(-Q_0 a_0)}^{-NNP_0}\overbrace{u'(c_0)}^{MUC_0}}_{\text{Time-0 optimal inflation}(\psi\pi_0)} \underbrace{\frac{Q_t}{Q_0} e^{\left[-\int_0^t (\bar{r}+\delta+\pi_s-\rho)ds\right]}}_{\text{Dynamic evolution}},$$

where the first term on the right-hand side accounts for the optimal initial inflation and the second term describes its dynamics. Notice how the expression determining initial inflation, $x'(\pi_0) = -NNP_0 MUC_0 > 0$, is just a particular instance of equation (23) when only the cross-border net nominal position motive is present.

Inflation decreases according to $Q_t/Q_0 \exp\left[-\int_0^t (\bar{r}+\delta+\pi_s-\rho)\, ds\right]$. This term captures how the gradual amortization of initially outstanding bonds makes future inflation progressively ineffective in terms of time-0 redistribution. In the limit as time goes to infinity inflation converges to zero $\lim_{t\to\infty} \pi_t = 0$. This is so even if the discount factor of households and investors differ: since the liquidity channel is shut down, the central bank has no incentive to manipulate inflation in the long-run given its negligible effect on the time-0 bond price.

4 Numerical analysis

4.1 Numerical solution

Despite its tight characterization in Proposition 1, a closed-form solution to the Ramsey problem remains elusive and hence we need to resort to numerical techniques to compute the equilibrium. In Appendix C we propose an algorithm to find the solution. The general idea is to guess a path for inflation and use it to find an equilibrium of this economy. We then use the equilibrium objects to compute the path of the costate μ_t from equation (21) and, given the latter, the optimal inflation path using equation (19). If the latter does not coincide with the initial guess, we update it and then iterate again until numerical convergence is reached. In order to find the equilibrium at each iteration, we have to solve the households' HJB equation and the KF equation. To this end, we apply a finite difference method similar to the ones employed in Achdou et al. (2017) or Nuño and Moll (2018). In summary, the numerical solution of the model just amounts to solving a heterogeneous-agent model with one extra forward looking variable, the costate μ_t. The optimality condition (19) then pins down optimal inflation.

It is worthwhile to highlight here that the assumption of continuous time improves the efficiency of the numerical solution in several dimensions. First, the HJB equation (5) can be solved using fast sparse-matrix methods. Second, the solution of the HJB equation makes straightforward the computation of the dynamics of the distribution (equation 10) as the operators in both equations are adjoint. This efficiency is crucial as the computation of the optimal policies requires several iterations along the complete time-path of the distribution. In a home PC, the Ramsey problem presented here can be solved in a few minutes. In any case, optimal monetary policy can be analyzed in discrete time following a similar approach, as discussed in an earlier version of this paper (Nuño and Thomas, 2016).

4.2 Calibration

The calibration is intended to be mainly *illustrative*, given the model's simplicity and parsimoniousness. We calibrate the model to replicate some relevant features of a prototypical European small open economy. We will focus for illustration on the UK, Sweden, and the Baltic countries (Estonia, Latvia, Lithuania). We choose these countries because they (separately) feature desirable properties for the purpose at hand.[10] Let the time unit be one year. For the calibration, we consider that the initial distribution coincides with the steady state implied by a zero inflation policy. This squares reasonably well with the experience of our target economies, which have displayed low and stable inflation for most of the recent past. When integrating across households, we therefore use the stationary wealth distribution associated to such steady state.[11]

We assume the following functional forms for preferences: $u(c) = \log(c)$, $x(\pi) = \frac{\psi}{2}\pi^2$. As mentioned before, the quadratic specification for the inflation utility cost, $\frac{\psi}{2}\pi_t^2$, can be micro-founded by modeling firms explicitly and allowing them to set prices subject to standard quadratic price adjustment costs *à la* Rotemberg (1982); see Appendix A for further details. We set the scale parameter ψ such that the slope of the inflation equation in a Rotemberg pricing setup replicates that in a Calvo pricing setup for reasonable calibrations of price adjustment frequencies and demand curve elasticities. The slope of the continuous-time New Keynesian Phillips curve in the Calvo model is given by $\chi(\chi + \rho)$, where χ is the price adjustment rate. In the Rotemberg model the slope is given by $\frac{\varepsilon-1}{\psi}$, where ε is the elasticity of firms' demand curves and ψ is the scale parameter in the quadratic price adjustment cost function in that

[10] On the one hand, UK and Sweden are two prominent examples of relatively small open economies that retain an independent monetary policy, like the economy in our framework. This is unlike the Baltic states, who recently joined the euro. However, historically the latter states have been relatively large debtors against the rest of the World, which makes them appealing for our analysis (UK and Sweden have also remained net debtors in basically each quarter for the last 20 years, but on average their net balance has been much closer to zero).

[11] The wealth dimension is discretized by using 1000 equally-spaced grid points from $a = \phi$ to $a = 10$. The upper bound is needed only for operational purposes but is fully innocuous, because the stationary distribution places essentially zero mass for wealth levels above $a = 8$.

model. It follows that, for the slope to be the same in both models, we need $\psi = \frac{\varepsilon-1}{\chi(\chi+\rho)}$.Setting ε to 11 (such that the gross markup $\varepsilon/(\varepsilon-1)$ equals 1.10) and χ to 4/3 (such that price last on average for 3 quarters), and given our calibration for ρ, we obtain $\psi = 5.5$.

We jointly set households' discount rate ρ and borrowing limit ϕ such that the steady-state net international investment position (NIIP) over GDP ($\bar{a}/\bar{y}$) and gross household debt to GDP ($\bar{b}/\bar{y}$) replicate those in our target economies. According to Eurostat, the NIIP/GDP ratio averaged minus 48.6% across the Baltic states in 2016:Q1, and only minus 3.8% across UK-Sweden. We thus target a NIIP/GDP ratio of minus 25%, which is about the midpoint of both values. Regarding gross household debt, we use BIS data on 'total credit to households', which averaged 85.9% of GDP across Sweden-UK in 2015:Q4 (data for the Baltic countries are not available). We thus target a 90% household debt to GDP ratio.

We target an average bond duration of 4.5 years, as in Auclert (2019). In our model, the Macaulay bond duration equals $1/(\delta+\bar{r})$. We set the world real interest rate $\bar{r}$ to 3 percent. Our duration target then implies an amortization rate of $\delta = 0.19$.

The idiosyncratic income process parameters are calibrated as follows. We follow Huggett (1993) in interpreting states 1 and 2 as 'unemployment' and 'employment', respectively. The transition rates between unemployment and employment (λ_1, λ_2) are chosen such that (i) the unemployment rate $\lambda_2/(\lambda_1+\lambda_2)$ is 10 percent and (ii) the job finding rate is 0.1 at monthly frequency or $\lambda_1 = 0.72$ at annual frequency.[12] These numbers describe the 'European' labor market calibration in Blanchard and Galí (2010). We normalize average income $\bar{y} = \frac{\lambda_2}{\lambda_1+\lambda_2}y_1 + \frac{\lambda_1}{\lambda_1+\lambda_2}y_2$ to 1. We also set y_1 equal to 71 percent of y_2, as in Hall and Milgrom (2008). Both targets allow us to solve for y_1 and y_2. Table 1 summarizes our baseline calibration.

Table 1. Baseline calibration

Parameter	Value	Description	Source/Target
$\bar{r}$	0.03	world real interest rate	standard
ψ	5.5	scale inflation disutility	slope NKPC in Calvo model
δ	0.19	bond amortization rate	Macaulay duration = 4.5 yrs
λ_1	0.72	transition rate unemp-to-employment	monthly job finding rate 0.1
λ_2	0.08	transition rate emplo-to-unemployment	unemployment rate 10%
y_1	0.73	income in unemployment state	Hall&Milgrom(2008)
y_2	1.03	income in employment state	$E(y) = 1$
ρ	0.0302	subjective discount rate	NIIP -25% of GDP
ϕ	-3.6	borrowing limit	HH debt/GDP 90%

4.3 Optimal transitional dynamics

We are ready to analyze optimal transitional dynamics. As explained in Section 3, the optimal policy paths depend on the initial (time-0) distribution of net wealth and income across house-

[12] Analogously to Blanchard and Galí (2010; see their footnote 20), we compute the equivalent annual rate λ_1 as $\lambda_1 = \sum_{i=1}^{12}(1-\lambda_1^m)^{i-1}\lambda_1^m$,where λ_1^m is the monthly job finding rate.

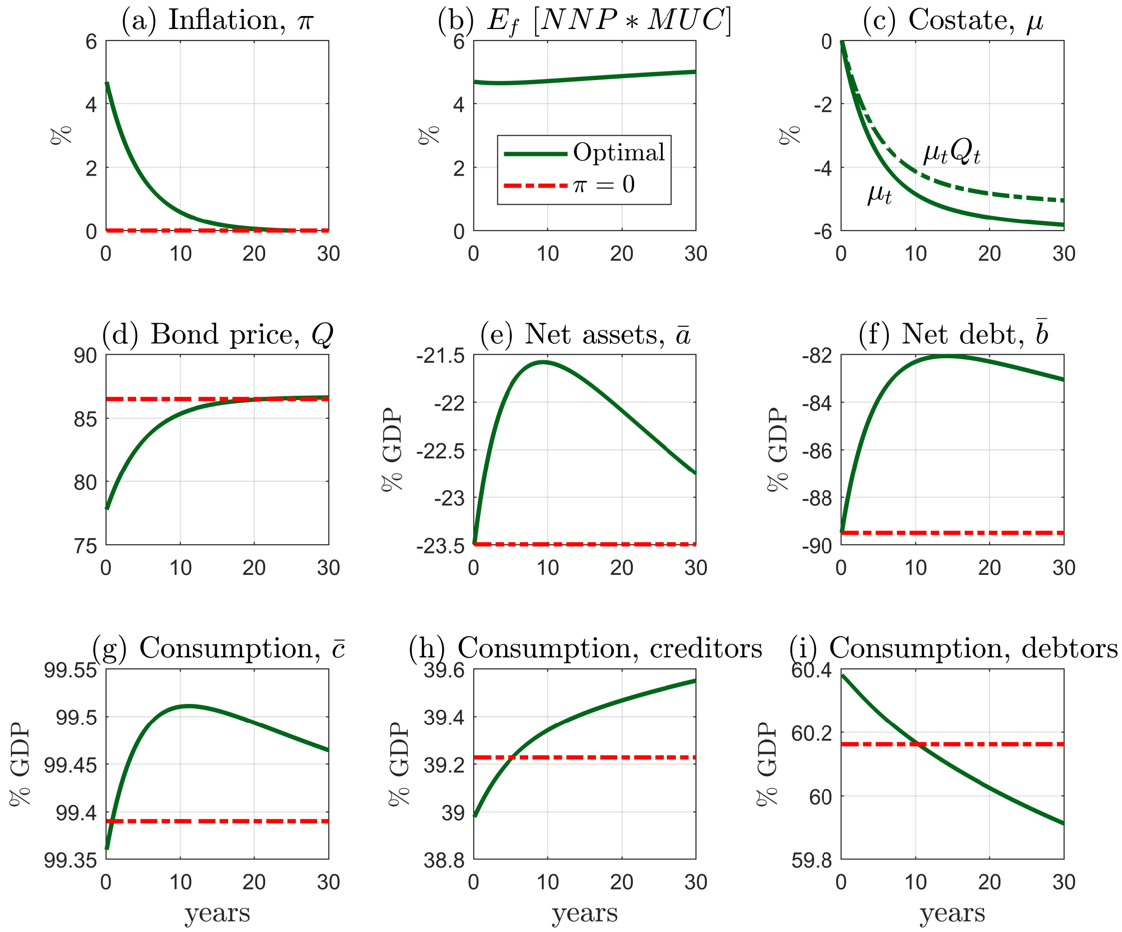

Figure 1: Transitional dynamics.

holds, $f_0(a, y)$, which is an (infinite-dimensional) primitive in our model. For the purpose of illustration, we consider the stationary distribution under zero inflation as the initial distribution. We thus assume $f_0(a, y_i) = f_{\pi=0}^{a|y}(a \mid y_i) f^y(y_i), i = 1, 2$, where $f^y(y_i) = \lambda_{j \neq i}/(\lambda_1 + \lambda_2)$, $i, j = 1, 2$, and $f_{\pi=0}^{a|y}$ is the stationary conditional density of net wealth under zero inflation. Notice that aggregate income is constant at $\bar{y}_t = \frac{\lambda_2}{\lambda_1+\lambda_2} y_1 + \frac{\lambda_1}{\lambda_1+\lambda_2} y_2 = 1$, given our calibration of $\{y_i\}_{i=1,2}$. Later we will analyze the robustness of our results to a wide range of alternative initial distributions.

The optimal paths are shown by the green solid lines in Figure 1 whereas the red dashed lines display the (invariant) paths under the zero-inflation policy, $\pi_t = 0$ for all $t \geq 0$. We simulate 800 years of data at monthly frequency and display the first 30 years. It follows from equation (23) and the fact that $\mu_0 = 0$ (no pre-commitments at time zero) that initial optimal inflation is

$$\pi_0 = \frac{1}{\psi} \left\{ \text{cov}_{f_0(\cdot)} \left[-NNP_0(\cdot), MUC_0(\cdot) \right] + \mathbb{E}_{f_0(\cdot)} \left[-NNP_0(\cdot) \right] \mathbb{E}_{f_0(\cdot)} \left[MUC_0(\cdot) \right] \right\}. \tag{27}$$

Therefore, the time-0 inflation rate, of about 4.6 percent, reflects exclusively the redistributive inflationary bias discussed in Proposition 2. As explained in Section 3, the redistributive motive has both a cross-border and a domestic dimension, where the latter arises from the fact that

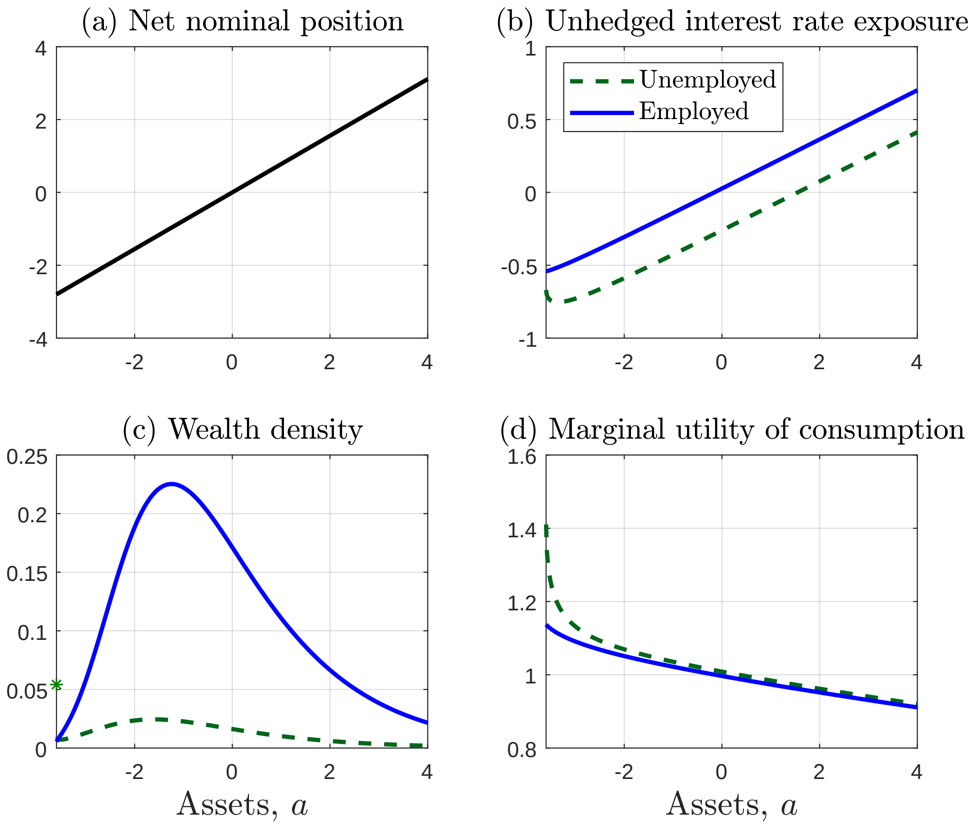

Figure 2: Time-0 equilibrium objects.

households with different nominal net positions have different marginal utilities of consumption. To illustrate this last dimension, panels (a), (d) and (c) in Figure 2 display respectively the NNP and MUC as functions of net wealth and (in the second case) income, as well as the net wealth-income distribution at time zero. As shown there, agents with lower net wealth and hence lower NNP have a higher MUC, implying a positive covariance between both. This gives the central bank an incentive to inflate so as to redistribute domestically. This inflationary bias is reinforced by the fact that the economy as a whole is initially a net debtor *vis-à-vis* the rest of the World ($\mathbb{E}_{f_0(\cdot)}[-NNP_0] > 0$) under our calibration.

From time zero onwards, Ramsey optimal inflation follows (23). As shown in the figure, inflation gradually declines towards its long-run level; the latter equals -0.05 per cent, i.e. very close to zero. This is consistent with Proposition 3 and the fact ρ and $\bar{r}$ are very close to each other in our calibration ($\bar{r} = 0.03$ and $\rho = 0.0302$). Panels (b) and (c) show why inflation declines over time: while the net nominal position redistributive motive to inflate (the first two right-hand-side terms in equation 23) remains roughly stable, the costate μ_t becomes more and more *negative* over time. As explained in Section 3, this captures the increasing weight of the interest rate exposure redistributive motive over time. Panels (b) and (d) in Figure 2 show that those households that issue new bonds (i.e. those with negative URE_t) have lower net wealth and hence higher marginal utility than bond-purchasing households ($URE_t > 0$). This implies a negative covariance between URE and MUC, giving the central bank an incentive to gradually *disinflate* over time. This disinflationary motive is reinforced by the fact that the country is a net issuer of new bonds at all times ($\mathbb{E}_{f_t(\cdot)}[-URE_t] > 0$) and therefore a commitment to lower future inflation benefits domestic households as a whole *vis-à-vis* international investors.

In summary, the central bank *front-loads* inflation in order to redistribute net wealth towards indebted households, but commits to gradually reducing inflation as the initially outstanding bonds mature. In the long-run inflation is slightly negative, reflecting the desire of the central bank to provide a certain degree of extra liquidity to households even at the expense of incurring in some disutility costs.

Finally, panels (e) and (g) in Figure 1 show that the optimal policy succeeds at reducing the country's net liabilities with the rest of the World and at temporarily increasing aggregate consumption, despite an initial spell of consumption below the zero-inflation counterfactual. This aggregate behavior however masks important redistributive effects across households, to which we turn next.

4.4 Redistributive effects of optimal inflation

Having analyzed the aggregate dynamics, one may now ask to what extent the central bank succeeds at redistributing consumption and welfare across households.

