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Block-Recursive Equilibria in Heterogeneous-Agent Models

Abstract

Equilibrium models with heterogeneous agents and aggregate uncertainty are difficult to analyze since policy functions and market prices depend on the cross-sectional distribution over agents' state variables which is generally a high-dimensional object. This paper develops and applies a general model framework in which this problem does not arise. If sufficiently many agents enter the economy in every aggregate state of the world, policy functions and prices depend only on the exogenous aggregate state but are independent of the distribution over idiosyncratic states. The first part of this paper proves existence results for such block-recursive equilibria and derives an ergodic property which is useful for their computation. The second part applies this equilibrium concept to models of firm dynamics with competitive or frictional input markets and to incomplete-market economies with endogenous asset market participation.

JEL-Codes: C620, D500, E320.

Keywords: block-recursive equilibrium, dynamic general equilibrium, heterogeneous agents.

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1 Introduction

The analysis of general-equilibrium models with heterogeneous households or firms is complicated by a number of well-known difficulties. On the theoretical side, the existence of a *recursive equilibrium*, where policy functions and market prices depend on a minimal set of state variables, cannot be guaranteed in general.[1] Applied research uses advanced numerical techniques to compute approximate equilibria, without specifying how close these approximations are to a recursive equilibrium with rational expectations (conditional on the existence of the latter). On the other hand, in the absence of aggregate uncertainty, the existence of a *stationary equilibrium* can often be established. In fact, the comparative statics of stationary equilibria is standard practice for policy analyses,[2] which are sometimes complemented by numeric approximations of transition paths after one-time policy changes. Yet, the (local or global) stability of a stationary equilibrium is typically unknown, which casts doubts on its validity as a long-run model outcome.

This paper analyzes and applies a general infinite-horizon model framework in which these problems do not arise. It has heterogeneous agents who are subject to idiosyncratic and aggregate risk and who decide in each period about trading a finite number I of commodities. Typical examples are incomplete-markets models where households have stochastic income and trade different assets, or dynamic models of firms which are subject to stochastic productivity shocks and decide about the adjustment of production inputs. The central difficulty in most standard models is that commodity prices are generally not only functions of the exogenous aggregate state but that they depend on the cross-sectional distribution over exogenous and endogenous idiosyncratic states which is a high-dimensional object. This feature is important because the agents' demands for commodities respond differently to price changes; hence it is essential to know the cross-sectional distribution in order to determine equilibrium prices.

The framework analyzed in this paper does not have this feature. It deviates from many standard models in that there is endogenous entry of possibly different agent types. In models of firm dynamics, these entrants are startups which differ in firm-specific characteristics, such as initial productivity. In incomplete-markets economies, they can be hand-to-mouth households with different income realizations or preferences deciding to participate in asset market trade. Provided that there are I different entry types entering the economy in all aggregate states, then the prices of the I commodities are pinned down by the entry conditions of these I entrant

[1] Kubler and Schmedders (2002), Kubler and Polemarchakis (2004) and Santos (2002) give examples of dynamic general equilibrium economies for which no recursive equilibrium exists. Under restrictive assumptions, existence of recursive equilibrium has been shown in overlapping generations models (Citanna and Siconolfi, 2010, 2012) and in incomplete-markets models with infinitely-lived households (Brumm et al., 2017). With appropriate extensions of the state space, existence results for rational-expectations equilibria have been obtained for different incomplete-market economies (e.g. Duffie et al., 1994; Miao, 2006; Cao, 2020).

[2] See Acemoglu and Jensen (2015) for existence and comparative-statics properties of stationary equilibria in a large class of heterogeneous-agent models.

types. As a result, commodity prices depend only on the *exogenous* aggregate state but are independent of the distribution over idiosyncratic states. On the other hand, entry ensures that commodity markets are in equilibrium in all states of the world. Following earlier literature, I label such an outcome a *block-recursive equilibrium* (BRE): The first block of equilibrium objects, namely value functions, policy functions and market prices, is determined without knowledge of the cross-sectional distribution. Using these prices and policy functions, market clearing in commodity markets determines the dynamics of the cross-sectional distribution together with market entry (i.e., the second equilibrium block).