Consumption redistribution. Figure 3 shows how the time-0 distribution of consumption across households, $f_0^c(c,y) \equiv f_0(c^{-1}(c,y),y)\,/\,(\partial c/\partial a)$, is affected in the optimal inflationary regime *vis-à-vis* the zero inflation regime.[13] Clearly, the Fisher channel of optimal inflation policy succeeds in raising consumption for consumption-poor households (which are also the wealth-poor households, given the monotonically increasing relationship between net wealth and consumption) and vice versa for consumption-rich households, thus narrowing the consumption distribution relative to the zero inflation regime.

Panels (h) and (i) in Figure 1 offer a dynamic perspective on consumption redistribution after time 0. The gradual recovery in bond prices (panel d), together with the decreasing path of inflation, progressively reverses the time-0 consumption redistribution, as creditors increase their consumption whereas debtors reduce it. Notwithstanding, the speed of this adjustment is heterogeneous across households, with creditors increasing consumption at a faster rate. The result is an initial increase in aggregate consumption (panel g), which peaks after around 10 years, followed by a steady decline towards its long-run value.

Welfare. We now turn to the welfare analysis. Aggregate welfare is defined as

$$\mathbb{E}_{f_0(a,y)}[v_0(a,y)] = \int_0^\infty e^{-\rho t}\mathbb{E}_{f_t(a,y)}[u(c_t(a,y)) - x(\pi_t)]\,dt \equiv W[c],$$

Table 2 displays the welfare losses of the suboptimal zero-inflation policy *vis-à-vis* the Ramsey optimal allocation. We express welfare losses as a permanent consumption equivalent, i.e. the

[13]Asterisks in the figure account for the mass of the Dirac delta at the borrowing limit of the unemployed households, as explained in Achdou et al. (2017).

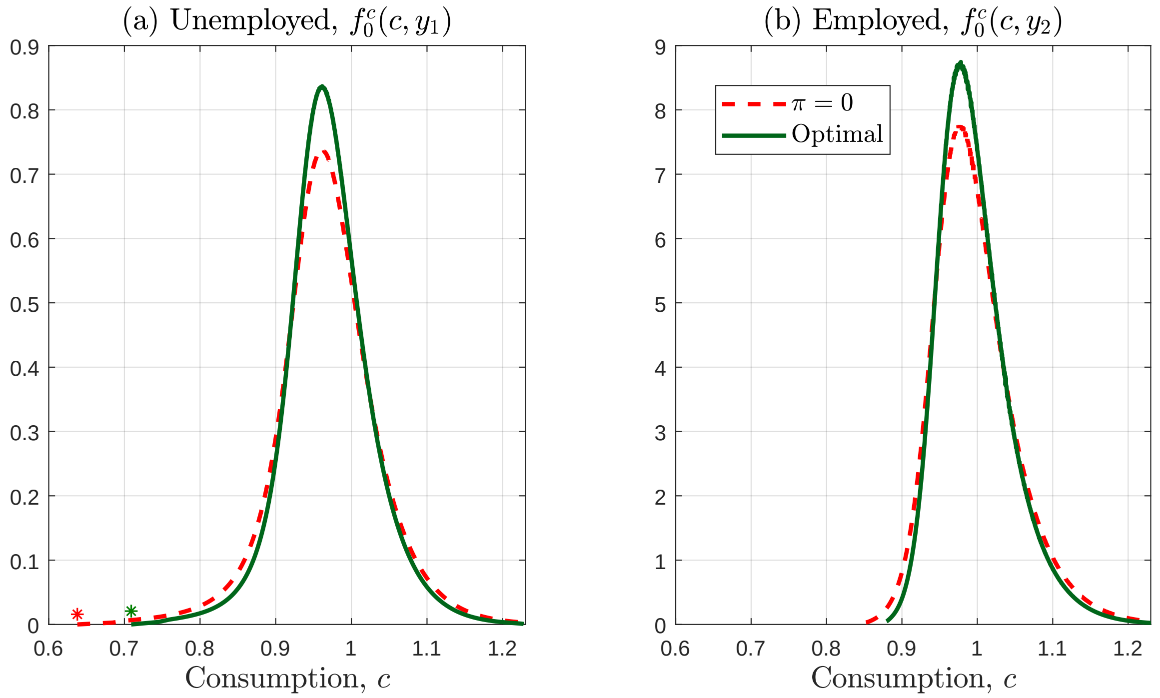

Figure 3: Consumption density at time 0.

number Θ (in %) that satisfies in each case $W^R\left[c^R\right] = W^{\pi=0}\left[(1+\Theta)\,c\right]$, where R denotes the Ramsey allocation. Under our assumed separable preferences with log consumption utility, it is possible to show that $\Theta = \exp\left\{\rho\left(W^R\left[c^R\right] - W\left[c\right]\right)\right\} - 1$. The table also displays the welfare losses incurred respectively by lending and indebted households.[14] The aggregate welfare loss from the suboptimal policy equals 0.05 percent of permanent consumption. This aggregate effect masks a substantial welfare redistribution between groups: the absence of inflation produces welfare losses of 0.22 percent for indebted households and welfare gains of 0.17 percent for lending households.

Table 2. Welfare losses of a zero-inflation policy relative to the optimal commitment

Economy-wide	Creditor HHs	Debtor HHs
0.05	-0.17	0.22

Note: welfare losses are expressed as a % of permanent consumption

4.5 Robustness

Bond maturity. We now assess how sensitive the baseline results are to alternative calibrations. Arguably a key parameter is the average maturity of nominal liabilities. In our baseline

[14]That is, we report $\Theta^{a>0}$ and $\Theta^{a<0}$, where $\Theta^{a>0} = \exp\left[\rho\left(W^{R,a>0} - W^{\pi=0,a>0}\right)\right] - 1$, with $\Theta^{a<0}$ defined analogously, and where for each policy regime we have defined $W^{a>0} \equiv \int_0^{\infty}\sum_{i=1}^{2} v_{i0}\left(a\right) f_{it}(a)da$, $W^{a<0} \equiv \int_{\phi}^{0}\sum_{i=1}^{2} v_{i0}\left(a\right) f_{it}(a)da$. Notice that $\Theta^{a>0}$ and $\Theta^{a>0}$ do not exactly add up to Θ, as the exponential function is not a linear operator. However, Θ is sufficiently small that $\Theta \approx \Theta^{a>0} + \Theta^{a>0}$.

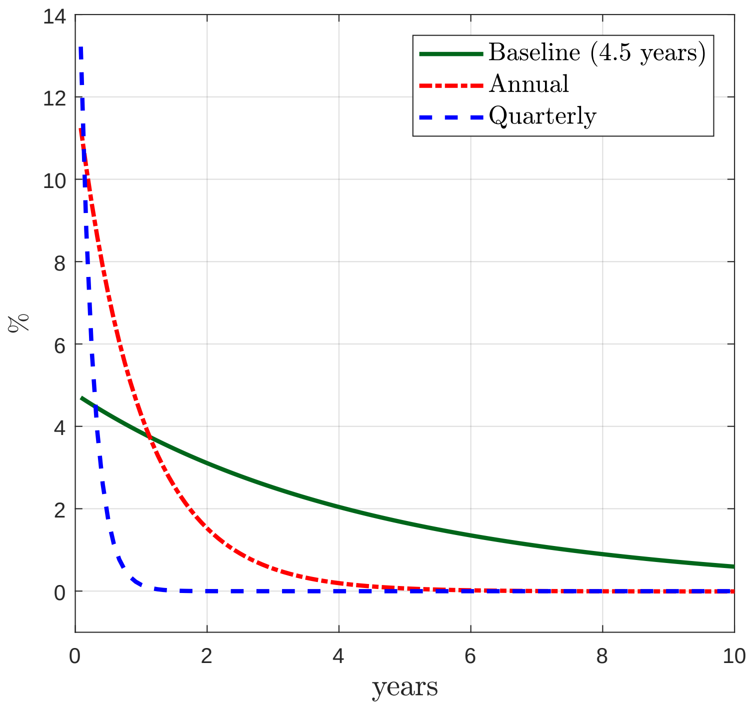

Figure 4: Optimal inflation under different debt durations.

calibration we are assuming an average duration of 4.5 years. By contrast, and with the exception of Auclert (2019), existing analyses of monetary policy with heterogeneous agents and nominal debt consider one-period (quarterly or annual) debt (e.g., Luetticke, 2020; or Bhandari et al., 2020). In order to analyze how debt maturity affects optimal policy, in Figure 4 we show the optimal path of inflation when δ is recalibrated to match an average duration of 1 year (red dashed-dotted line) or 1 quarter (0.25 years, blue dashed lines) and compare them with the baseline (green solid line).

As debt duration is reduced, the front-loading of inflation discussed above is amplified. For instance, with an average bond duration of one quarter, initial inflation jumps over 13 percent but reverts back to (near) zero in less than a year. The intuition is as follows. As explained above, the strength of the time-0 Fisher redistributive motive depends crucially on the decrease in the initial bond price, Q_0. As debt duration is reduced, bond prices become *less* responsive to future inflation. This forces the central bank to front-load inflation more aggressively. In the limit as duration goes to zero (δ goes to infinity) debt becomes instantaneous and optimal inflation is zero at all times, as discussed in section 3.3.

Discount rates and initial wealth dispersion. Appendix G contains two additional robustness exercises analyzing (i) the sensitivity of optimal steady-state inflation to the gap between domestic households' and foreign investors' discount rates ($\rho - \bar{r}$), and (ii) the sensitivity of initial inflation π_0 to the initial net wealth distribution. The results can be summarized as follows. First, optimal steady-state inflation decreases approximately linearly with the gap $\rho - \bar{r}$, because the central bank's incentive to protect households with negative URE –by committing to lower future inflation– becomes more and more dominant relative to its incentive to redistribute resources towards currently indebted households –by raising current inflation.

Second, initial inflation increases with the dispersion of the initial net wealth distribution (while holding constant the initial net foreign asset position), reflecting a stronger net nominal position redistributive motive. This exercise also reveals that both the domestic and cross-border redistributive motives are quantitatively important for explaining initial inflation, with contributions of about one third and two thirds, respectively. In particular, even if the country is assumed to have a zero NNP *vis-à-vis* the rest of the World at time 0, the purely domestic net nominal position motive is enough to justify an optimal initial inflation of π_0 about 1.5 percent.

4.6 Optimal response to shocks from a timeless perspective

So far we have restricted our analysis to the transitional dynamics, given the initial state of the economy, while abstracting from aggregate shocks. We now extend our analysis to allow for aggregate disturbances. For the purpose of illustration we consider a so-called "MIT shock" to aggregate income. In particular, let individual income now be given by $\{y_1 Y_t, y_2 Y_t\}$, such that aggregate income equals Y_t (given our assumption that $\bar{y} = 1$).[15] Assume a one-time, unanticipated decrease of 1 percentage point in Y_t, after which it returns gradually to its steady-state value $Y_{ss} = 1$ according to

$$dY_t = \eta_Y (1 - Y_t)\, dt,$$

with $\eta_Y = 0.5$. As discussed by Boppart, Krusell and Mittman (2018) and Auclert, Bardóczy and Rognlie (2019), this is equivalent to solving a model with an aggregate stochastic process $dY_t = \eta_Y (1 - Y_t)\, dt + \sigma dZ_t$, with $\sigma = 0.01$ and Z_t a Brownian motion, around the deterministic steady state using a first order approximation as in the method of Ahn et al. (2017). The methodology in Boppart, Krusell and Mittman (2018) would also allow us to compute aggregate moments, a feature that we do not exploit in this paper.

An issue that arises here is how long after 'time 0' (the implementation date of the Ramsey optimal commitment) the aggregate shock is assumed to take place. Since we do not want to take a stand on this dimension, we consider the limiting case in which the Ramsey optimal commitment has been going on for a sufficiently long time that the economy rests at its stationary equilibrium by the time the shock arrives. This can be viewed as an example of optimal policy 'from a *timeless perspective*', in the sense of Woodford (2003). In practical terms, it requires solving the optimal commitment problem analyzed in Section 3 with two modifications (apart of course from the time variation in Y_t): (i) the initial wealth distribution is the stationary distribution implied by the optimal commitment itself, and (ii) the initial condition $\mu_0 = 0$ (absence of precommitments) is replaced by $\mu_0 = \mu_\infty$, where the latter object is the stationary value of the costate in the commitment case. Both modifications guarantee that the central bank behaves as if it had been following the time-0 optimal commitment for an arbitrarily long

[15] This is isomorphic to a TFP shock in a model with linear technology and fixed-labor supply.

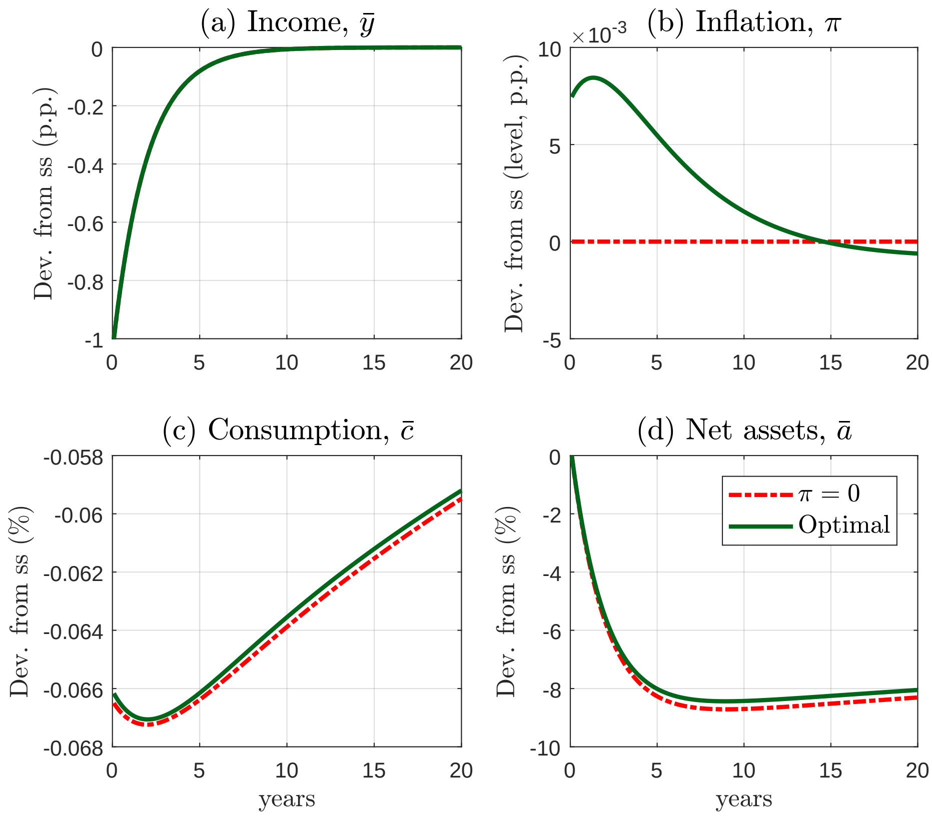

Figure 5: Generalized impulse response function of an aggregate income shock.

time.

The dashed red lines in Figure 5 display the responses to the shock under a zero inflation policy. The shock leads households to reduce their consumption on impact and to progressively accumulate more debt (negative wealth) *vis-à-vis* the rest of the World. The solid green lines display the economy's response under the optimal commitment policy. Inflation rises slightly on impact, as the central bank tries to partially counteract the negative effect of the shock on household consumption. However, the inflation reaction is an order of magnitude smaller than that of the shock itself. Intuitively, the value of sticking to past commitments to keep inflation near zero weighs more in the central bank's decision than the value of using inflation transitorily so as to stabilize consumption in response to an unforeseen event.

The main conclusion of this exercise is that, despite the strong incentives to redistribute in the absence of precommitments, once the central bank has committed to an optimal (near) zero inflation path it does *not* exploit in a significant way the redistributive channels of monetary policy to accommodate aggregate shocks. Naturally this conclusion depends on the specific nature of the shock, but it is robust to other standard shocks such as an increase in the World interest rate (analyzed in Nuño and Thomas, 2016).

5 A limit on the market value of debt

Our baseline model assumes an exogenous limit on the real *face* value of net liabilities, $(-)a_t$. It is also interesting to study instead the case if a borrowing limit affecting the real *market* value of net debt, $(-)Q_t a_t$. Consider thus the alternative of placing an exogenous limit on the real market value of net liabilities: $Q_t a_t \geq \phi^m$, or equivalently

$$a_t^m \geq \phi^m. \tag{28}$$

with $\phi^m \leq 0$.[16] This is the case assumed e.g. by Auclert (2019).[17] This differs from the market-value representation of our baseline borrowing limit (eq. 14) in that the central bank can *no longer* tighten or relax it by affecting bond prices. In other words, this alternative borrowing constraint effectively shuts down the liquidity channel of monetary policy operating in our baseline analysis. The Fisher channel still operates in this case: monetary policy can only affect the net wealth distribution (in market value terms) through its impact on time-0 bond prices.

In order to assess the impact of abstracting from the liquidity channel, we solve the optimal monetary policy problem in this alternative model. Appendix E explains in detail the derivations. This case is quite tractable because the domain of the net wealth distribution in market value terms, $[\phi^m, \infty)$, is policy invariant. The optimal inflation is now determined by

$$\psi\pi_t = \underbrace{\frac{Q_t}{Q_0} e^{\left[-\int_0^t (\bar{r}+\delta+\pi_s-\rho)ds\right]}}_{\text{Dynamic evolution}} \underbrace{(-Q_0) \sum_{i=1}^{2} \int_{\phi^m}^{\infty} \overbrace{v_{i0}\left(\frac{a^m}{Q_0}\right)}^{\text{time-0 value function}} \frac{d}{dQ_0} \overbrace{\left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)\right]}^{\text{initial distribution in market value}} da^m}_{\text{Time-0 optimal inflation}(\psi\pi_0)}. \tag{29}$$

According to equation (29), the marginal disutility of inflation $x'(\pi_t) = \psi\pi_t$ must equal its marginal benefit. The latter is represented as the product of optimal time-0 inflation (times ψ) and a term capturing the dynamic evolution of optimal inflation after time 0. Optimal time-0 inflation follows a formula similar to the one in the baseline case (equation 25) but where both the domain $[\phi^m, \infty)$ and the value function $v_{i0}(\cdot)$ do not depend on bond prices, consistently with the fact that the liquidity channel does not operate in this case. Inflation dynamics follow the expression $Q_t/Q_0 \exp\left[-\int_0^t (\bar{r}+\delta+\pi_s-\rho)\, ds\right]$ as in the representative agent case, reflecting the diminishing role of future inflation in terms of time-0 redistribution.

Notice that only the *initial* net wealth distribution $f_0^m(a^m)$ matters for the optimal inflation path. However, solving for that path is still complicated by the fact that such initial distribution is *endogenous* to the policy-maker's actions. In fact, the inflation rate at *any* time $t \geq 0$ affects

[16] In our numerical analysis of this specification, we maintain the same calibration described in section 4, with $\phi^m = Q^{\pi=0}\phi$, where $Q^{\pi=0} = \delta/(\bar{r}+\delta)$ is the nominal bond price in the case of zero inflation. This allows both specifications to produce the same results in the particular case of zero inflation.

[17] Our environment is very stylized, so it is hard to disentangle which borrowing limit is more realistic. A detailed discussion of borrowing limits in a model featuring housing can be found in Greenwald (2018), for instance.

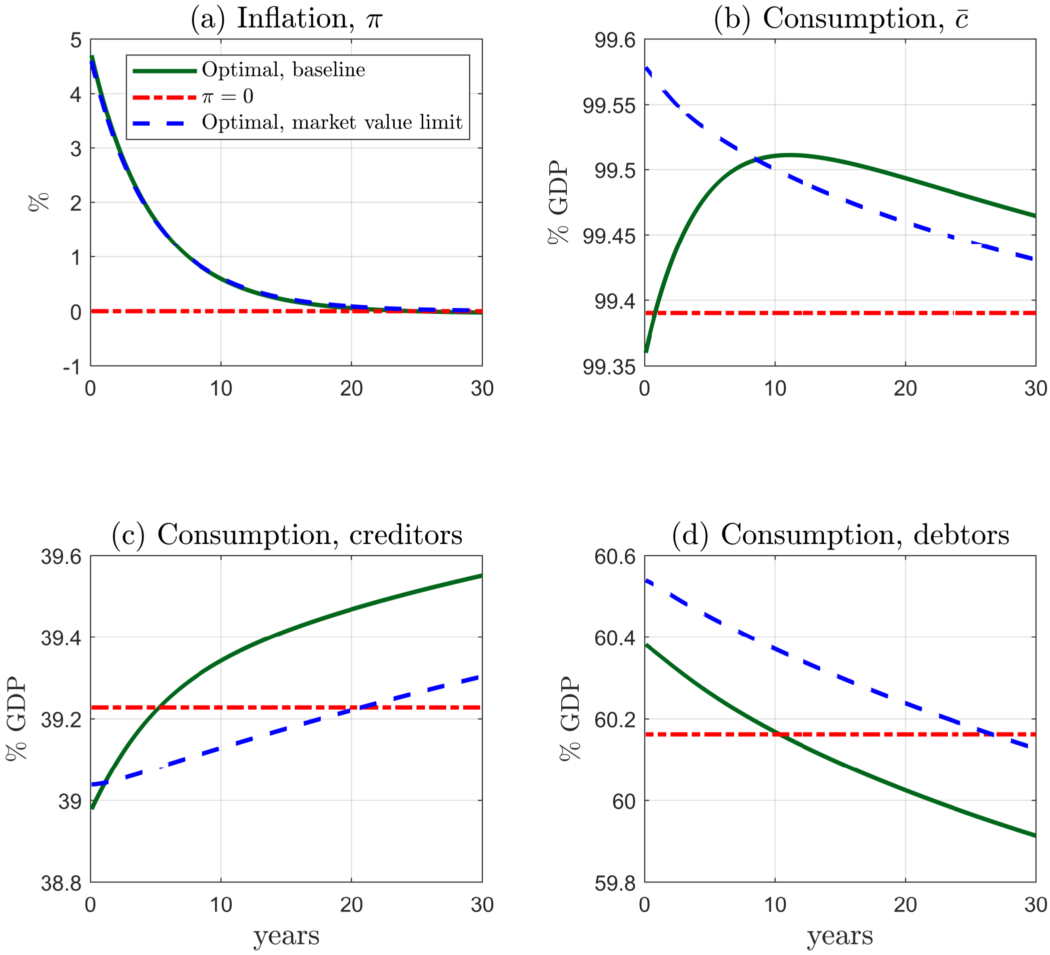

Figure 6: Transitional dynamics with alternative exogenous debt limits.

the time-0 distribution of the market value of net wealth through its effect on Q_0 (eq. 26). In Appendix C.3, we sketch the algorithm that we use to solve the problem.

Figure 6 displays the transitional dynamics of the model in the case of a limit on the market value of debt, compared with the baseline (already shown in Figure 1). Notice first that the optimal inflation path is barely affected by the specification of the borrowing limit. We will return to this issue below. However, the responses of consumption are markedly different across both specifications. As in the baseline case, time-0 consumption is lower for creditors and higher for debtors, compared to the zero inflation case.[18] However, in the case of a market-value debt limit, debtor's initial consumption gain is much larger than under the face-value debt limit, and hence time-0 aggregate consumption is higher too.

To understand this difference, Figure 7 displays the time-0 consumption policies and the wealth distribution, both as functions of market-value net wealth, a^m. Panels (c) and (d) show how, given the same initial decline in bond prices −a consequence of almost identical inflation paths−, the wealth distribution in market value terms practically coincides in both cases. The difference lies in the behavior of the consumption policy function: whereas it remains unaltered (relative to zero inflation) in the case of a debt limit in market value, it

[18]Notice that under zero inflation it does not matter whether the debt limit is on the face or market value of debt, since bond prices are constant.

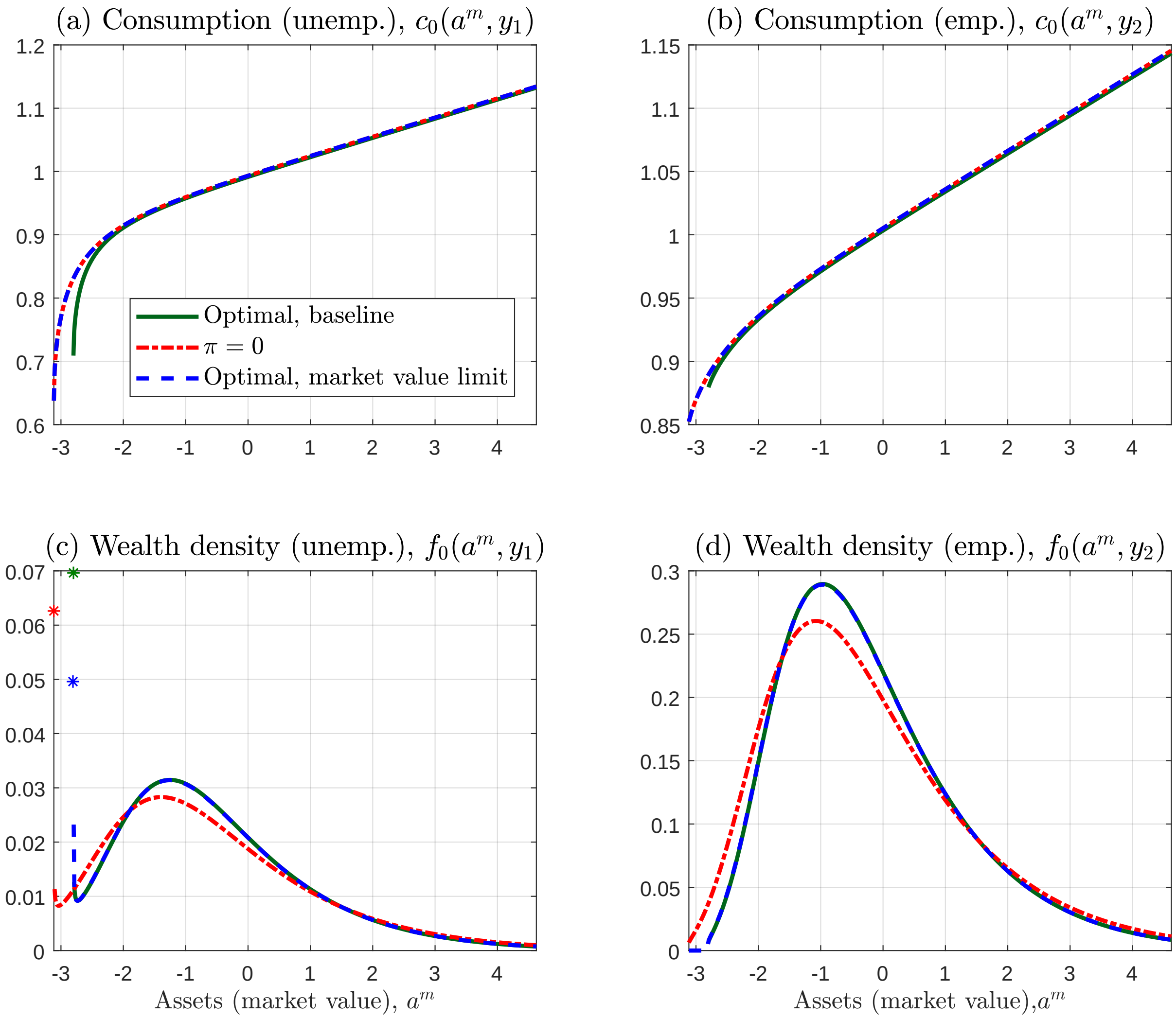

Figure 7: Time-0 redistribution with alternative exogenous debt limits.

varies in the baseline model, because the initial fall in bond prices *tightens* the borrowing limit through equation (14). This explains why debtors' initial consumption increases (relative to zero inflation) by a smaller amount in the baseline model, thus producing the initial decline in aggregate consumption discussed in Section 4.3.

Why is optimal inflation so similar in both models despite the different consumption dynamics? The difference between the two models is due exclusively to the working of the liquidity channel. As there is no mass in the wealth distribution below ϕQ_0 in the case with a debt limit in the market value of debt (as shown in Panel (c) of Figure 7), the effect of the time-0 bond price on the lower limits plays no role. The difference between the time-0 value functions is quantitatively small, as displayed in Figure 10 in Appendix F, which explains the similarity in the optimal inflation paths.

To summarize, while the debt limit specification (face vs. market value) matters for the transmission channels of monetary policy and the dynamics of aggregate consumption, this appears to have a relatively limited quantitative impact on the optimal inflation policy in our framework. An open question is whether the same would hold true in a richer, more complex, environment such as those considered in the recent HANK literature (e.g., Kaplan, Moll and Violante, 2018; or Gornemann, Kuester and Nakajima, 2016) expanded with long-term debt.

6 Conclusion

We have analyzed optimal monetary policy in a continuous-time, small-open-economy version of a standard incomplete-markets model extended to allow for nominal, long-term claims and costly inflation. Our analysis sheds light on a recent policy and academic debate on the consequences that wealth heterogeneity across households should have for the appropriate conduct of monetary policy.

Our first contribution relates to our normative results. Our model features two prominent transmission channels of monetary policy: the classic Fisher channel, and a liquidity channel. Under incomplete markets and standard concave preferences, indebted households have a higher marginal utility than lending ones, giving the central bank an incentive to use inflation in order to redistribute wealth from the latter to the former. The result is an initial inflationary bias. This bias is counteracted over time by a disinflationary motive as initial bonds mature. In the long-run, the Fisher channel plays no role and optimal inflation is negative: the central bank raises asset prices in order to provides some additional liquidity to households The optimal commitment policy is found indeed to imply inflation 'front-loading', with a gradual undoing of the initial inflationary stance.

Our second contribution is methodological: we solve for a fully dynamic optimal policy problem in an incomplete-markets model with uninsurable idiosyncratic risk. While models of this kind have been established as a workhorse for policy analysis in macro models with heterogeneous agents, the fact that in such models the infinite-dimensional, endogenously-evolving wealth distribution is a state in the policy-maker's problem has made it difficult to make progress in the analysis of fully optimal policy problems. Our analysis proposes a novel methodology, based on infinite dimensional calculus, for dealing with problems of this kind.

Finally, our methodology can be extended to the analysis of the standard closed-economy New Keynesian model with heterogeneous agents and exogenous borrowing limits. As discussed in Nuño and Moll (2018), in this case the value function of each household does not coincide with the social value assigned to it by the central bank. This is a consequence of the pecuniary externality present in the closed-economy version of the model. The difference between private and social valuations gives rise to a new redistributive motive that will affect the optimal conduct of monetary policy. Furthermore, the standard New Keynesian transmission channels will also interact with the Fisher and liquidity ones analyzed in this paper. We leave the study of this model for future research.

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Online appendix (not for publication)

A. An economy with costly price adjustment

In this appendix, we lay out a model economy with the following characteristics: (i) firms are explicitly modeled, (ii) a subset of them are price-setters but incur a convex cost for changing their nominal price, and (iii) the social welfare function and the equilibrium conditions constraining the central bank's problem are the same as in the model economy in the main text.

A.1. Final good producer

In the model laid out in the main text, we assumed that output of the consumption good was exogenous. Consider now an alternative setup in which the consumption good is produced by a representative, perfectly competitive final good producer with the following Dixit-Stiglitz technology,

$$y_t = \left(\int_0^1 y_{jt}^{(\varepsilon-1)/\varepsilon} dj \right)^{\varepsilon/(\varepsilon-1)}, \tag{30}$$

where $\{y_{jt}\}$ is a continuum of intermediate goods and $\varepsilon > 1$. Let P_{jt} denote the nominal price of intermediate good $j \in [0,1]$. The firm chooses $\{y_{jt}\}$ to maximize profits, $P_t y_t - \int_0^1 P_{jt} y_{jt} dj$, subject to (30). The first order conditions are

$$y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} y_t, \tag{31}$$

for each $j \in [0,1]$. Assuming free entry, the zero profit condition and equations (31) imply $P_t = (\int_0^1 P_{jt}^{1-\varepsilon} dj)^{1/(1-\varepsilon)}$.

A.2. Intermediate goods producers

Each intermediate good j is produced by a monopolistically competitive intermediate-good producer, which we will refer to as 'firm j' henceforth for brevity. Firm j operates a linear production technology,

$$y_{jt} = n_{jt}, \tag{32}$$

where n_{jt} is labor input. At each point in time, firms can change the price of their product but face quadratic price adjustment cost as in Rotemberg (1982). Letting $\dot{P}_{jt} \equiv dP_{jt}/dt$ denote the change in the firm's price, price adjustment costs in units of the final good are given by

$$\Psi_t \left(\frac{\dot{P}_{jt}}{P_{jt}} \right) \equiv \frac{\psi}{2} \left(\frac{\dot{P}_{jt}}{P_{jt}} \right)^2 \tilde{C}_t, \tag{33}$$

where $\tilde{C}_t$ is aggregate consumption. Let $\pi_{jt} \equiv \dot{P}_{jt}/P_{jt}$ denote the rate of increase in the firm's price. The instantaneous profit function in units of the final good is given by

$$\begin{aligned}\Pi_{jt} &= \frac{P_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Psi_t(\pi_{jt}) \\ &= \left(\frac{P_{jt}}{P_t} - w_t\right)\left(\frac{P_{jt}}{P_t}\right)^{-\varepsilon} y_t - \Psi_t(\pi_{jt}),\end{aligned} \tag{34}$$

where w_t is the perfectly competitive real wage and in the second equality we have used (31) and (32). Without loss of generality, firms are assumed to be risk neutral and have the same discount factor as households, ρ. Then firm j's objective function is

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \Pi_{jt} dt,$$

with Π_{jt} given by (34). The state variable specific to firm j, P_{jt}, evolves according to $dP_{jt} = \pi_{jt} P_{jt} dt$. The aggregate state relevant to the firm's decisions is simply time: t. Then firm j's *value function* $V(P_{jt}, t)$ must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho V(P_j, t) = \max_{\pi_j} \left\{ \left(\frac{P_j}{P_t} - w_t\right)\left(\frac{P_j}{P_t}\right)^{-\varepsilon} y_t - \Psi_t(\pi_j) + \pi_j P_j \frac{\partial V}{\partial P_j}(P_j, t) \right\} + \frac{\partial V}{\partial t}(P_j, t).$$

The first order and envelope conditions of this problem are (we omit the arguments of V to ease the notation),

$$\psi \pi_{jt} \tilde{C}_t = P_j \frac{\partial V}{\partial P_j}, \tag{35}$$

$$\rho \frac{\partial V}{\partial P_j} = \left[\varepsilon w_t - (\varepsilon - 1)\frac{P_j}{P_t}\right]\left(\frac{P_j}{P_t}\right)^{-\varepsilon} \frac{y_t}{P_j} + \pi_j \left(\frac{\partial V}{\partial P_j} + P_j \frac{\partial^2 V}{\partial P_j^2}\right).$$

In what follows, we will consider a symmetric equilibrium in which all firms choose the same price: $P_j = P, \pi_j = \pi$ for all j. After some algebra, it can be shown that the above conditions imply the following pricing Euler equation,

$$\left[\rho - \frac{d\tilde{C}_t}{dt}\frac{1}{\tilde{C}_t}\right]\pi_t = \frac{\varepsilon - 1}{\psi}\left(\frac{\varepsilon}{\varepsilon - 1} w_t - 1\right)\frac{1}{\tilde{C}_t} + \frac{d\pi_t}{dt}. \tag{36}$$

Equation (36) determines the market clearing wage w_t.

A.3. Households

The preferences of household $k \in [0,1]$ are given by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \log(\tilde{c}_{kt}) dt,$$

where $\tilde{c}_{kt}$ is household consumption of the final good. We now define the following object,

$$c_{kt} \equiv \tilde{c}_{kt} + \frac{\tilde{c}_{kt}}{\tilde{C}_t} \int_0^1 \Psi_t(\pi_{jt})\, dj,$$

i.e. household k's consumption plus a fraction of total price adjustment costs ($\int \Psi_t(\cdot)\, dj$) equal to that household's share of total consumption ($\tilde{c}_{kt}/\tilde{C}_t$). Using the definition of Ψ_t (eq. 33) and the symmetry across firms in equilibrium ($\dot{P}_{jt}/P_{jt} = \pi_t, \forall j$), we can write

$$c_{kt} = \tilde{c}_{kt} + \tilde{c}_{kt} \frac{\psi}{2} \pi_t^2 = \tilde{c}_{kt} \left(1 + \frac{\psi}{2} \pi_t^2\right). \tag{37}$$

Therefore, household k's instantaneous utility can be expressed as

$$\begin{aligned} \log(\tilde{c}_{kt}) &= \log(c_{kt}) - \log\left(1 + \frac{\psi}{2}\pi_t^2\right) \\ &= \log(c_{kt}) - \frac{\psi}{2}\pi_t^2 + o\left(\left\|\frac{\psi}{2}\pi_t^2\right\|^2\right), \end{aligned} \tag{38}$$

where $o(\|x\|^2)$ denotes terms of order second and higher in x. Expression (38) is the same as the utility function in the main text, up to a first order approximation of $\log(1+x)$ around $x = 0$, where $x \equiv \frac{\psi}{2}\pi^2$ represents the percentage of aggregate spending that is lost to price adjustment. For our baseline calibration ($\psi = 5.5$), the latter object is relatively small even for relatively high inflation rates, and therefore so is the approximation error in computing the utility losses from price adjustment. Therefore, the utility function used in the main text provides a fairly accurate approximation of the welfare losses caused by inflation in the economy with costly price adjustment described here.

Households can be in one of two idiosyncratic states. Those in state $i = 1$ do not work. Those in state $i = 2$ work and provide z units of labor inelastically. As in the main text, the instantaneous transition rates between both states are given by λ_1 and λ_2, and the share of households in each state is assumed to have reached its ergodic distribution; therefore, the fraction of working and non-working households is $\lambda_1/(\lambda_1+\lambda_2)$ and $\lambda_2/(\lambda_1+\lambda_2)$, respectively. Hours per worker z are such that total labor supply $\frac{\lambda_1}{\lambda_1+\lambda_2} z$ is normalized to 1.