It is far from trivial under what conditions a BRE exists. The main difficulty is to ensure that commodity markets are in equilibrium with positive entry in all aggregate states of the world. This includes not only the exogenous aggregate state, but in particular the cross-sectional distribution over idiosyncratic states. Section 2 introduces the general model framework, defines recursive and block-recursive equilibria, and derives precise condition under which BRE exist (Theorems 1 and 2). The first theorem starts from a given stationary equilibrium (without aggregate shocks) and uses continuity arguments to derive the existence of BRE in the presence of small aggregate shocks. Equilibrium distributions in these equilibria remain close to the stationary distribution in a weak topology sense. Theorem 2 derives a similar result for the strong topology, using stronger assumptions but no continuity arguments. Both theorems show that the dynamics of equilibrium distributions follows an *iterated function system*, cf. Barnsley and Demko (1985), on a bounded metric space of distribution measures.

Nearly the same assumptions that ensure existence of BRE also permit a statement about their ergodic behavior: Theorem 3 shows that the iterated function system converges to a unique invariant distribution. This distribution is a high-dimensional object, namely a probability measure on the infinite-dimensional aggregate state space. Nonetheless, the ergodic property permits a straightforward computation via simulations of the iterated function system which can be implemented at low computational cost. The assumptions underlying these theorems also require that a stationary equilibrium is locally stable in the corresponding topology which justifies comparative statics experiments and which facilitates the computation of transitional dynamics.

Section 3 illustrates the usefulness of this equilibrium concept for different economic models. In Section 3.1, I consider models of firm dynamics following Hopenhayn (1992). It is not a new insight of this paper that such models can have block-recursive solutions. Both Hopenhayn (1992) and Hopenhayn and Rogerson (1993) describe how the *stationary* equilibrium in their models, with quasi-linear utility of the representative household in general equilibrium, can be solved in a block-recursive way: The equilibrium price (equivalently, the equilibrium real wage) in their models is pinned down by an entry condition without knowledge of the cross-sectional firm distribution.[3] Nonetheless, almost all of the macroeconomic literature which

[3]The general results derived in Section 2 also borrow from the insights developed in Hopenhayn (1992).

studies firm dynamics with aggregate uncertainty has not utilized this property and instead resorts to approximate aggregation techniques.[4] An important exception is the work of Lee and Mukoyama (2007, 2018) who use block recursivity to solve their model of entry and exit dynamics over the business cycle. As in much of this literature, they have one variable input of production (and hence one equilibrium price to be determined), and they do not specify the conditions for equilibrium existence. This paper applies block recursivity to models of firm dynamics with multiple production inputs and multiple entry types and it derives rigorous conditions for the existence of BRE and their asymptotic properties. In some examples, BRE may not exist, either because the steady state is unstable[5] or because aggregate uncertainty is too large. I consider a calibrated version of a model with two types of labor inputs, production workers and managers, where the latter input enhances the span-of-control of the firm's production technology. The composition of entrant types, namely small businesses and firms with growth potential, is endogenous in this model, which gives rise to an endogenous propagation of cohort-level employment, similar to Sedláček and Sterk (2017).

Section 3.2 considers models with heterogeneous multi-worker firms and search frictions in the labor market which are useful for the analysis of job and worker flows over the business cycle. Much of the recent search-and-matching literature builds on *directed search* which is known to provide tractable solutions due to block recursivity (see Shi, 2009; Gonzalez and Shi, 2010; Kaas and Kircher, 2015; Menzio and Shi, 2010, 2011; Schaal, 2017). In models with search on-the-job, directed search is essential for block-recursive solutions, cf. the discussion in Menzio and Shi (2011).[6] However, in business-cycle models with heterogeneous multi-worker firms and without on-the-job-search, directed search is by no means necessary for this feature. Hawkins (2011) demonstrates that the business-cycle model of Elsby and Michaels (2013) with random search and intrafirm bargaining is block-recursive when entry of firms is allowed for. This model is one application of the framework developed in this paper, as is the directed search model with multi-worker firms of Kaas and Kircher (2015). Neither Hawkins (2011) nor Kaas and Kircher (2015) prove the existence of BRE in their respective models in the presence of aggregate shocks.

In Section 3.3 I study an incomplete-markets model with exogenous idiosyncratic labor

[4]These contributions either rule out entry altogether (Khan and Thomas, 2008; Bachmann et al., 2013), they assume it to be exogenous (e.g. Veracierto, 2002), they make potential entrants ex-ante heterogeneous (e.g. Clementi and Palazzo, 2016) or entry costs dependent on the aggregate entry rate (e.g. Samaniego, 2008; Sedláček and Sterk, 2017), all of which rule out block recursivity, as does a utility function of the representative household which is not quasi-linear; see the discussion in Section 3.1.