An exogenous government insurance scheme imposes a (total) lump-sum transfer τ_t from working to non-working households. All households receive, in a lump-sum manner, an equal share of aggregate firm profits *gross* of price adjustment costs, which we denote by $\hat{\Pi}_t \equiv P_t^{-1} \int_0^1 P_{jt} y_{jt} dj - w_t \int_0^1 n_{jt} dj$. Therefore, disposable income (gross of price adjustment costs) for non-working and working households are given respectively by

$$I_{1t} \equiv \frac{\tau_t}{\lambda_2/(\lambda_1+\lambda_2)} + \hat{\Pi}_t,$$

$$I_{2t} \equiv w_t z - \frac{\tau_t}{\lambda_1/(\lambda_1+\lambda_2)} + \hat{\Pi}_t.$$

We assume that the transfer τ_t is such that gross disposable income for households in state i equals a constant level y_i, $i = 1, 2$, with $y_1 < y_2$. As in our baseline model, both income levels satisfy the normalization

$$\frac{\lambda_2}{\lambda_1 + \lambda_2} y_1 + \frac{\lambda_1}{\lambda_1 + \lambda_2} y_2 = 1.$$

Also, later we show that in equilibrium gross income equals one: $\hat{\Pi}_t + w_t \frac{\lambda_1}{\lambda_1+\lambda_2} z = 1$. It is then easy to verify that implementing the gross disposable income allocation $I_{it} = y_i$, $i = 1, 2$, requires a transfer equal to $\tau_t = \frac{\lambda_2}{\lambda_1+\lambda_2} y_1 - \frac{\lambda_2}{\lambda_1+\lambda_2} \hat{\Pi}_t$. Finally, total price adjustment costs are assumed to be distributed in proportion to each household's share of total consumption, i.e. household k incurs adjustment costs in the amount $(\tilde{c}_{kt}/\tilde{C}_t)(\frac{\psi}{2}\pi_t^2 \tilde{C}_t) = \tilde{c}_{kt}\frac{\psi}{2}\pi_t^2$. Letting $I_{kt} \equiv y_{kt} \in \{y_1, y_2\}$ denote household k's gross disposable income, the law of motion of that household's real net wealth is thus given by

$$\begin{aligned} da_{kt} &= \left[\left(\frac{\delta}{Q_t} - \delta - \pi_t \right) a_{kt} + \frac{I_{kt} - \tilde{c}_{kt} - \tilde{c}_{kt}\psi\pi_t/2}{Q_t} \right] dt \\ &= \left[\left(\frac{\delta}{Q_t} - \delta - \pi_t \right) a_{kt} + \frac{y_{kt} - c_{kt}}{Q_t} \right] dt, \end{aligned} \tag{39}$$

where in the second equality we have used (37). Equation (39) is exactly the same as its counterpart in the main text, equation (3). Since household's welfare criterion is also the same, it follows that so is the corresponding maximization problem.

A.4. Aggregation and market clearing

In the symmetric equilibrium, each firm's labor demand is $n_{jt} = y_{jt} = \bar{y}_t$. Since labor supply $\frac{\lambda_1}{\lambda_1+\lambda_2} z = 1$ equals one, labor market clearing requires

$$\int_0^1 n_{jt} dj = \bar{y}_t = 1.$$

Therefore, in equilibrium aggregate output is equal to one. Firms' profits gross of price adjustment costs equal

$$\hat{\Pi}_t = \int_0^1 \frac{P_{jt}}{P_t} y_{jt} dj - w_t \int_0^1 n_{jt} dj = \bar{y}_t - w_t,$$

such that gross income equals $\hat{\Pi}_t + w_t = \bar{y}_t = 1$.

A.5. Central bank and monetary policy

We have shown that households' welfare criterion and maximization problem are as in our baseline model. Thus the dynamics of the net wealth distribution continue to be given by equation (10). Foreign investors can be modeled exactly as in Section 2. Therefore, the central bank's optimal policy problems, both under commitment and discretion, are exactly as in our baseline model.

B. Proofs

B.1. Mathematical preliminaries

First we need to introduce some mathematical notation. Given the stochastic process a_t in (3), we define the operator $\mathcal{A}$,

$$\mathcal{A}v \equiv \begin{pmatrix} s_1(t,a)\frac{\partial v_1(t,a)}{\partial a} + \lambda_1\left[v_2(t,a) - v_1(t,a)\right] \\ s_2(t,a)\frac{\partial v_2(t,a)}{\partial a} + \lambda_2\left[v_1(t,a) - v_2(t,a)\right] \end{pmatrix}, \tag{40}$$

so that the HJB equation (5) can be expressed as

$$\rho v = \frac{\partial v}{\partial t} + \max_c \left\{u\left(c\right) - x\left(\pi\right) + \mathcal{A}v\right\},$$

where $v \equiv \binom{v_1(t,a)}{v_2(t,a)}$ and $u\left(c\right) - x\left(\pi\right) \equiv \binom{u(c_1)-x(\pi)}{u(c_2)-x(\pi)}$.[19]

Let $\Phi \equiv \{1,2\} \times \mathbb{R}$ and $\hat{\Phi} = [0,\infty) \times \Phi$, we employ the notation

$$\langle f,g\rangle_\Phi = \sum_{i=1}^{2}\int_\Phi f_i g_i da = \int_\Phi f^{\mathbf{T}} g da,\ \forall f,g \in L^2\left(\Phi\right), \tag{41}$$

$$(f,g) = \int_0^\infty e^{-\rho t} f g dt,\ \forall f,g \in L^2[0,\infty) \tag{42}$$

$$(f,g)_\Phi = \int_0^\infty e^{-\rho t}\langle f,g\rangle_\Phi dt = \left\langle e^{-\rho t} f, g\right\rangle_{\hat{\Phi}},\ \forall f,g \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}, \tag{43}$$

The spaces of Lebesgue-integrable functions $L^2\left(\Phi\right)$ and $L^2[0,\infty)$ with the inner products (41) and (42), respectively, are Hilbert spaces (see Luenberger, 1969; or Brezis, 2011). The space $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ with the inner product (43) is also a Hilbert space (See Nuño and Moll, 2018).[20]

Given an operator $\mathcal{A}$, its *adjoint* is an operator $\mathcal{A}^*$ such that $\langle f,\mathcal{A}v\rangle_\Phi = \langle \mathcal{A}^* f, v\rangle_\Phi$. In the case of the operator defined by (40) its adjoint is the operator

$$\mathcal{A}^* f \equiv \begin{pmatrix} -\frac{\partial(s_1 f_1)}{\partial a} - \lambda_1 f_1 + \lambda_2 f_2 \\ -\frac{\partial(s_2 f_2)}{\partial a} - \lambda_2 f_2 + \lambda_1 f_1 \end{pmatrix}, \tag{44}$$

such that the KF equation (10) results in

$$\frac{\partial f}{\partial t} = \mathcal{A}^* f, \tag{45}$$

[19] The *infinitesimal generator* of the process is thus $\frac{\partial v}{\partial t} + \mathcal{A}v$.

[20] To be more precise, we should work in the Sobolev space $H^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$, defined as the space of functions such that

$$\int_{\hat{\Phi}} e^{-\rho t}\left|f\right|^2 + \left|f'\right|^2 + \left|f''\right|^2 < \infty.$$

However we stick to $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ as all the results coincide and the proofs would be more cumbersome.

for $f \equiv \binom{f_1(t,a)}{f_2(t,a)}$. We can see that $\mathcal{A}$ and $\mathcal{A}^*$ are adjoints as

$$\begin{aligned}
\langle \mathcal{A}v, f \rangle_\Phi &= \int_\Phi (\mathcal{A}v)^\mathbf{T} f da = \sum_{i=1}^{2} \int_\Phi \left[s_i \frac{\partial v_i}{\partial a} + \lambda_i \left[v_j - v_i \right] \right] f_i da \\
&= \sum_{i=1}^{2} v_i s_i f_i |_{-\infty}^{\infty} + \sum_{i=1}^{2} \int_\Phi v_i \left[-\frac{\partial}{\partial a} \left(s_i f_i \right) - \lambda_i f_i + \lambda_j f_j \right] da \\
&= \int_\Phi v^\mathbf{T} \mathcal{A}^* f da = \langle v, \mathcal{A}^* f \rangle_\Phi .
\end{aligned}$$

The Gâteaux and Frechet derivatives generalize the standard derivative to infinite-dimensional spaces (See Luenberger, 1969, Gelfand and Fomin,1991, or Sagan, 1992). In particular, if $W[f]$ is a functional in function space such as $L^2(\Phi)$ or $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ and h is and arbitrary function in that space, the *Gâteaux derivative* of W at f with increment h is defined as

$$\delta W[f;h] = \lim_{\alpha \to 0} \frac{W[f+\alpha h] - W[f]}{\alpha} = \frac{d}{d\alpha} W[f+\alpha h] |_{\alpha=0}, \tag{46}$$

Nuño and Moll (2018) discuss the key theorems for the application of these concepts to the solution of infinite dimensional dynamic programming problems. We summarize their findings by stating that, if $W[f]$ is a functional in a function space such as $L^2(\Phi)$ and H is a mapping from the same functional space into $\mathbb{R}^p$ −where p is a natural number−, then a necessary condition for W to have a maximum (or minimum) at f under the constraint $H[f] = 0$ is that there exists another function $\eta \in L^2(\Phi)$ (the Lagrange multiplier) such that the Gâteaux derivative of the Lagrangian $\mathcal{L}[f] \equiv W[f] + \langle \eta, H[f] \rangle_\Phi$ is zero when evaluated at f for any increment h in the same space:

$$\delta \mathcal{L}[f;h] = 0, \quad \text{for any } h \in L^2(\Phi).$$

And equivalently for the spaces $L^2[0,\infty)$ or $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$.

B.2. Proof of Proposition 1. Solution to the Ramsey problem

The idea of the proof is to construct a Lagragian in a Hilbert function space and to obtain the first-order conditions by taking the Gâteaux derivatives.

Statement of the problem. The problem of the central bank is given by

$$W[f_0(\cdot)] = \max_{\{\pi_t, Q_t, v_t(\cdot), c_t(\cdot), f_t(\cdot)\}_{t\geq 0}} \int_0^\infty e^{-\rho t} \left[\sum_{i=1}^{2} \int_\Phi \left(u(c_t) - x(\pi_t) \right) f_{it}(a) da \right] dt,$$

subject to the law of motion of the distribution (10), the bond pricing equation (9) and the individual HJB equation (5).

The Lagragian. From now on, for compactness we use the operator $\mathcal{A}$, its adjoint operator $\mathcal{A}^*$, and the inner product $\langle \cdot, \cdot \rangle$ defined in expressions (40), (44), and (41), respectively. The

Lagrangian is defined in $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ as

$$\begin{aligned}
\mathcal{L}\left[\pi, Q, f, v, c\right] \equiv & \int_0^\infty e^{-\rho t}\left\langle u\left(c_t\right) - x\left(\pi_t\right), f_t\right\rangle_\Phi dt + \int_0^\infty \left\langle e^{-\rho t}\zeta_t, \mathcal{A}^* f_t - \frac{\partial f}{\partial t}\right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t}\mu_t\left(Q\left(\bar{r} + \pi_t + \delta\right) - \delta - \dot{Q}_t\right) dt \\
& + \int_0^\infty \left\langle e^{-\rho t}\theta_t, u\left(c_t\right) - x\left(\pi_t\right) + \mathcal{A}v_t + \frac{\partial v}{\partial t} - \rho v_t\right\rangle_\Phi dt \\
& + \int_0^\infty \left\langle e^{-\rho t}\eta_t, u'\left(c_t\right) - \frac{1}{Q_t}\frac{\partial v}{\partial a}\right\rangle_\Phi dt
\end{aligned}$$

where ζ_t,η_t, $\theta_t \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ and $\mu_t \in L^2[0,\infty)$ are the Lagrange multipliers associated to equations (10), (7), (5) and (9), respectively. The Lagragian can be expressed as

$$\begin{aligned}
\mathcal{L} = & \int_0^\infty e^{-\rho t}\left\langle u\left(c_t\right) - x\left(\pi_t\right) + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta_t - \rho\zeta_t + \mu_t\left(Q_t\left(\bar{r} + \pi_t + \delta\right) - \delta - \dot{Q}_t\right), f_t\right\rangle_\Phi dt \\
& + \int_0^\infty e^{-\rho t}\left(\left\langle \theta_t, u\left(c_t\right) - x\left(\pi_t\right)\right\rangle_\Phi + \left\langle \mathcal{A}^*\theta_t - \frac{\partial \theta}{\partial t}, v_t\right\rangle_\Phi + \left\langle \eta_t, u'\left(c_t\right) - \frac{1}{Q_t}\frac{\partial v}{\partial a}\right\rangle_\Phi\right) dt \\
& + \left\langle \zeta_0\left(\cdot\right), f_0\right\rangle_\Phi - \lim_{T\to\infty}\left\langle e^{-\rho T}\zeta_T, f_T\right\rangle_\Phi \\
& + \lim_{T\to\infty}\left\langle e^{-\rho T}\theta_T, v_T\right\rangle_\Phi - \left\langle \theta_0, v_0\right\rangle_\Phi + \int_0^\infty e^{-\rho t}\sum_{i=1}^{2} v_{it}\left(a\right) s_{it}\left(a\right)\theta_{it}\left(a\right)\Big|_{-\infty}^{\infty} dt,
\end{aligned}$$

where we have applied the definition of adjoint operator

$$\begin{aligned}
\left\langle \zeta_t, \mathcal{A}^* f_t\right\rangle_\Phi &= \left\langle \mathcal{A}\zeta_t, f_t\right\rangle_\Phi, \\
\left\langle \theta_t, \mathcal{A}v_t\right\rangle_\Phi &= \left\langle \mathcal{A}^*\theta_t, v_t\right\rangle_\Phi + \sum_{i=1}^{2} v_{it}\left(a\right) s_{it}\left(a\right)\theta_{it}\left(a\right)\Big|_{-\infty}^{\infty}
\end{aligned}$$

and integrated by parts

$$\begin{aligned}
\int_0^\infty \left\langle e^{-\rho t}\zeta_t, -\frac{\partial f}{\partial t}\right\rangle_\Phi dt &= -\sum_{i=1}^{2}\int_0^\infty \int_\Phi e^{-\rho t}\zeta_{it}(a)\frac{\partial f_i}{\partial t} dadt \\
&= -\sum_{i=1}^{2}\int_\Phi f_{it}(a)\, e^{-\rho t}\zeta_{it}(a)\big|_0^\infty da \\
&\quad +\sum_{i=1}^{2}\int_0^\infty \int_\Phi f_{it}(a)\frac{\partial}{\partial t}\left(e^{-\rho t}\zeta_{it}(a)\right) dadt \\
&= \sum_{i=1}^{2}\int_\Phi f_{i0}(a)\,\zeta_{i0}(a)\, da - \lim_{T\to\infty}\sum_{i=1}^{2}\int_\Phi e^{-\rho T} f_{iT}(a)\,\zeta_{iT}(a)\, da \\
&\quad +\sum_{i=1}^{2}\int_0^\infty \int_\Phi e^{-\rho t} f_{it}\left(\frac{\partial \zeta_i}{\partial t} - \rho\zeta_{it}\right) dadt \\
&= \langle \zeta_0, f_0\rangle_\Phi - \lim_{T\to\infty}\left\langle e^{-\rho T}\zeta_T, f_T\right\rangle_\Phi \\
&\quad +\int_0^\infty e^{-\rho t}\left\langle \frac{\partial \zeta}{\partial t} - \rho\zeta_t, f_t\right\rangle_\Phi dt,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \left\langle e^{-\rho t}\theta_t, \frac{\partial v}{\partial t} - \rho v_t\right\rangle dt &= \sum_{i=1}^{2}\int_0^\infty \int_\Phi e^{-\rho t}\theta_{it}(a)\left(\frac{\partial v_i}{\partial t} - \rho v_{it}(a)\right) dadt \\
&= \sum_{i=1}^{2}\int_\Phi \theta_{it}(a)\, e_{it}^{-\rho t} v_{it}(a)\big|_0^\infty da \\
&\quad -\sum_{i=1}^{2}\int_0^\infty \int_\Phi v_i\left[\frac{\partial}{\partial t}\left(e^{-\rho t}\theta_{it}(a)\right) + \rho\theta_{it}(a)\right] dadt \\
&= \lim_{T\to\infty}\sum_{i=1}^{2}\int_\Phi e^{-\rho T} v_{iT}(a)\,\theta_{iT}(a)\, da - \sum_{i=1}^{2}\int_\Phi v_{i0}(a)\,\theta_{i0}(a)\, da \\
&\quad -\sum_{i=1}^{2}\int_0^\infty \int_\Phi e^{-\rho t} v_{it\}}(a)\frac{\partial \theta_i}{\partial t} dadt \\
&= \lim_{T\to\infty}\left\langle e^{-\rho T}\theta_T, v_T\right\rangle_\Phi - \langle \theta_0, v_0\rangle_\Phi \\
&\quad +\int_0^\infty e^{-\rho t}\left\langle -\frac{\partial \theta}{\partial t}, v_t\right\rangle_\Phi dt,
\end{aligned}$$

Step 3: Necessary conditions. In order to find the maximum, we need to take the Gâteaux derivatives with respect to the controls f, π, Q, v and c.

The Gâteaux derivative with respect to f is

$$\begin{aligned}
\frac{d}{d\alpha}\mathcal{L}\left[\pi, Q, f+\alpha h, v, c\right]|_{\alpha=0} &= \langle \zeta_0, h_0\rangle_\Phi - \lim_{T\to\infty}\left\langle e^{-\rho T}\zeta_T, h_T\right\rangle_\Phi \\
&\quad -\int_0^\infty e^{-\rho t}\left\langle u(c_t) - x(\pi_t) + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta_t - \rho\zeta_t, h_t\right\rangle_\Phi dt,
\end{aligned}$$

which should equal zero for any perturbation $h \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ such that $h_0(\cdot) = 0$, as the initial value of $f_0(\cdot)$. We obtain

$$\rho\zeta_t = u(c_t) - x(\pi_t) + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta_t, \quad \text{for } t \geq 0 \tag{47}$$

Given that $\zeta \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$, we obtain the transversality condition $\lim_{T\to\infty} e^{-\rho T}\zeta_T(a) = 0$. Equation (47) is the same as the individual HJB equation (5). The boundary conditions are also the same (state constraints on the domain Φ) and therefore their solutions should coincide: $\zeta_t(\cdot) = v_t(\cdot)$.

In the case of c, the Gâteaux derivative is

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{L}[\pi, Q, f, v, c + \alpha h]|_{\alpha=0} &= \int_0^\infty e^{-\rho t}\left\langle \left(u'(c_t) - \frac{1}{Q_t}\frac{\partial \zeta}{\partial a}\right)h_t, f_t\right\rangle_\Phi dt \\ &+ \int_0^\infty e^{-\rho t}\left(\left\langle \theta_t, \left(u'(c_t) - \frac{1}{Q_t}\frac{\partial v}{\partial a}\right)h_t\right\rangle_\Phi + \langle \eta_t, u''(c_t)h_t\rangle_\Phi\right)dt, \end{aligned}$$

where $\frac{\partial}{\partial a}(\mathcal{A}\zeta_t) = -\frac{1}{Q_t}\frac{\partial \zeta}{\partial a}$. The Gâteaux derivative should be zero for any function $h_t \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$. Due to the first order conditions (7) and to the fact that $\zeta(\cdot) = v(\cdot)$ this expression reduces to

$$\int_0^\infty e^{-\rho t}\langle \eta_t, u''(c_t)h_t\rangle_\Phi dt = 0.$$

As $u(\cdot)$ is strictly concave, $u''(\cdot) < 0$ and hence $\eta = 0$ for all t, that is, the first order condition (7) is not binding as its associated Lagrange multiplier is zero.

In the case of v, the Gâteaux derivative is

$$\begin{aligned} \frac{d}{d\alpha}\mathcal{L}[\pi, Q, f, v + \alpha h, c]|_{\alpha=0} &= \int_0^\infty e^{-\rho t}\left(\left\langle \mathcal{A}^*\theta_t - \frac{\partial \theta}{\partial t}, h_t\right\rangle_\Phi\right)dt \\ &+ \lim_{T\to\infty}\langle e^{-\rho T}\theta_T, h_T\rangle_\Phi - \langle \theta_0, h_0\rangle_\Phi \\ &+ \int_0^\infty e^{-\rho t}\sum_{i=1}^{2} h_{it}(a)\, s_{it}(a)\, \theta_{it}(a)|_{-\infty}^{\infty}\, dt, \end{aligned}$$

where we have already taken into account the fact that $\eta = 0$. Given that $\theta \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$, we obtain the transversality condition $\lim_{T\to\infty} e^{-\rho T}\theta_T = 0$. As the Gâteaux derivative should be zero at the maximum for any suitable h, we obtain a Kolmogorov forward equation in θ

$$\frac{\partial \theta}{\partial t} = \mathcal{A}^*\theta_t, \quad \text{for } a > \phi, \ t > 0, \tag{48}$$

with boundary conditions

$$\begin{aligned} \lim_{a\to-\infty} s_{it}(a)\,\theta_{it}(a) &= \lim_{a\to\infty} s_{it}(a)\,\theta_{it}(a) = 0, \ i = 1, 2, \\ \theta_{i0}(\cdot) &= 0, \ i = 1, 2. \end{aligned}$$

This is a KF equation with an initial density of $\theta_0(\cdot) = 0$.[21] Therefore, the distribution at any point in time should be zero $\theta = 0$. Both the Lagrange multiplier of the households' HJB equation θ and that of the first-order condition η are zero, reflecting the fact that the HJB equation is slack, that is, that the monetary authority would choose the same consumption as the households. This would not be the case in a closed economy, in which some externalities may arise, as discussed, for instance, in Nuño and Moll (2018).

The Gâteaux derivative in the case of π is

$$\frac{d}{d\alpha}\mathcal{L}[\pi + \alpha h, Q, f, v, c]|_{\alpha=0} = \int_0^\infty e^{-\rho t}\left\langle -x'(\pi_t) - a\left(\frac{\partial v}{\partial a}\right) + \mu_t Q_t, f_t\right\rangle_\Phi h_t dt,$$

where we have already taken into account the fact that $\theta(\cdot) = \eta(\cdot) = 0.$ and $\zeta(\cdot) = v(\cdot)$. As the Gâteaux derivative should be zero for any $h_t \in L^2[0,\infty)$, the optimality condition then results in

$$\mu_t Q_t = \sum_{i=1}^{2}\int_\Phi \left(a\frac{\partial v_{it}}{\partial a} + x'(\pi_t)\right) f_{it}(a)\, da, \tag{49}$$

where we have applied the normalization condition: $\langle 1, f\rangle_\Phi = 1$.

In the case of Q the Gâteaux derivative is

$$\frac{d}{d\alpha}\mathcal{L}[\pi, Q + \alpha h, \cdot]|_{\alpha=0} = \int_0^\infty e^{-\rho t}\left\langle -\frac{\delta h_t}{Q_t^2}a\frac{\partial v}{\partial a} - \frac{(y - c_t)\,h}{Q_t^2}\frac{\partial v}{\partial a} + \mu_t\left[h(\bar{r} + \pi_t + \delta) - \dot{h}_t\right], f_t\right\rangle_\Phi dt,$$

where we have also taken into account the fact that $\zeta = v$ and $\theta = \eta = 0$. Integrating by parts

$$\begin{aligned}
\int_0^\infty e^{-\rho t}\left\langle -\mu_t \dot{h}_t, f_t\right\rangle_\Phi dt &= -\int_0^\infty e^{-\rho t}\mu_t \dot{h}_t \langle 1, f_t\rangle_\Phi dt = -\int_0^\infty e^{-\rho t}\mu_t \dot{h}_t dt \\
&= -\left. e^{-\rho t}\mu_t h_t\right|_0^\infty + \int_0^\infty e^{-\rho t}(\dot{\mu}_t - \rho\mu_t)\, h_t dt \\
&= \mu_0 h_0 + \int_0^\infty e^{-\rho t}\langle(\dot{\mu}t - \rho\mu_t)\, h_t, f_t\rangle_\Phi dt.
\end{aligned}$$

Therefore, the optimality condition in this case is

$$\int_0^\infty e^{-\rho t}\left\langle -\frac{\delta}{Q_t^2}a\frac{\partial v}{\partial a} - \frac{(y - c_t)}{Q_t^2}\frac{\partial v}{\partial a} + \mu_t(\bar{r} + \pi_t + \delta - \rho) + \dot{\mu}_t, f_t\right\rangle_\Phi h_t dt + \mu_0 h_0 = 0.$$

The Gâteaux derivative should be zero for any $h_t \in L^2[0,\infty)$. Thus we obtain

$$\begin{aligned}
\left\langle -\frac{\delta}{Q_t^2}a\frac{\partial v}{\partial a} - \frac{(y - c_t)}{Q_t^2}\frac{\partial v}{\partial a}, f_t\right\rangle_\Phi + \mu_t(\bar{r} + \pi_t + \delta - \rho) + \dot{\mu}_t &= 0,\ t > 0, \\
\mu_0 &= 0.
\end{aligned}$$

[21] Notice that if we denote $g_t \equiv \left\langle \mathcal{A}^*\theta_t - \frac{\partial\theta}{\partial t}, 1\right\rangle_\Phi$ and $G_t \equiv \int_t^\infty e^{-\rho s}g_s ds$ then the fact that $\mathcal{A}^*\theta_t - \frac{\partial\theta}{\partial t} = 0$, for $a \geq \phi$, $t > 0$, implies that $G_t = 0$, for $t > 0$. As G_t is differentiable, then it is continuous and hence $G_0 = 0$ so that the condition $G_0 + \langle\theta_0, h_0\rangle_\Phi = 0$ for any $h_0 \in L^2(\Phi)$ requires $\theta_0 = 0$. A similar argument can be employed to analyzed the boundary conditions in Φ.

or equivalently,

$$\begin{aligned}\frac{d\mu}{dt} &= (\rho - \bar{r} - \pi_t - \delta)\,\mu_t + \sum_{i=1}^{2}\int_{\Phi}\frac{\partial v_{it\}}}{\partial a}\frac{\delta a + (y - c_t)}{Q_t^2} f_{it\}}(t,a)\,da,\ t > 0, \qquad (50)\\ \mu_0 &= 0.\end{aligned}$$

Finally, using the household's first order condition $\frac{\partial v_{it}}{\partial a} = Q_t u'(c_{it})$ to substitute for $\frac{\partial v_{it}}{\partial a}$ in equations (49) and (50) yields the expressions in the main text.

B.3. Proof of Proposition 2: Inflationary bias

As the value function is strictly concave in a by Lemma 2, it satisfies

$$\frac{\partial v_{it}(\tilde{a})}{\partial a} < \frac{\partial v_{it}(0)}{\partial a} < \frac{\partial v_{it}(\hat{a})}{\partial a},\ \text{for all } \tilde{a} \in (0,\infty),\ \hat{a} \in (\phi, 0),\ t \geq 0,\ i = 1,2. \qquad (51)$$

In addition, Assumption 1 (the country is a always a net debtor: $\bar{a}_t \leq 0$) implies

$$\sum_{i=1}^{2}\int_{0}^{\infty}(a)\,f_{it}(a)da \leq \sum_{i=1}^{2}\int_{-\infty}^{0}(-a)\,f_{it}(a)da,\ \forall t \geq 0. \qquad (52)$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{2}\int_{0}^{\infty} a f_{it}(a)\frac{\partial v_{it}(a)}{\partial a}da &< \frac{\partial v_{it}(0)}{\partial a}\sum_{i=1}^{2}\int_{0}^{\infty} a f_{it}(a)\,da \leq \frac{\partial v_{it}(0)}{\partial a}\sum_{i=1}^{2}\int_{-\infty}^{0}(-a)\,f_{it}(a)da\\ &< \sum_{i=1}^{2}\int_{-\infty}^{0}(-a)\,f_{it}(a)\frac{\partial v_{it}(a)}{\partial a}da, \qquad (53)\end{aligned}$$

where we have applied (51) in the first and last inequalities and (52) in the intermediate one.[22] The optimal inflation (23) at time-0 ($\mu_0 = 0$) satisfies

$$\sum_{i=1}^{2}\int_{\phi}^{\infty} a f_{i0}\frac{\partial v_{i0}}{\partial a}da + x'(\pi_0) = 0.$$

Combining this expression with (53) we obtain

$$x'(\pi_0) = \sum_{i=1}^{2}\int_{\phi}^{\infty}(-a)\,Q_0 u'(c_0)\,f_{i0}da = \sum_{i=1}^{2}\int_{\phi}^{\infty}(-a)\,\frac{\partial v_{i0}}{\partial a}f_{i0}da > 0.$$

Finally, taking into account the fact that $x'(\pi_0) = \pi_0/\psi > 0$ only for $\pi_0 > 0$, we have that $\pi_0 > 0$.

[22] We have also used the fact that $af(a) > 0$ for all $a > 0$ and $(-a)f(a) > 0$ for all $a < 0$, as well as $\partial v_{it}(0)/\partial a > 0$ (which follows from the household first order condition and the assumption that $u' > 0$).

B.4. Proof of Proposition 3: Optimal long-run inflation in the limit as $\bar{r} \to \rho$

In the steady state, equations (24) and (23) in the main text become

$$(\rho - \bar{r} - \pi - \delta)\,\mu + \frac{1}{Q^2}\sum_{i=1}^{2}\int \frac{\partial v_i}{\partial a}\left[\delta a + (y_i - c_i)\right] f_i(a)\,da = 0,$$

$$\mu Q = x'(\pi) + \sum_{i=1}^{2}\int a\frac{\partial v_i}{\partial a} f_i(a)\,da,$$

respectively. Notice that we have replaced $Qu'(c_i)$ by $\frac{\partial v_i}{\partial a}$. Consider now the limiting case $\rho \to \bar{r}$, and guess that $\pi \to 0$. The above two equations then become

$$\begin{aligned}\mu Q &= \frac{1}{\delta Q}\sum_{i=1}^{2}\int \frac{\partial v_i}{\partial a}\left[\delta a + (y_i - c_i)\right] f_i(a)\,da,\\ \mu Q &= \sum_{i=1}^{2}\int a\frac{\partial v_i}{\partial a} f_i(a)\,da,\end{aligned}$$

as $x'(0) = 0$. Combining both equations, and using the fact that in the zero-inflation steady state the bond price equals $Q = \frac{\delta}{\delta + \bar{r}}$, we obtain

$$\sum_{i=1}^{2}\int \frac{\partial v_i}{\partial a}\left(\bar{r}a + \frac{y_i - c_i}{Q}\right) f_i(a)\,da = 0. \tag{54}$$

In the zero inflation steady state, the value function v satisfies the HJB equation

$$\rho v_i(a) = u(c_i(a)) + \left(\bar{r}a + \frac{y_i - c_i(a)}{Q}\right)\frac{\partial v_i}{\partial a} + \lambda_i\left[v_j(a) - v_i(a)\right], \quad i = 1, 2, \; j \neq i, \tag{55}$$

where we have used $x(0) = 0$. We also have the first-order condition

$$u'(c_i(a)) = Q\frac{\partial v_i}{\partial a} \Rightarrow c_i(a) = u'^{-1}\left(Q\frac{\partial v_i}{\partial a}\right).$$

We guess and verify a solution of the form $v_i(a) = \kappa_i a + \vartheta_i$, so that $u'(c_i) = Q\kappa_i$. Using our guess in (55), and grouping terms that depend on a and those that do not, we have that

$$\rho\kappa_i = \bar{r}\kappa_i + \lambda_i(\kappa_j - \kappa_i), \tag{56}$$

$$\rho\vartheta_i = u\left(u'^{-1}(Q\kappa_i)\right) + \frac{y_i - u'^{-1}(Q\kappa_i)}{Q}\kappa_i + \lambda_i(\vartheta_j - \vartheta_i), \tag{57}$$

for $i, j = 1, 2$ and $j \neq i$. In the limit as $\bar{r} \to \rho$, equation (56) results in $\kappa_j = \kappa_i \equiv \kappa$, so that consumption is the same in both states. The value of the slope κ can be computed from the boundary conditions.[23] We can solve for $\{\vartheta_i\}_{i=1,2}$ from equations (57),

$$\vartheta_i = \frac{1}{\rho} u\left(u'^{-1}(Q\kappa)\right) + \frac{y_i - u'^{-1}(Q\kappa)}{\rho Q}\kappa + \frac{\lambda_i (y_j - y_i)}{\rho(\lambda_i + \lambda_j + \rho) Q}\kappa,$$

for $i, j = 1, 2$ and $j \neq i$. Substituting $\frac{\partial v_i}{\partial a} = \kappa$ in (54), we obtain

$$\sum_{i=1}^{2} \int_{\phi}^{\infty} \left(\bar{r}a + \frac{y_i - c_i}{Q}\right) f_i(a)\, da = 0. \tag{58}$$

Equation (58) is exactly the zero-inflation steady-state limit of equation (11) in the main text (once we use the definitions of $\bar{a}$, $\bar{y}$ and $\bar{c}$), and is therefore satisfied in equilibrium. We have thus verified our guess that $\pi \to 0$.

B.5. Proof of Proposition 4. Solution to the Ramsey problem with a borrowing limit in the market value of wealth

The problem of the central bank is to find inflation and bond prices paths $\{\pi_t, Q_t\}_{t\geq 0}$ that maximize the Lagragian in $L^2[0,\infty)$:

$$\begin{aligned}\mathcal{L}[\pi, Q] &= \sum_{i=1}^{2} \int_{\phi^m}^{\infty} \left\{ v_i^c(a^m) - \int_0^{\infty} e^{-\rho t} x(\pi_t)\, dt \right\} \left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)\right] da^m \\ &\quad + \int_0^{\infty} e^{-\rho t} \mu_t \left[Q_t(\bar{r} + \pi_t + \delta) - \delta - \dot{Q}_t\right] dt,\end{aligned}$$

where $\mu_t \in L^2[0,\infty)$ is the Lagrange multipliers associated to equation (9).

The first order condition with respect to inflation is obtained by computing the Gâteaux derivative with respect to a perturbation h_t :

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}[\pi + \alpha h, Q]\,|_{\alpha=0} &= \frac{d}{d\alpha} \sum_{i=1}^{2} \int_{\phi^m}^{\infty} \left\{ v_i^c(a^m) - \int_0^{\infty} e^{-\rho t} x(\pi_t + \alpha h_t)\, dt \right\} \left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)\right] da^m \Bigg|_{\alpha=0} \\ &\quad + \frac{d}{d\alpha} \int_0^{\infty} e^{-\rho t} \mu_t \left[Q_t(\bar{r} + \pi_t + \alpha h_t + \delta) - \delta - \dot{Q}_t\right] dt \Bigg|_{\alpha=0} \\ &= \sum_{i=1}^{2} \int_{\phi^m}^{\infty} \left\{ v_i^c(a^m) - \int_0^{\infty} e^{-\rho t} x'(\pi_t) h_t dt \right\} \left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)\right] da^m \\ &\quad + \int_0^{\infty} e^{-\rho t} \mu_t Q_t h_t dt\end{aligned}$$

[23] The condition that the drift should be positive at the borrowing constraint, $s_i(\phi) \geq 0$, $i = 1, 2$, implies that

$$s_1(\phi) = \bar{r}\phi + \frac{y_1 - u'^{-1}(Q\kappa)}{Q} = 0,$$

and

$$\kappa = \frac{u'(\bar{r}\phi Q + y_1)}{Q}.$$

In the case of state $i = 2$, this guarantees $s_2(\phi) > 0$.