[5]To my knowledge, instability of the (unique) stationary equilibrium in the Hopenhayn (1992) model has not been noticed before.

[6]The models of Shi (2009) and Menzio and Shi (2010, 2011) do not belong to the general model framework described in this paper. They have one firm type entering many different markets at constant cost such that the aggregate entry rate is always positive. This paper has multiple entry types who may or may not enter in equilibrium, and a BRE fails to exist if entry is zero in some aggregate states.

income (Huggett, 1993) and aggregate uncertainty. I add an endogenous asset participation margin to this model: Households are born as hand-to-mouth consumers and may acquire the financial skills to become active in asset market trade at a given fixed cost. When new traders become active in every aggregate state, equilibrium is block-recursive. I illustrates the properties of such equilibria on the basis of a simple example. If asset prices are moderately procyclical, a block-recursive equilibrium exists such that asset market participation is also procyclical. On the other hand, existence fails if asset prices are either too volatile or acyclical.

The framework developed in this paper only encompasses models with exogenous aggregate states whereas it does not capture economies where aggregate state variables evolve endogenously over time, such as the aggregate capital stock in a growth model. In the concluding Section 4, I briefly discuss possible extensions in this direction. As long as the endogenous aggregate state variables do not enter the agents' payoff functions, the analytical results carry over with relatively minor modifications. Examples are incomplete-markets economies where some assets can be produced and accumulated. On the other hand, when an endogenous aggregate state enters the agents' payoffs, such as in macroeconomic incomplete-markets models as in Aiyagari (1994), the formal analysis of block-recursive equilibria becomes a more intricate problem.

2 Theory

This section analyzes a general environment that describes equilibrium outcomes in infinite-horizon, discrete-time models with heterogeneous agents and aggregate uncertainty. The agents in this framework may represent households or firms, and their payoffs stand for the utility or profit values which depend on idiosyncratic and aggregate exogenous state variables, as well as on the agents' decisions about trading a finite-dimensional commodity bundle. In the applications of the next section, these commodities stand for asset holdings of households or production inputs of firms, albeit other applications are conceivable. Payoffs further depend on market prices which are determined endogenously so that all commodity markets are in equilibrium.

The terminology used in this section resembles the one of dynamic general equilibrium models with competitive markets. The applications that I consider in Sections 3.1 and 3.3 belong to this class of models. However, this framework also covers economies with search frictions where search is random or directed. In those applications, the mathematical structure is identical but the terminology (in particular, "prices" and "market clearing") is inappropriate; see Section 3.2 for clarification.

2.1 The Environment

Exogenous aggregate state. The exogenous aggregate state is denoted z and follows a Markov process on the finite set $Z = \{z_1, \ldots, z_N\}$. All values z_n, $n = 1, \ldots, N$, are elements of a compact subset $\bar{Z}$ of an Euclidean space with non-empty interior.

Exogenous idiosyncratic state. The idiosyncratic state is denoted $x \in X$ where X is a finite set. It follows a Markov process with transition probabilities $\pi_{xx'}(z)$, $x, x' \in X$, which depend continuously on the aggregate state z.

Commodities. There is a finite number I of commodities. An agent's trade of commodity i is denoted $a^i \in A_i$, $i = 1, \ldots, I$, where A_i is a compact interval of the real numbers containing zero. $A \equiv A_1 \times \cdots \times A_I \subset \mathrm{I\!R}^I$ denotes the set of feasible commodity bundles.

Prices. The price of commodity i is denoted $p^i \in P_i$, where the set of feasible prices P_i is a compact interval of the real numbers with non-empty interior. $P \equiv P_1 \times \cdots \times P_I \subset \mathrm{I\!R}^I$ is the set of commodity price vectors.

Agents. There is a continuum of agents which may be either active (engaged in commodity trade) or inactive (no commodity trade). The total mass of agents $\bar{M}$ can be finite or infinite.[7]

Preferences. All agents discount future payoffs with factor $\beta < 1$. The period payoff of an inactive agent depends only on exogenous state variables and is denoted $\bar{u}(x, z)$. The period payoff of an active agent is $u(a', a, p, x, z)$, where $a' \in A$ ($a \in A$) denotes the agent's commodity bundle at the end (at the beginning) of the period, and $p \in P$ is the price vector. Both $\bar{u}$ and u are continuous in $z \in \bar{Z}$. In addition, u is continuous in (a', a, p) and concave in (a', a), with strict concavity in a'.