As the Gâteaux derivative should be zero for any h_t, the optimality condition then results in

$$x'(\pi_t) = \mu_t Q_t.$$

where we have applied the normalization condition: $\sum_{i=1}^{2}\int_{\phi^m}^{\infty}\frac{1}{Q_0}f_{i0}\left(\frac{a^m}{Q_0}\right)da^m = 1.$

Similarly, the first order condition with respect to bond prices is given by

$$\begin{aligned}
\frac{d}{d\alpha}\mathcal{L}\left[\pi, Q+\alpha h\right]|_{\alpha=0} &= \frac{d}{d\alpha}\sum_{i=1}^{2}\int_{\phi^m}^{\infty}\left\{v_i^c(a^m) - \int_0^{\infty}e^{-\rho t}x(\pi_t)\,dt\right\}\left[\frac{f_{i0}\left(\frac{a^m}{Q_0+\alpha h_0}\right)}{Q_0+\alpha h_0}\right]da^m\Bigg|_{\alpha=0} + \\
&\quad + \frac{d}{d\alpha}\int_0^{\infty}e^{-\rho t}\mu_t\left[(Q_t+\alpha h_t)(\bar{r}+\pi_t+\delta) - \delta - \left(\dot{Q}_t+\alpha\dot{h}_t\right)\right]dt\Bigg|_{\alpha=0} \\
&= \sum_{i=1}^{2}\int_{\phi^m}^{\infty}\left\{v_i^c(a^m) - \int_0^{\infty}e^{-\rho t}x'(\pi_t)\,h_t dt\right\}\frac{d}{dQ_0}\left[\frac{1}{Q_0}f_{i0}\left(\frac{a^m}{Q_0}\right)\right]h_0 da^m \\
&\quad + \int_0^{\infty}e^{-\rho t}\mu_t\left[(\bar{r}+\pi_t+\delta)\,h_t - \dot{h}_t\right]dt.
\end{aligned}$$

Integrating by parts

$$\begin{aligned}
\int_0^{\infty}e^{-\rho t}\mu_t\left(-\dot{h}_t\right)dt. &= -\,e^{-\rho t}\mu_t h_t\big|_0^{\infty} + \int_0^{\infty}e^{-\rho t}(\dot{\mu}_t - \rho\mu_t)\,h_t dt \\
&= \mu_0 h_0 - \lim_{t\to\infty}e^{-\rho t}\mu_t h_t + \int_0^{\infty}e^{-\rho t}(\dot{\mu}_t - \rho\mu_t)\,h_t dt.
\end{aligned}$$

Therefore, the optimality condition in this case is

$$\begin{aligned}
\dot{\mu}_t &= \mu_t(\rho - \bar{r} - \pi_t - \delta), \\
\mu_0 &= -\sum_{i=1}^{2}\int_{\phi^m}^{\infty}\left\{v_i^c(a^m) - \int_0^{\infty}e^{-\rho t}x'(\pi_t)\,h_t dt\right\}\frac{d}{dQ_0}\left[\frac{1}{Q_0}f_{i0}\left(\frac{a^m}{Q_0}\right)\right]h_0 da^m.
\end{aligned}$$

B.6. Lemma 1

$$\mathbb{E}_{f_0(a,y)}\left[v_0(a,y)\right] = \int_0^{\infty}e^{-\rho t}\mathbb{E}_{f_t(a,y)}\left[u(c_t(a,y)) - x(\pi_t)\right]dt.$$

Proof. We consider the case in which borrowing limit is defined in the face value of debt, as in equation (4).Given the welfare criterion defined in equation (16), we have

$$\begin{aligned} W_0 &= \int_\phi^\infty \sum_{i=1}^2 v_0(a,y_i) f_0(a,y_i) da \\ &= \int_\phi^\infty \sum_{i=1}^2 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left[u\left(c_t\right) - x\left(\pi_t\right)\right] dt | a_0 = a,\ y_0 = y_i \right] f_{i0}(a) da \\ &= \int_\phi^\infty \sum_{i=1}^2 \left[\sum_{j=1}^2 \int_\phi^\infty \int_0^\infty e^{-\rho t} \left[u(c_{jt}\left(\tilde{a}\right)) - x\left(\pi_t\right)\right] f_t(\tilde{a}, \tilde{y}_j; a, y_i) dt d\tilde{a} \right] f_{i0}(a) da \\ &= \int_0^\infty \sum_{j=1}^2 e^{-\rho t} \int_\phi^\infty \left[u(c_{jt}\left(\tilde{a}\right)) - x\left(\pi_t\right)\right] \left[\sum_{i=1}^2 \int_\phi^\infty f_t(\tilde{a}, \tilde{y}_j; a, y_i) f_{i0}(a) da \right] d\tilde{a} dt \\ &= \int_0^\infty e^{-\rho t} \sum_{j=1}^2 \int_\phi^\infty \left[u(c_{jt}\left(\tilde{a}\right)) - x\left(\pi_t\right)\right] f_t(\tilde{a}, \tilde{y}_j) d\tilde{a} dt, \end{aligned}$$

where $f_t(\tilde{a}, \tilde{y}_j; a, y_i)$ is the transition probability from $a_0 = a$, $y_0 = y_i$ to $a_t = \tilde{a}$, $y_t = \tilde{y}_j$ and in the last equality we have used the Chapman–Kolmogorov equation,

$$f_t(\tilde{a}, \tilde{y}_j) = \sum_{i=1}^2 \int_\phi^\infty f_t(\tilde{a}, \tilde{y}_j; a, y_i) f_0(a, y_i) da.$$

B.7. Lemma 2

$$\partial u' / \partial a < 0$$

Proof. In order to prove the concavity of the value function we express the model in discrete time for an arbitrarily small Δt. The Bellman equation of a household is

$$\begin{aligned} v_t^{\Delta t}\left(a, y\right) &= \max_{a' \in \Gamma(a,y)} \left[u\left(\frac{Q_t}{\Delta t} \left[\left(1 + \left(\frac{\delta}{Q_t} - \delta - \pi_t \right) \Delta t \right) a + \frac{y \Delta t}{Q_t} - a' \right] \right) - x\left(\pi_t\right) \right] \Delta t \\ &\quad + e^{-\rho \Delta t} \sum_{i=1}^2 v_{t+\Delta t}^{\Delta t}\left(a', y_i\right) \mathbb{P}\left(y' = y_i | y\right), \end{aligned}$$

where $\Gamma\left(a, y\right) = \left[\phi, \left(1 + \left(\frac{\delta}{Q_t} - \delta - \pi_t \right) \Delta t \right) a + \frac{y \Delta t}{Q_t} \right]$, and $\mathbb{P}\left(y' = y_i | y\right)$ are the transition probabilities of a two-state Markov chain. The Markov transition probabilities are given by $\lambda_1 \Delta t$ and $\lambda_2 \Delta t$.

We verify that this problem satisfies the conditions of Theorem 9.8 of Stokey and Lucas (1989): (i) Φ is a convex subset of $\mathbb{R}$, (ii) the Markov chain has a finite number of values; (iii) the correspondence $\Gamma\left(a, y\right)$ is nonempty, compact-valued and continuous; (iv) the function u is bounded, concave and continuous and $e^{-\rho \Delta t} \in (0, 1)$; and (v) the set $A^y = \{(a, a') \text{ such that } a' \in \Gamma\left(a, y\right)\}$ is convex. We conclude that $v_t^{\Delta t}\left(a, y\right)$ is strictly concave for any $\Delta t > 0$. Finally, for any a_1, $a_2 \in \Phi$

$$\begin{aligned} v_t^{\Delta t}(\omega a_1 + (1-\omega) a_2, y) &> \omega v_t^{\Delta t}(a_1, y) + (1-\omega) v_t^{\Delta t}(a_2, y), \\ \lim_{\Delta t \to 0} v_t^{\Delta t}(\omega a_1 + (1-\omega) a_2, y) &> \lim_{\Delta t \to 0} \left[\omega v_t^{\Delta t}(a_1, y) + (1-\omega) v_t^{\Delta t}(a_2, y)\right], \\ v_t(\omega a_1 + (1-\omega) a_2, y) &> \omega v_t(a_1, y) + (1-\omega) v_t(a_2, y), \end{aligned}$$

so that $v(t, a, y)$ is strictly concave.

B.8. Lemma 3

$$\begin{aligned} \pi_0 &= -\frac{Q_0}{\psi} \sum_{i=1}^{2} \int_{\phi Q_0}^{\infty} v_{i0}\left(\frac{a^m}{Q_0}\right) \frac{d}{dQ_0}\left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)\right] da^m = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-Q_0 a) u'(c_{it}(a)) f_{i0}(a)\, da \\ &= \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-Q_0 a) u'(c_{it}(a)) f_{i0}(a)\, da \end{aligned}$$

Proof. Here we show how equation (25),

$$\pi_0 = -\frac{Q_0}{\psi} \sum_{i=1}^{2} \int_{\phi Q_0}^{\infty} v_{i0}\left(\frac{a^m}{Q_0}\right) \frac{d}{dQ_0}\left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)\right] da^m$$

coincides with equation (23) when $t = 0$. First, we compute the derivative $\frac{d}{dQ_0}\left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)\right]$:

$$\pi_0 = \frac{1}{\psi Q_0} \sum_{i=1}^{2} \int_{\phi Q_0}^{\infty} v_{i0}\left(\frac{a^m}{Q_0}\right) \left[f_{i0}\left(\frac{a^m}{Q_0}\right) + \frac{a^m}{Q_0} f'_{i0}\left(\frac{a^m}{Q_0}\right)\right] da^m,$$

and we change variables to $a = a^m / Q_0$:

$$\begin{aligned} \pi_0 &= \tfrac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} v_{i0}(a) \left[f_{i0}(a) + a f'_{i0}(a)\right] da \\ &= \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} v_{i0}(a) \frac{d}{da}\left[a f_{i0}(a)\right] da. \end{aligned}$$

Integrating by parts and taking into account the first order condition for consumption (7), we get

$$\pi_0 = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-Q_0 a) u'(c_{it}(a)) f_{i0}(a)\, da,$$

which is exactly equation (23) in the main text evaluated at $t = 0$ (given that $\mu_0 = 0$). Finally, if we make again a change of variable $a^m = Q_0 a$:

$$\pi_0 = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi Q_0}^{\infty} (-a^m) u'(c_{it}(a^m)) f_{i0}^m(a^m)\, da^m.$$

B.9. Lemma 4

$$\lim_{\delta\to\infty} \pi_t = 0,\ t > 0.$$

Proof. The price of bonds in the limit is

$$\lim_{\delta\to\infty} Q_t = \lim_{\delta\to\infty} \int_t^\infty \delta e^{-(\bar{r}+\delta)(s-t)-\int_t^s \pi_u du} ds = \lim_{\delta\to\infty} \int_t^\infty \delta e^{-\delta(s-t)} ds = 1,$$

that is, the price of an instantaneous bond is always unity. The optimality conditions of the central bank in the limit are

$$\lim_{\delta\to\infty} \pi_t = \lim_{\delta\to\infty} \frac{1}{\psi} \mathbb{E}_{f_t(a,y)} \left[-Q_t a u' \left(c_t \left(a,y\right)\right)\right] + \lim_{\delta\to\infty} \frac{1}{\psi} \mu_t Q_t,$$

and, for $t > 0$

$$\begin{aligned}
\lim_{\delta\to\infty} \mu_t &= \lim_{\delta\to\infty} \int_0^t e^{-\int_s^t (\bar{r}+\pi_z+\delta-\rho)dz} \frac{1}{Q_s} \mathbb{E}_{f_s(a,y)} \left[\left(\delta a + y - c_s\left(a,y\right)\right) u'\left(c_s\left(a,y\right)\right)\right] ds \\
&= \lim_{\delta\to\infty} \int_0^t e^{-\int_s^t \delta dz} \frac{1}{Q_s} \mathbb{E}_{f_s(a,y)} \left[\delta a u'\left(c_s\left(a,y\right)\right)\right] ds \\
&= \lim_{\delta\to\infty} \int_0^t e^{-\delta(t-s)} \delta \mathbb{E}_{f_s(a,y)} \left[a u'\left(c_s\left(a,y\right)\right)\right] ds \\
&= \mathbb{E}_{f_t(a,y)} \left[a u'\left(c_s\left(a,y\right)\right)\right].
\end{aligned}$$

where in the last step we have applied the fact that the limit of $e^{-\delta(t-s)}\delta$ is a Dirac delta function centered at t. Combining both expressions, we get

$$\lim_{\delta\to\infty} \pi_t = \frac{1}{\psi} \mathbb{E}_{f_t(a,y)} \left[Q_t a u'\left(c_t\left(a,y\right)\right)\right] - \frac{1}{\psi} \mathbb{E}_{f_t(a,y)} \left[Q_t a u'\left(c_t\left(a,y\right)\right)\right] = 0.$$

B.10. Lemma 5

Consumption in the case of a representative agent is $c_t = \rho(a_0^m + y/\bar{r})e^{-(\rho-\bar{r})t}$, and optimal inflation

$$\psi\pi_t = \underbrace{\frac{Q_t}{Q_0} e^{\left[-\int_0^t (\bar{r}+\delta+\pi_s-\rho)ds\right]}}_{\text{Dynamic evolution}} \underbrace{\left(-Q_0 a_0\right) u'\left(c_0\right)}_{\text{Time-0 optimal inflation}(\psi\pi_0)}.$$

.

Proof. The problem of the representative household, expressed in market value of wealth, is

$$\max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-\rho t} \log(c_t) dt$$

subject to

$$\dot{a}_t^m = \bar{r} a_t^m + y - c_t.$$

The HJB equation is

$$\rho V\left(a^{m}\right)=\max_{c}\left\{\log(c)-\frac{\psi}{2}\pi_{t}^{2}+\left(\bar{r}a^{m}+y-c\right)V'\left(a^{m}\right)\right\},$$

where we guess and verify $V\left(a^{m}\right)=\frac{1}{\rho}\log(a^{m}+y/\bar{r})+\kappa_{t}$, where κ_t only depends on time. Hence $c_t=\rho(a_t^m+y/\bar{r})$. Taking derivatives at both sides, we get

$$\dot{c}_t=\rho\dot{a}_t^m=\rho\left(\bar{r}a_t^m+y-c_t\right)=\left(\bar{r}-\rho\right)c_t,$$

which can be solved as

$$c_t=c_0e^{-(\rho-\bar{r})t}=\rho(a_0^m+y/\bar{r})e^{-(\rho-\bar{r})t}.$$

(Face-value) assets evolve according to

$$a_t=\frac{1}{Q_t}\left[\frac{1}{\rho}c_t-y/\bar{r}\right]=(a_0+\frac{y}{\bar{r}Q_t})e^{-(\rho-\bar{r})t}-\frac{y}{\bar{r}Q_t}.$$

The problem of the central bank is

$$\max_{\{\pi_t\}_{t\geq 0}}\int_0^{\infty}e^{-\rho t}\left\{\log\left[\rho(Q_0a_0+y/\bar{r})e^{-(\rho-\bar{r})t}\right]-\frac{\psi}{2}\pi_t^2\right\}dt$$

subject to

$$\dot{Q}_t=Q_t\left(\bar{r}+\pi_t+\delta\right)-\delta.$$

The first order condition with respect to π is

$$\begin{aligned}0 &= \frac{d}{d\alpha}\int_0^{\infty}e^{-\rho t}\left\{\log\left[\rho(Q_0a_0+y/\bar{r})e^{-(\rho-\bar{r})t}\right]-\frac{\psi}{2}\left(\pi_t+\alpha h_t\right)^2\right\}dt|_{\alpha=0}\\ &+ \frac{d}{d\alpha}\int_0^{\infty}e^{-\rho t}\mu_t\left[Q_t\left(\bar{r}+\pi_t+\alpha h_t+\delta\right)-\delta-\dot{Q}_t\right]dt\Big|_{\alpha=0},\end{aligned}$$

which yields

$$\begin{aligned}0 &= \int_0^{\infty}e^{-\rho t}\left\{-\psi\pi_t+\mu_tQ_t\right\}h_tdt\\ \psi\pi_t &= \mu_tQ_t.\end{aligned}$$

The objective function can be simplified as $\int_0^{\infty}e^{-\rho t}\left\{\log\left[\rho(Q_0+\alpha h_0a_0+y/\bar{r})e^{-(\rho-\bar{r})t}\right]-\frac{\psi}{2}\pi_t^2\right\}dt$, which equals

$$\begin{aligned}&\int_0^{\infty}e^{-\rho t}\left\{\log\left[\rho(Q_0+\alpha h_0a_0+y/\bar{r})\right]-\left(\rho-\bar{r}\right)t-\frac{\psi}{2}\pi_t^2\right\}dt\\ =&\ \frac{1}{\rho}\log\left[\rho(Q_0+\alpha h_0a_0+y/\bar{r})\right]-\int_0^{\infty}e^{-\rho t}\left\{\left(\rho-\bar{r}\right)t+\frac{\psi}{2}\pi_t^2\right\}dt,\end{aligned}$$

and the first order condition with respect to Q is

$$
\begin{aligned}
0 &= \frac{d}{d\alpha}\left\{\frac{1}{\rho}\log\left[\rho(Q_0+\alpha h_0 a_0+y/\bar{r})\right]\right\}dt|_{\alpha=0} \\
&+ \frac{d}{d\alpha}\int_0^\infty e^{-\rho t}\mu_t\left[\left(Q_t+\alpha h_t\right)\left(\bar{r}+\pi_t+\delta\right)-\delta-\left(\dot{Q}_t+\alpha\dot{h}_t\right)\right]dt\Big|_{\alpha=0},
\end{aligned}
$$

thus,

$$
0 = u'(c_0)\,h_0 a_0+\int_0^\infty e^{-\rho t}\mu_t\left[h_t\left(\bar{r}+\pi_t+\delta\right)-\left(\dot{h}_t\right)\right]dt
$$

Integrating by parts (as in Proposition 4):Integrating by parts

$$
\begin{aligned}
\int_0^\infty e^{-\rho t}\mu_t\left(-\dot{h}_t\right)dt. &= -\,e^{-\rho t}\mu_t h_t|_0^\infty+\int_0^\infty e^{-\rho t}\left(\dot{\mu}_t-\rho\mu_t\right)h_t dt \\
&= \mu_0 h_0-\lim_{t\to\infty}e^{-\rho t}\mu_t h_t+\int_0^\infty e^{-\rho t}\left(\dot{\mu}_t-\rho\mu_t\right)h_t dt.
\end{aligned}
$$

Hence

$$
\begin{aligned}
-u'(c_0)\,a_0 &= \mu_0, \\
\dot{\mu}_t &= \mu_t\left(\rho-\bar{r}-\pi_t-\delta\right).
\end{aligned}
$$

Optimal inflation is thus

$$
\psi\pi_t = \frac{Q_t}{Q_0}e^{\left[-\int_0^t(\bar{r}+\delta+\pi_s-\rho)ds\right]}\left(-Q_0 a_0\right)u'(c_0).
$$

C. Computational method

We describe here the numerical algorithm to compute the Ramsey allocation. First we describe how to compute the optimal steady state (Proposition 1). Then, we illustrate how to compute the optimal transition dynamics, given an initial income-wealth distribution $f_0(a,y)$. Finally we discuss how to modify the algorithm for the case of of the borrowing limit in the market value of wealth.

C.1. Optimal steady state

We describe the numerical algorithm used to jointly solve for the equilibrium objects in steady state. The algorithm proceeds in 3 steps. We describe each step in turn. We assume that there is an upper bound arbitrarily large $\varkappa$ such that $f(a,y)=0$ for all $a>\varkappa$. In steady state this can be proved in general following the same reasoning as in Proposition 2 of Achdou et al. (2017). Alternatively, we may assume that there is a maximum constraint in asset holding such that $a\leq\varkappa$, and that this constraint is so large that it does not affect to the results. In any case, let $[\phi,\varkappa]$ be the valid domain.

Solution to the Hamilton-Jacobi-Bellman equation Given π, the bond pricing equation (9) is trivially solved in this case:

$$Q = \frac{\delta}{\bar{r} + \pi + \delta}. \tag{59}$$

The HJB equation is solved using an *upwind finite difference* scheme similar to Achdou et al. (2017). It approximates the value function $v(a)$ on a finite grid with step $\Delta a : a \in \{a_1, ..., a_J\}$, where $a_j = a_{j-1} + \Delta a = a_1 + (j-1)\,\Delta a$ for $2 \leq j \leq J$. The bounds are $a_1 = \phi$ and $a_I = \varkappa$, such that $\Delta a = (\varkappa - \phi)\,/\,(J-1)$. We use the notation $v_{i,j} \equiv v_i(a_j)$, $i = 1, 2$, and similarly for the policy function $c_{i,j}$.

Notice first that the HJB equation involves first derivatives of the value function, $v_i'(a)$ and $v_i''(a)$. At each point of the grid, the first derivative can be approximated with a forward (F) or a backward (B) approximation,

$$v_i'(a_j) \approx \partial_F v_{i,j} \equiv \frac{v_{i,j+1} - v_{i,j}}{\Delta a}, \tag{60}$$

$$v_i'(a_j) \approx \partial_B v_{i,j} \equiv \frac{v_{i,j} - v_{i,j-1}}{\Delta a}. \tag{61}$$

In an upwind scheme, the choice of forward or backward derivative depends on the sign of the *drift function* for the state variable, given by

$$s_i\,(a) \equiv \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{(y_i - c_i\,(a))}{Q}, \tag{62}$$

for $\phi \leq a \leq 0$, where

$$c_i\,(a) = \left[\frac{v_i'(a)}{Q}\right]^{-1/\gamma}. \tag{63}$$

Let superscript n denote the iteration counter. The HJB equation is approximated by the following upwind scheme,

$$\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} + \rho v_{i,j}^{n+1} = \frac{(c_{i,j}^{n})^{1-\gamma} - 1}{1-\gamma} - \frac{\psi}{2}\pi^2 + \partial_F v_{i,j}^{n+1} s_{i,j,F}^{n} \mathbf{1}_{s_{i,j,F}^{n}>0} + \partial_B v_{i,j}^{n+1} s_{i,j,B}^{n} \mathbf{1}_{s_{i,j,B}^{n}<0} + \lambda_i \left(v_{-i,j}^{n+1} - v_{i,j}^{n+1}\right),$$

for $i = 1, 2$, $j = 1, ..., J$, where $\mathbf{1}\,(\cdot)$ is the indicator function and

$$s_{i,,jF}^{n} = \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{y_i - \left[\frac{Q}{\partial_F v_{i,j}^{n}}\right]^{1/\gamma}}{Q}, \tag{64}$$

$$s_{i,j,B}^{n} = \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{y_i - \left[\frac{Q}{\partial_B v_{i,j}^{n}}\right]^{1/\gamma}}{Q}. \tag{65}$$

We consider a CRRA utility function for generality, in the case of the main we restrict it to logarithmic preferences. Therefore, when the drift is positive ($s_{i,,jF}^{n} > 0$) we employ a forward approximation of the derivative, $\partial_F v_{i,j}^{n+1}$; when it is negative ($s_{i,j,B}^{n} < 0$) we employ a backward approximation, $\partial_B v_{i,j}^{n+1}$. The term $\frac{v_{i,j}^{n+1} - v_{i,j}^{n}}{\Delta} \to 0$ as $v_{i,j}^{n+1} \to v_{i,j}^{n}$. Moving all terms involving v^{n+1}

to the left hand side and the rest to the right hand side, we obtain

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta}+\rho v_{i,j}^{n+1}=\frac{(c_{i,j}^{n})^{1-\gamma}-1}{1-\gamma}-\frac{\psi}{2}\pi^{2}+v_{i,j-1}^{n+1}\alpha_{i,j}^{n}+v_{i,j}^{n+1}\beta_{i,j}^{n}+v_{i,j+1}^{n+1}\xi_{i,j}^{n}+\lambda_{i}v_{-i,j}^{n+1}, \quad (66)$$

where

$$\begin{aligned}\alpha_{i,j}^{n} &\equiv -\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a},\\ \beta_{i,j}^{n} &\equiv -\frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a}+\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a}-\lambda_{i},\\ \xi_{i,j}^{n} &\equiv \frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a},\end{aligned}$$

for $i=1,2$, $j=1,...,J$. Notice that the state constraints $\phi\leq a\leq\varkappa$ mean that $s_{i,1,B}^{n}=s_{i,J,F}^{n}=0$.

In equation (66), the optimal consumption is set to

$$c_{i,j}^{n}=\left(\frac{\partial v_{i,j}^{n}}{Q}\right)^{-1/\gamma}. \quad (67)$$

where

$$\partial v_{i,j}^{n}=\partial_{F}v_{i,j}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}+\partial_{B}v_{i,j}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}+\partial\bar{v}_{i,j}^{n}\mathbf{1}_{s_{i,F}^{n}\leq 0}\mathbf{1}_{s_{i,B}^{n}\geq 0}.$$

In the above expression, $\partial\bar{v}_{i,j}^{n}=Q(\bar{c}_{i,j}^{n})^{-\gamma}$ where $\bar{c}_{i,j}^{n}$ is the consumption level such that $s\left(a_{i}\right)\equiv s_{i}^{n}=0$:

$$\bar{c}_{i,j}^{n}=\left(\frac{\delta}{Q}-\delta-\pi\right)a_{j}Q+y_{i}.$$

Equation (66) is a system of $2\times J$ linear equations which can be written in matrix notation as:

$$\frac{1}{\Delta}\left(\mathbf{v}^{n+1}-\mathbf{v}^{n}\right)+\rho\mathbf{v}^{n+1}=\mathbf{u}^{n}+\mathbf{A}^{n}\mathbf{v}^{n+1}$$

where the matrix $\mathbf{A}^{n}$ and the vectors v^{n+1} and $\mathbf{u}^{n}$ are defined by

$$\mathbf{A}^{n}=-\begin{bmatrix}\beta_{1,1}^{n} & \xi_{1,1}^{n} & 0 & 0 & \cdots & 0 & \lambda_{1} & 0 & \cdots & 0\\ \alpha_{1,2}^{n} & \beta_{1,2}^{n} & \xi_{1,2}^{n} & 0 & \cdots & 0 & 0 & \lambda_{1} & \ddots & 0\\ 0 & \alpha_{1,3}^{n} & \beta_{1,3}^{n} & \xi_{1,3}^{n} & \cdots & 0 & 0 & 0 & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_{1,J-1}^{n} & \beta_{1,J-1}^{n} & \xi_{1,J-1}^{n} & 0 & \cdots & \lambda_{1} & 0\\ 0 & 0 & \cdots & 0 & \alpha_{1,J}^{n} & \beta_{1,J}^{n} & 0 & 0 & \cdots & \lambda_{1}\\ \lambda_{2} & 0 & \cdots & 0 & 0 & 0 & \beta_{2,1}^{n} & \xi_{2,1}^{n} & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & 0 & \lambda_{2} & 0 & \cdots & \alpha_{2,J}^{n} & \beta_{2,J}^{n}\end{bmatrix},\quad \mathbf{v}^{n+1}=\begin{bmatrix}v_{1,1}^{n+1}\\ v_{1,2}^{n+1}\\ v_{1,3}^{n+1}\\ \vdots\\ v_{1,J-1}^{n+1}\\ v_{1,J}^{n+1}\\ v_{2,1}^{n+1}\\ \vdots\\ v_{2,J}^{n+1}\end{bmatrix} \quad (68)$$

$$\mathbf{u}^n = \begin{bmatrix} \frac{(c_{1,1}^n)^{1-\gamma}-1}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \frac{(c_{1,2}^n)^{1-\gamma}-1}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \vdots \\ \frac{(c_{1,J}^n)^{1-\gamma}-1}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \frac{(c_{2,1}^n)^{1-\gamma}-1}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \vdots \\ \frac{(c_{2,J}^n)^{1-\gamma}-1}{1-\gamma} - \frac{\psi}{2}\pi^2 \end{bmatrix}.$$

The system in turn can be written as

$$\mathbf{B}^n \mathbf{v}^{n+1} = \mathbf{d}^n \tag{69}$$

where , $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right) \mathbf{I} - \mathbf{A}^n$ and $\mathbf{d}^n = \mathbf{u}^n + \frac{1}{\Delta}\mathbf{v}^n$.

The algorithm to solve the HJB equation runs as follows. Begin with an initial guess $\{v_{i,j}^0\}_{j=1}^J$, $i = 1, 2$. Set $n = 0$. Then:

1. Compute $\{\partial_F v_{i,j}^n, \partial_B v_{i,j}^n\}_{j=1}^J$, $i = 1, 2$ using (60)-(61).
2. Compute $\{c_{i,j}^n\}_{j=1}^J$, $i = 1, 2$ using (63) as well as $\{s_{i,j,F}^n, s_{i,j,B}^n\}_{j=1}^J$, $i = 1, 2$ using (64) and (65).
3. Find $\{v_{i,j}^{n+1}\}_{j=1}^J$, $i = 1, 2$ solving the linear system of equations (69).
4. If $\{v_{i,j}^{n+1}\}$ is close enough to $\{v_{i,j}^{n+1}\}$, stop. If not set $n := n + 1$ and proceed to 1.

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}^n$.

Solution to the Kolmogorov Forward equation The stationary distribution of debt-to-GDP ratio, $f(a, y)$, satisfies the Kolmogorov Forward equation:

$$0 = -\frac{d}{da}\left[s_i(a) f_i(a)\right] - \lambda_i f_i(a) + \lambda_{-i} f_{-i}(a), \; i = 1, 2. \tag{70}$$

$$1 = \sum_{i=1}^{2} \int_{\phi}^{\infty} f_i(a) da. \tag{71}$$

We also solve this equation using an finite difference scheme. We use the notation $f_{i,j} \equiv f_i(a_j)$. The system can be now expressed as

$$\begin{aligned} 0 \;=\; & -\frac{f_{i,j} s_{i,j,F} \mathbf{1}_{s_{i,j,F}^n > 0} - f_{i,j-1} s_{i,j-1,F} \mathbf{1}_{s_{i,j-1,F}^n > 0}}{\Delta a} - \frac{f_{i,j+1} s_{i,j+1,B} \mathbf{1}_{s_{i,j+1,B}^n < 0} - f_{i,j} s_{i,,jB} \mathbf{1}_{s_{i,,jB}^n < 0}}{\Delta a} \\ & -\lambda_i f_{i,j} + \lambda_{-i} f_{-i,j}, \end{aligned}$$

or equivalently

$$f_{i,j-1}\xi_{i,j-1} + f_{i,j+1}\alpha_{i,j+1} + f_{i,j}\beta_{i,j} + \lambda_{-i} f_{-i,j} = 0, \tag{72}$$

then (72) is also a system of $2 \times J$ linear equations which can be written in matrix notation as:

$$\mathbf{A^T f} = \mathbf{0}, \tag{73}$$

where $\mathbf{A^T}$ is the transpose of $\mathbf{A} = \lim_{n\to\infty} \mathbf{A}^n$. Notice that $\mathbf{A}^n$ is the approximation to the operator $\mathcal{A}$ and $\mathbf{A^T}$ is the approximation of the adjoint operator $\mathcal{A}^*$. In order to impose the normalization constraint (71) we replace one of the entries of the zero vector in equation (73) by a positive constant.[24] We solve the system (73) and obtain a solution $\hat{\mathbf{f}}$. Then we renormalize as

$$f_{i,j} = \frac{\hat{f}_{i,j}}{\sum_{j=1}^{J}\left(\hat{f}_{1,j} + \hat{f}_{2,j}\right)\Delta a}.$$

Complete algorithm In order to find the value of the inflation and of the costate that satisfy conditions (21) and (23) in steady-state, we consider an initial guess of inflation, $\pi^{(1)} = 0$. Set $m := 1$. Then:

Step 1: *Solution to the Hamilton-Jacobi-Bellman equation.* Given $\pi^{(m)}$, compute the bond price $Q^{(m)}$ using (59) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}^{(m)}$ and of the matrix $\mathbf{A}^{(m)}$.

Step 2: *Solution to the Kolmogorov Forward equation*. Given $\mathbf{A}^{(m)}$ find the aggregate distribution $\mathbf{f}^{(m)}$.

Step 3: *Finding the Lagrange multiplier.* Given $\mathbf{f}^{(m)}$, $\mathbf{v}^{(m)}$,compute the Lagrange multiplier $\mu^{(m)}$ using condition (23) as

$$\mu^{(m)} = \left[\sum_{i=1}^{2}\sum_{j=1}^{J} a_j f_{i,j}^{(m)}\left(c_{i,j}^{(m)}\right)^{-\gamma}\Delta a + \frac{1}{Q^{(m)}}\psi\pi^{(m)}\right].$$

Step 4: *Optimal inflation*. Given $\mathbf{f}^{(m)}$, $\mathbf{v}^{(m)}$ and $\mu^{(m)}$, iterate steps 1-3 until $\pi^{(m)}$ satisfies

$$\left(\rho - \bar{r} - \pi^{(m)} - \delta\right)\mu^{(m)} + \frac{1}{\left(Q^{(m)}\right)}\left[\sum_{i=1}^{2}\sum_{j=2}^{J-1}\left(\delta a_j + y_i - c_{i,j}^{(m)}\right) f_{i,j}^{(m)}\left(c_{i,j}^{(m)}\right)^{-\gamma}\Delta a\right] = 0.$$

C.2. Optimal transitional dynamics

We describe now the numerical algorithm to analyze the transitional dynamics, similar to the one described in Achdou et al. (2017). We define T as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and discretize it in N intervals of length

$$\Delta t = \frac{T}{N}.$$

[24] In particular, we have replaced the entry 2 of the zero vector in (73) by 0.1.

The asymptotic steady-state The asymptotic steady-state distribution of the model can be computed following the steps described in Section C.1. The result is a stationary distribution $\mathbf{f}_N$, a matrix $\mathbf{A}_N$, a bond price Q_N and a inflation rate π_N defined at the final time $T = N\Delta t$.

Solution to the bond pricing equation The dynamic bond pricing equation (9) can be approximated backwards as

$$(\bar{r} + \pi_n + \delta) Q_n = \delta + \frac{Q_{n+1} - Q_n}{\Delta t}, \Longleftrightarrow Q_n = \frac{Q_{n+1} + \delta \Delta t}{1 + \Delta t (\bar{r} + \pi_n + \delta)}, \quad n = N-1, .., 0, \tag{74}$$

where Q_N is the asymptotic bond price from Step 0.

Solution to the Hamilton-Jacobi-Bellman equation The dynamic HJB equation (5) can approximated using an upwind approximation as

$$\rho \mathbf{v}^n = \mathbf{u}^{n+1} + \mathbf{A}_{n+1} \mathbf{v}^n + \frac{(\mathbf{v}^{n+1} - \mathbf{v}^n)}{\Delta t},$$

where $\mathbf{A}^n$ is constructing backwards in time using a procedure similar to the one described in Step 1 of Section B. By defining $\mathbf{B}^{n+1} = \left(\frac{1}{\Delta t} + \rho\right) \mathbf{I} - \mathbf{A}_{n+1}$ and $\mathbf{d}^{n+1} = \mathbf{u}^{n+1} + \frac{\mathbf{V}^{n+1}}{\Delta t}$, we have

$$\mathbf{v}^n = \left(\mathbf{B}^{n+1}\right)^{-1} \mathbf{d}^{n+1}. \tag{75}$$

Solution to the Kolmogorov Forward equation Let $\mathbf{A}_n$ defined in (68) be the approximation to the operator $\mathcal{A}$. Using a finite difference scheme similar to the one employed in the Step 2 of Section A, we obtain:

$$\frac{\mathbf{f}_{n+1} - \mathbf{f}_n}{\Delta t} = \mathbf{A}_n^{\mathbf{T}} \mathbf{f}_{n+1}, \Longleftrightarrow \mathbf{f}_{n+1} = \left(\mathbf{I} - \Delta t \mathbf{A}_n^{\mathbf{T}}\right)^{-1} \mathbf{f}_n, \quad n = 1, .., N \tag{76}$$

where $\mathbf{f}_0$ is the discretized approximation to the initial distribution $f_0(a)$.

Complete algorithm The algorithm proceeds as follows:

Step 0: *Steady-state***.** Compute the stationary distribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N and inflation rate π_N. Set $\pi^{(0)} \equiv \{\pi_n^{(0)}\}_{n=0}^{N-1} = \pi_N$ and $k := 1$.

Step 1: *Bond pricing.* Given $\pi^{(k-1)}$, compute the bond price path $Q^{(k)} \equiv \{Q_n^{(k)}\}_{n=0}^{N-1}$ using (74).

Step 2: *Hamilton-Jacobi-Bellman equation.* Given $\pi^{(k-1)}$ and $Q^{(k)}$ solve the HJB equation (75) backwards to obtain an estimate of the value function $\mathbf{v}^{(k)} \equiv \{\mathbf{v}_n^{(k)}\}_{n=0}^{N-1}$ and of the matrix $\mathbf{A}^{(k)} \equiv \{\mathbf{A}_n^{(k)}\}_{n=0}^{N-1}$.

Step 3: *Kolmogorov Forward equation***.** Given $\mathbf{A}^{(k)}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (76).

Step 4: *Lagrange multiplier.* Given $\mathbf{f}^{(k)}$ and $\mathbf{v}^{(k)}$, compute the Lagrange multiplier $\mu^{(k)} \equiv \{\mu_n^{(k)}\}_{n=0}^{N-1}$ using (21):

$$\begin{aligned}\mu_{n+1}^{(k)} &= \mu_n^{(k)}\left[1+\Delta t\left(\rho-\bar{r}-\pi_n^{(k)}-\delta\right)\right] \\ &+\frac{\Delta t}{Q_n^{(k)}}\left[\sum_{i=1}^{2}\sum_{j=1}^{J}\left(\delta a_j+y_i-c_{n,i,j}^{(k)}\right)f_{n,i,j}^{(k+1)}\left(c_{n,i,j}^{(k)}\right)^{-\gamma}\Delta a\right],\end{aligned}$$

with $\mu_0^{(k)}=0$.

Step 5: *Optimal inflation*. Given $\mathbf{f}^{(k)}$, $\mathbf{v}^{(k)}$ and $\mu^{(k)}$ iterate steps 1-4 until $\pi^{(k)}$ satisfies

$$\Theta_n^{(k)} \equiv \sum_{i=1}^{2}\sum_{j=1}^{J} a_j f_{n,i,j}^{(k)} Q_n^{(k)}\left(c_{n,i,j}^{(k)}\right)^{-\gamma}\Delta a+\psi\pi_n^{(k)}-Q_n^{(k)}\mu_n^{(k)}=0.$$

This is done by iterating $\pi_n^{(k)}=\pi_n^{(k-1)}-\xi\Theta_n^{(k)}$, with constant $\xi=0.05$.