Choice sets. An active agent chooses $a' \in \mathcal{A}(a, p, x, z)$ where the choice set $\mathcal{A}(.)$ is a non-empty, compact-valued and continuous correspondence which satisfies the convexity requirement that $\lambda a'_1 + (1-\lambda)a'_0 \in \mathcal{A}(\lambda a_1 + (1-\lambda)a_0, p, x, z)$ whenever $a'_0 \in \mathcal{A}(a_0, p, x, z)$ and $a'_1 \in \mathcal{A}(a_1, p, x, z)$, for all $\lambda \in [0, 1]$ and $a_0, a_1 \in A$.

Entry. At the beginning of a period, an inactive agent in state (x, z) may enter at cost $c(x, z)$ which is then subtracted from the period payoff. This agent enters with zero commodities (i.e., $a = 0 \in A$) and draws the initial idiosyncratic state x_0 from distribution $\pi_0(x_0|x)$.[8]

Exit. Exit follows an exogenous process. I assume that every agent dies at the end of a period with probability $\xi(x, z) \in [0, 1]$ which depends continuously on z. Exiting agents are replaced by newborn inactive agents with the same productivity. Write $\beta_0(x, z) \equiv \beta[1-\xi(x, z)]$ for the effective discount factor of an agent in state (x, z).

Commodity supply. While heterogeneous agents optimally decide about their commodity trades, there may be another side of the market which supplies these commodities. In

[7] $\bar{M} = \infty$ is a common assumption in many models of firm dynamics where an unbounded mass of potential firms may enter in every period. In contrast, a finite measure of agents is appropriate in models with household heterogeneity.

[8] In applications with firm heterogeneity, this specification encompasses the common assumption that entrant firms draw initial productivity stochastically.

applications of this framework, the supply side can be modeled as the outcome of decisions of rational agents (for instance, labor supply of a representative household in a model of firm dynamics). In the general framework of this section, I describe the supply side simply by a continuous function $S : P \times \bar{Z} \to \mathbb{R}^I_+$ which maps the price vector $p \in P$ and the aggregate state $z \in \bar{Z}$ into a vector of commodities.

Commodity markets. In applications with frictionless markets (Sections 3.1 and 3.3), the commodity markets are in equilibrium if aggregate supply S equals the aggregation of commodity trades a' over all active agents. However, this framework also encompasses models with search frictions (see Section 3.2) where an agent's trade absorbs additional resources (e.g. the commodities of unmatched sellers). To include such models, define the absorption function $\hat{A} : A \times A \times P \to \mathbb{R}^I_+$ which maps (a', a, p) into the absorption of commodities when an agent adjusts the commodity vector a to a' at price vector p. This function is Lipschitz continuous in all arguments and satisfies $\hat{A}(a', a, p) \geq a'$. If commodity markets are frictionless, the absorption function is simply $\hat{A}(a', a, p) = a'$.

2.2 Recursive and Block-Recursive Equilibria

A recursive equilibrium describes a situation where agents decide optimally about commodity trades and entry, while commodity markets are in equilibrium. Each period, prices, value functions and policy functions depend on a minimal set of relevant state variables. Generally, these state variables must include information about the distribution of active agents over idiosyncratic states $(a, x) \in A \times X$ since this distribution feeds into commodity demand and hence determines equilibrium prices.

Let $\mathcal{M}_+(A \times X)$ be the set of bounded Borel measures[9] μ on $A \times X$, describing the measure of *active* agents at the beginning of a period (before entry) over idiosyncratic states (a, x). In the case of a finite total measure of agents ($\bar{M} < \infty$), one also needs to keep track of the distribution of all agents (*active* and *inactive*) over idiosyncratic states x, denoted by $\mu_X(.)$ which is an element of $\mathcal{M}_+(X, \bar{M})$, the set of measures μ_X on finite set X (i.e., non-negative vectors of dimension card(X) summing up to $\bar{M}$). The aggregate state vector is the collection $(z, \mu, \mu_X) \in Z \times \mathcal{M}_+(A \times X) \times \mathcal{M}_+(X, \bar{M})$.

To simplify the exposition, the following analysis considers recursive equilibria where entry is never constrained above by the available inactive agents. In such situations, value functions, policy functions, and prices depend on (z, μ) but are independent of the measure μ_X.

Definition 1: A *recursive equilibrium (with unconstrained entry)* is a subset $\mathcal{M} \subset \mathcal{M}_+(A \times X)$, a subset $\mathcal{M}_X \subset \mathcal{M}_+(X, \bar{M})$, value functions $v : A \times X \times Z \times \mathcal{M} \to \mathbb{R}$, $\bar{v} : X \times Z \to \mathbb{R}$ for active and inactive agents, a policy function $\hat{a} : A \times X \times Z \times \mathcal{M} \to A$, a price function $\hat{p} : Z \times \mathcal{M} \to P$, an entry function $m : X \times Z \times \mathcal{M} \to \mathbb{R}_+$, and transition functions for distribution

[9] $A \subset \mathbb{R}^I$ is endowed with the Borel σ-algebra and finite set X has the σ-algebra generated by all elements.

measures $\Psi : \mathcal{M} \times Z \to \mathcal{M}$ (mapping μ into μ', given z) and $\Psi_X : \mathcal{M}_X \times Z \times \mathcal{M} \to \mathcal{M}_X$ (mapping μ_X into μ'_X, given (z, μ)) such that[10]

(a) Optimization of active agents: Their value function v solves the recursive problem

$$v(a, x, z, \mu) = \max_{a' \in \mathcal{A}(a,p,x,z)} u(a', a, p, x, z) + \beta_0(x, z)\mathbb{E}_{x,z} v(a', x', z', \mu')$$

given $p = \hat{p}(z, \mu)$ and $\mu' = \Psi(\mu, z)$, for all (a, x, z, μ). The policy function $\hat{a}(a, x, z, \mu)$ maximizes the objective function in this problem.

(b) Optimal entry: For all (x, z, μ), the measure of entrants $m(x, z, \mu)$ satisfies the complementary-slackness condition

$$m(x, z, \mu) \geq 0 \quad , \quad \mathbb{E}_x(v(0, x_0, z, \mu) - \bar{v}(x, z) - c(x, z)) \leq 0 \ ,$$

where the value function of inactive agents $\bar{v}$ satisfies

$$\bar{v}(x, z) = \bar{u}(x, z) + \beta_0(x, z)\mathbb{E}_{x,z}\bar{v}(x', z') \ , \quad \text{for all } (x, z).$$

(c) Market clearing: For all (z, μ),

$$S(\hat{p}(z, \mu), z) = \int \hat{A}\left(\hat{a}(a, x, z, \mu), a, \hat{p}(z, \mu)\right) d\tilde{\mu}(a, x) \ ,$$

$$\text{where} \quad \tilde{\mu} = \mu + \sum_{x \in X} \mu_x^0 \cdot m(x, z, \mu) \tag{1}$$

denotes the distribution of active agents *after entry* and $\mu_x^0 \in \mathcal{M}_+(A \times X)$ is the probability distribution measure defined by $\mu_x^0(\{0\} \times x_0) = \pi_0(x_0|x)$.

(d) Evolution of distribution measures:

(d1) For all (z, μ), $\mu' = \Psi(\mu, z)$ satisfies

$$\mu'(A_0 \times X_0) = \int \sum_{x' \in X_0} [1 - \xi(x, z)]\pi_{xx'}(z)\mathbb{I}\big(\hat{a}(a, x, z, \mu) \in A_0\big) d\tilde{\mu}(a, x) \ ,$$

for all Borel sets $A_0 \subset A$, $X_0 \subset X$, and with $\tilde{\mu}$ defined in (1).

(d2) For all (z, μ, μ_X), $\mu'_X = \Psi_X(\mu_X, z, \mu)$ satisfies

$$\mu'_X(x') = \sum_{x \in X} \pi_{xx'}(z)\Big[\mu_X(x) - m(x, z, \mu) + \sum_{\hat{x} \in X} m(\hat{x}, z, \mu)\pi_0(x|\hat{x})\Big] \ ,$$

for all $x' \in X$. Moreover, for all (z, μ, μ_X) and $x \in X$,

$$\mu_X(x) > \mu(A \times \{x\}) + m(x, z, \mu) \ .$$

[10]Simplifying notation, $\mathbb{E}_y g(y')$ denotes the expectation of $g(y')$ when y' is next period's value of a random variable, conditional on current value y. $\mathbb{I}$ denotes the indicator function.

The recursive equations in (a) and (b) specify value functions of active and inactive agents where only the former decide about commodity trade. The complementary-slackness condition in (b) says that positive entry of agents of type x in aggregate state (z, μ) requires that potential entrants are indifferent between remaining inactive and becoming active and paying the entry cost.[11] Condition (c) says that aggregate commodity supply is equal to the aggregate absorption of commodity trades of all agents after entry. Condition (d1) specifies how the distribution measure of active agents evolves from one period to the next, given the distribution $\tilde{\mu}$ after entry, policy function $\hat{a}$, exit rates ξ and transition probabilities $\pi_{xx'}$. Condition (d2), which is not needed if $\bar{M} = \infty$, describes the evolution of all agents (active and inactive) over time, taking into account that entrants draw new idiosyncratic states from distribution π_0.[12] The last requirement in (d2) says that there are sufficiently many inactive agents in every state $x \in X$ so that entry is never constrained from above.