C.3. Optimal transitional dynamics with a borrowing limit in the market value of wealth

In this case, a number of changes should be made to the previous algorithm to adapt it to the results in Proposition 4. First, the optimal steady state inflation is zero, and hence there is no need to iterate when computing the steady state. Second, we employ as a state the debt expressed in market value a^m, and hence the drift simplifies to

$$s_{i,,jF} = \bar{r}a^m+y_i-\left[\frac{1}{\partial_F v_{i,j}}\right]^{1/\gamma}, \tag{77}$$

$$s_{i,j,B} = \bar{r}a^m+y_i-\left[\frac{1}{\partial_B v_{i,j}}\right]^{1/\gamma}. \tag{78}$$

Third, as the initial distribution $f_0(a,y)$ is given in terms of the face value of wealth and we need to recompute it market value $f_0^m(a^m,y)=\frac{1}{Q_0}f_0(a^m/Q_0,y)$ we need to interpolate over a grid defined in the market value of wealth. Fourth, the Lagrange multiplier $\mu^{(k)} \equiv \{\mu_n^{(k)}\}_{n=0}^{N-1}$ using (85) and (86):

$$\mu_{n+1}^{(k)}=\mu_n^{(k)}\left[1+\Delta t\left(\rho-\bar{r}-\pi_n^{(k)}-\delta\right)\right],$$

with $\mu_0^{(k)}=-\sum_{i=1}^{2}\sum_{j=1}^{J}v_{0,i,j}^{(k)}\frac{\Delta f_{0,i,j}}{\Delta Q_0}\Delta a$ where $\Delta f_{0,i,j}$ is the difference between the initial density in market prices given a bond price $Q_0+\Delta Q_0$ and the density given a bond price Q_0. We set $\Delta Q_0=0.01$. Fifth, the optimal inflation is given by (84):

$$\Theta_n^{(k)} \equiv \psi\pi_n^{(k)}-Q_n^{(k)}\mu_n^{(k)}=0.$$

D. Robustness

Steady state Ramsey inflation. In Proposition 3, we established that the Ramsey optimal long-run inflation rate converges to zero as the central bank's discount rate ρ converges to that of foreign investors, $\bar{r}$. In our baseline calibration, both discount rates are indeed very close to each other, implying that Ramsey optimal long-run inflation is essentially zero. We now evaluate the sensitivity of Ramsey optimal steady state inflation to the difference between both discount rates. From equation (23), optimal steady state inflation is

$$\pi_\infty = \frac{1}{\psi}\mathbb{E}_{f_\infty(a,y)}\left[NNP_\infty(a,y)\,MUC_\infty(a,y)\right] + \frac{1}{\psi}\mu_\infty Q_\infty, \tag{79}$$

where from equation (24)

$$\mu_\infty = \frac{\mathbb{E}_{f_\infty(a,y)}\left[NNP_\infty(a,y)\,MUC_\infty(a,y)\right]}{Q_\infty\left[\pi_\infty + \delta - (\rho - \bar{r})\right]}. \tag{80}$$

Figure 8 displays π (left axis), as well as its two determinants (right axis) on the right-hand side of equation (79). Optimal inflation decreases approximately linearly with the gap $\rho - \bar{r}$. As the latter increases, two counteracting effects take place. On the one hand, it can be shown that as the households become more impatient relative to foreign investors, the net asset distribution shifts towards the left, i.e. more and more households become net borrowers and come close to the borrowing limit, where the marginal utility of wealth is highest. As shown in the figure, this increases the central bank's incentive to inflate for the purpose of redistributing wealth towards debtors. On the other hand, higher indebtedness implies also more issuance of new debt. Moreover, a higher gap $\rho - \bar{r}$ increases the extent to which the central bank internalizes the effect of trend inflation on the price of bond issuances. The latter two effects imply that in equation (80), *ceteris paribus*, the numerator increases and the denominator falls, respectively, such that μ_∞ becomes more *negative*. This gives the central bank an incentive to committing to *lower* long-run inflation. As shown by Figure 8, this second effect dominates the redistributive inflationary effect, such that in net terms optimal long-run inflation becomes more negative as the discount rate gap widens.

Initial inflation. As explained before, time-0 optimal inflation and its subsequent path depend on the initial net wealth distribution across households, which is an infinite-dimensional object. In our baseline numerical analysis, we set it equal to the stationary distribution in the case of zero inflation. We now investigate how initial inflation depends on such initial distribution. To make the analysis operational, we restrict our attention to the class of Normal distributions truncated at the borrowing limit ϕ. That is,

$$f_0(a) = \begin{cases} \phi(a;\mu,\sigma)\,/\,[1-\Phi(\phi;\mu,\sigma)]\,, & a \geq \phi \\ 0, & a < \phi \end{cases}, \tag{81}$$

where $\phi(\cdot;\mu,\sigma)$ and $\Phi(\cdot;\mu,\sigma)$ are the Normal pdf and cdf, respectively.[25] The parameters μ

[25] In these simulations, we assume that the initial net asset distribution conditional on income is the same

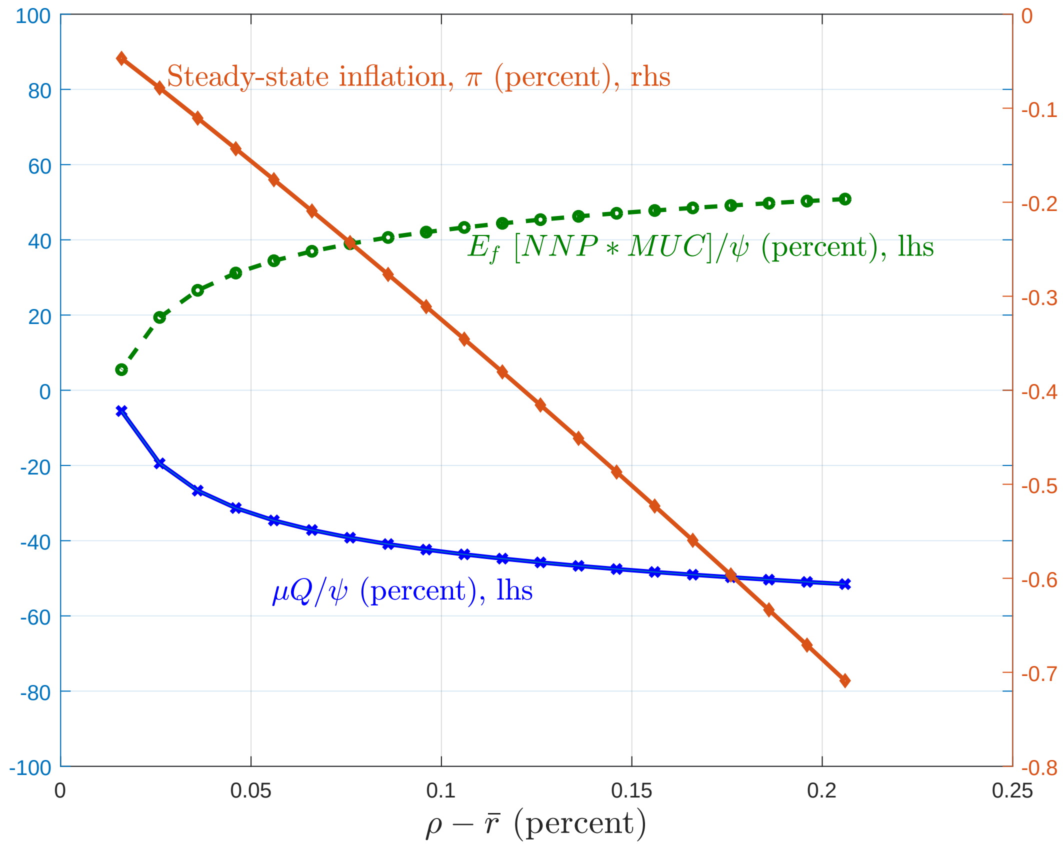

Figure 8: Sensitivity analysis to changes in $\rho - \bar{r}$.

and σ, the former not to be confused with the costate in equation (24), allow us to control both (i) the initial net foreign asset position and (ii) the domestic dispersion in household wealth, and hence to isolate the effect of each factor on the optimal inflation path. Notice also that optimal long-run inflation rates do *not* depend on $f_0(a)$ and are therefore exactly the same as in our baseline numerical analysis regardless of μ and σ. This allows us to focus here on inflation at time 0, while noting that the transition paths towards the respective long-run levels are isomorphic to those displayed in Figure 1.

Figure 9 displays optimal initial inflation rates for alternative initial net wealth distributions. In panels (a) and (b), we show the effect of increasing wealth dispersion while restricting the country to have a zero net position *vis-à-vis* the rest of the World, i.e. we increase σ and simultaneously adjust μ to ensure that $\bar{a}_0 = 0$. In the extreme case of a (quasi) degenerate initial distribution at zero net assets (solid black line in panel a), the central bank has no incentive to create inflation, and thus optimal initial inflation is zero. As the degree of initial wealth dispersion increases, so does optimal initial inflation.

Panels (c) and (d) in Figure 9 isolate instead the effect of increasing the liabilities with the rest of the World, while assuming at the same time $\sigma \simeq 0$, i.e. eliminating any wealth dispersion. That is, we approximate 'Dirac delta' distributions centered at different values of μ. Since such

for high- and low-income households: $f_0^{a|y}(a \mid y_2) = f_0^{a|y}(a \mid y_1) \equiv \tilde{f}_0(a)$. This implies that the marginal asset density coincides with its conditional density: $f_0^a(a) = \sum_{i=1,2} f_0^{a|y}(a \mid y_i) f^y(y_i) = \tilde{f}_0(a)$.

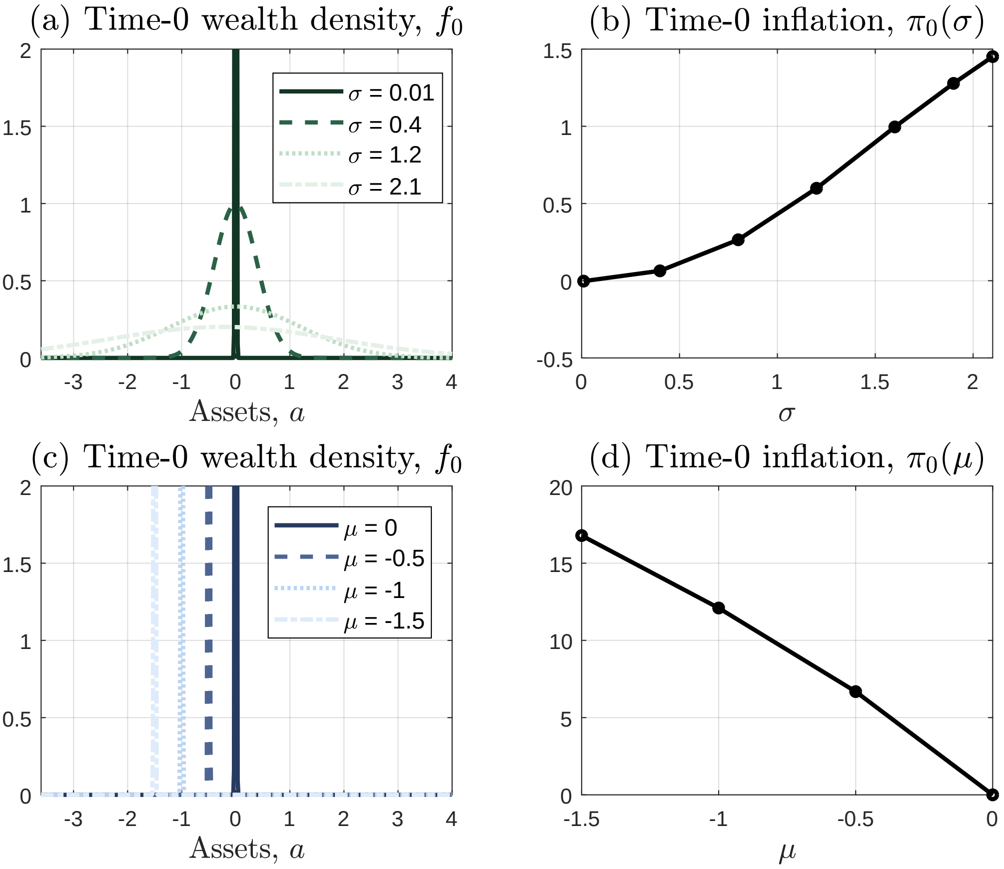

Figure 9: Optimal initial inflation for different initial net asset distributions.

distributions are not affected by the truncation at $a = \phi$, we have $\bar{a}(0) = \mu$, i.e. the net foreign asset position coincides with μ. As shown by the lower right panel, optimal inflation increases fairly quickly with the external indebtedness; for instance, an external debt-to-GDP ratio of 50 percent justifies an initial inflation of over 6 percent.

We can finally use Figure 9 to shed some light on the contribution of each redistributive motive (cross-border and domestic) to the initial optimal inflation rate, $\pi_0 = 4.6\%$, found in our baseline analysis. We may do so in two different ways. First, we note that the initial wealth distribution used in our baseline analysis implies a consolidated net foreign asset position of $\bar{a}_0 = -25\%$ of GDP ($\bar{y} = 1$). Using as initial condition a *degenerate* distribution at exactly that level (i.e. $\mu = -0.205$ and $\sigma \simeq 0$) delivers $\pi_0 = 3.1\%$ (see panel d). Therefore, the pure *cross-border* redistributive motive explains a significant part (about two thirds) but not all of the optimal time-0 inflation under the Ramsey policy. Alternatively, we may note that our baseline initial distribution has a standard deviation of 1.95. We then find the (σ, μ) pair such that the (truncated) normal distribution has the same standard deviation while ensuring that $\bar{a}_0 = 0$ (thus switching off the cross-border redistributive motive); this requires $\sigma = 2.1$, which delivers $\pi_0 = 1.5\%$ (panel b). We thus find again that the pure *domestic* redistributive motive explains about a third of the baseline optimal initial inflation. We conclude that both the cross-border and the domestic redistributive motives are quantitatively important for explaining the optimal inflation chosen by the monetary authority.

E. The case with the borrowing limit in the market value of debt

In this Appendix we consider the case of a exogenous borrowing limit in terms of the market value of wealth

$$Q_t a_t \geq \phi^m, \tag{82}$$

where $\phi^m \leq 0$.

In this case, the Ramsey problem (17) is then greatly simplified. If we separate the utility in consumption and inflation and express the value functional W_0 in terms of the market value of wealth, we get:

$$W_0 = \sum_{i=1}^{2} \int_{\phi^m}^{\infty} \left[\overbrace{v_i^c\left(a^m\right)}^{\text{value of consumption}} - \overbrace{\int_0^{\infty} e^{-\rho t} x\left(\pi_t\right) dt}^{\text{value of inflation}} \right] \left[\overbrace{\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right)}^{\text{initial distribution in market value}} \right] da^m,$$

where $v_i^c\left(a^m\right) = \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} u(c_t) dt$ is the value function of consumption as a function of initial market-value wealth. This value function does not depend on time as neither the drift in the law of motion of wealth (equation 12) nor the borrowing limit (82) depend on any aggregate variable. Therefore, we do not need to condition on the individual HJB equation. Similarly, the law of motion of the distribution is not a constraint in this case, as the central bank only cares about its impact on the initial distribution.

The Lagragian of the problem in this case simplifies to

$$\begin{aligned} \mathcal{L}_0 \equiv & \sum_{i=1}^{2} \int_{\phi^m}^{\infty} \left\{ v_i^c\left(a^m\right) - \int_0^{\infty} e^{-\rho t} x\left(\pi_t\right) dt \right\} \left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right) \right] da^m \\ & + \int_0^{\infty} e^{-\rho t} \mu_t \left[Q_t\left(\bar{r} + \pi_t + \delta\right) - \delta - \dot{Q}_t \right] dt. \end{aligned} \tag{83}$$

and we obtain the first-order conditions with respect to the functions π, Q by variational arguments. The following proposition characterizes the solution to this problem.

Proposition 5 (Optimal inflation - borrowing limit in market-value) *In addition to equations (10), (9), (5) and (7), if a solution to the Ramsey problem (17) exists, the inflation path* π_t *must satisfy*

$$x'\left(\pi_t\right) = \mu_t Q_t, \tag{84}$$

where μ_t *is the Lagrange multiplier associated to the bond pricing condition (9) with law of motion*

$$\frac{d\mu_t}{dt} = \left(\rho - \bar{r} - \pi_t - \delta\right) \mu_t, \tag{85}$$

and initial value

$$\mu_0 = -\frac{dW_0}{dQ_0} = -\sum_{i=1}^{2} \int_{\phi^m}^{\infty} \left\{ v_i^c\left(a^m\right) - \int_0^{\infty} e^{-\rho t} x\left(\pi_t\right) dt \right\} \frac{d}{dQ_0} \left[\frac{1}{Q_0} f_{i0}\left(\frac{a^m}{Q_0}\right) \right] da^m. \tag{86}$$

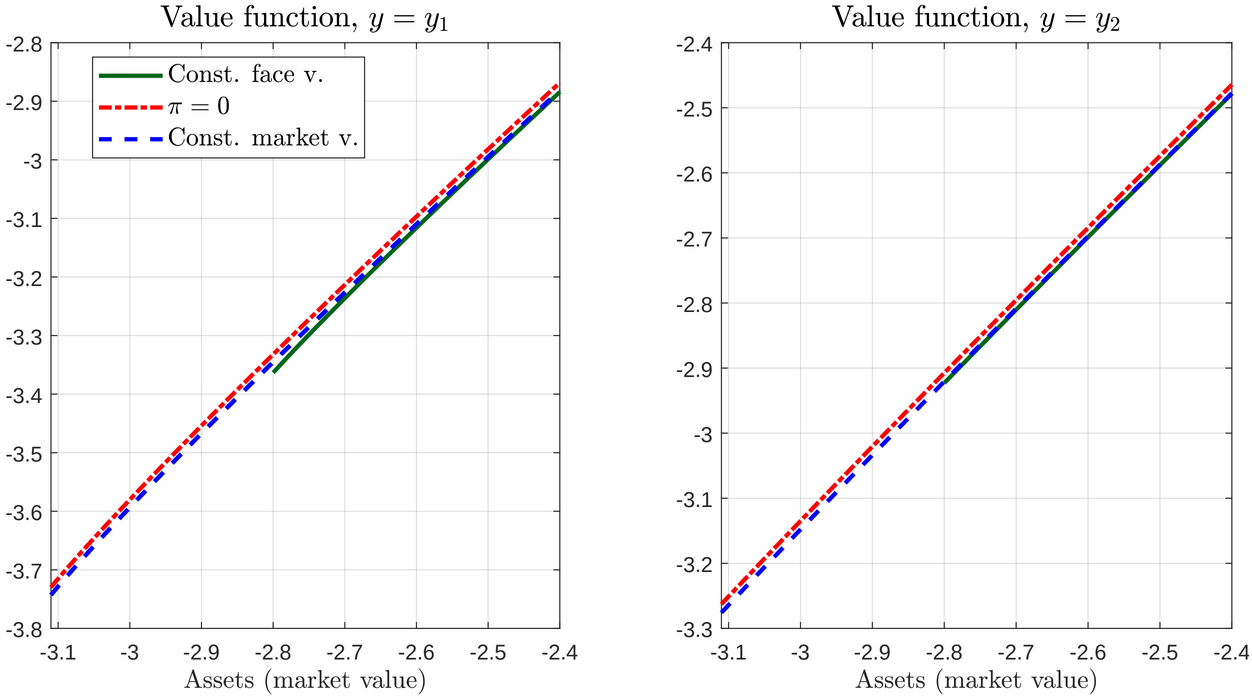

Figure 10: Time-0 value functions.

Notice how time-0 optimal inflation is only driven by the trade-off between the controlling time-0 bond prices to redistribute wealth and the disutility cost of inflation. As the market-valued borrowing limit ϕ^m is constant, the individual value function only depends on the future path of inflation

$$v_{i0}(a^m) = v_i^c(a^m) - \int_0^\infty e^{-\rho t} x(\pi_t)\, dt.$$

Optimal inflation is zero in the steady state. To check this, it is enough to verify that the steady-state value of the Lagrange multiplier μ is zero in equation (85) and hence inflation is zero in equation (84). Inflation in the distant future cannot affect current bond prices and hence the only relevant channel in the long run is the disutility cost of inflation, which calls for zero inflation.

Finally, if we solve equation (85) and combine it with equations (86) and (84) we obtain equation (29).

F. Additional figures