As is well known, the theoretical and numerical analysis of recursive equilibria is complicated by the fact that a high-dimensional state variable (here, the infinite-dimensional distribution measure μ) enters the agents' value and policy functions. This is required whenever there is aggregate uncertainty: Movements in z induce changes in the heterogeneous agents' commodity demands, and the extent to which this shifts aggregate demand depends on the cross-sectional distribution.

In the absence of aggregate uncertainty, the analysis of heterogeneous-agent models is typically confined to a *stationary equilibrium*, which is defined as follows:

Definition 2: A *stationary equilibrium* is a recursive equilibrium where $z_n = \bar{z}$ for all n, $\mathcal{M} = \{\bar{\mu}\}$ and $\mathcal{M}_X = \{\bar{\mu}_X\}$.

Note that the stationarity of distribution measures is embodied in this definition, due to Definition 1(d) and the requirement that the invariant sets $\mathcal{M}$ and $\mathcal{M}_X$ reduce to a singleton. Stationary equilibria are much easier to study and often used for comparative statics experiments. Nevertheless, the (local) stability of a stationary equilibrium is rarely examined. Thus it remains unclear whether a recursive equilibrium without aggregate shocks exists and converges to the stationary solution $\bar{\mu}$ if the initial distribution measure μ_0 differs from $\bar{\mu}$, even when μ_0 is close to $\bar{\mu}$ in an appropriate topological sense. In fact, a stationary equilibrium can be unstable even when it is unique, as I demonstrate with an example of a Hopenhayn model in Section 3.1.

The goal of this section is to establish conditions under which value and policy functions depend on *finite-dimensional* state variables so that these complications disappear. There is indeed hope for such a simplification, provided that sufficiently many agents enter the economy

[11]Therefore, continuation utility of inactive agents is $\mathbb{E}_{x,z}\bar{v}(x', z')$, regardless of their entry decision at the beginning of the next period.

[12]Note again that every exiting agent is replaced by a newborn agent of the same type which is why exit probabilities do not show up in the dynamics of μ_X.

in every aggregate state. To see this, suppose that there is a set of *entry types* $X^E \subset X$ (that is, inactive agents with idiosyncratic state $x \in X^E$) with cardinality equal to the number of commodities I. If these agents enter in every aggregate state, the value function can potentially be defined on the smaller state space $(a, x, z) \in A \times X \times Z$, and the price function also reduces to $\hat{p} : Z \to P$. These functions jointly solve the set of equations

$$v(a, x, z) = \max_{a' \in \mathcal{A}(a, \hat{p}(z), x, z)} u(a', a, \hat{p}(z), x, z) + \beta_0(x, z)\mathbb{E}_{x,z} v(a', x', z') \ , \tag{2}$$

$$\text{for } (a, x, z) \in A \times X \times Z \ ,$$

$$\bar{v}(x, z) = \mathbb{E}_x v(0, x_0, z) - c(x, z) \ , \text{ for } (x, z) \in X^E \times Z \ . \tag{3}$$

Equation (2) is a standard recursive problem, given the price function $\hat{p}$. This price function is a vector with $N \cdot I$ elements (I commodity prices in N aggregate states) which is potentially pinned down by the $N \cdot I$ entry conditions (3), i.e., I entry types $x \in X^E$ in N aggregate states $z \in Z$ are indifferent between remaining inactive and entering after paying the entry cost. In other words, all commodities in all states of the world are priced by market entrants.[13]

Following Shi (2009), I label equilibria with this feature *block-recursive*: Value functions, policy functions and market prices are pinned down by the "block" of equations (2) and (3) which correspond to conditions (a) and (b) of Definition 1. The second block of equilibrium conditions (conditions (c) and (d)) must be solved after the first block. In particular, the measures of entrants $m(x, z, \mu) \geq 0$, $x \in X^E$, depend on the distribution measure of incumbent agents μ and must be consistent with market clearing in all I commodity markets. A block-recursive equilibrium is defined as follows.

Definition 3: Let $X^E \subset X$ have cardinality I. A *block-recursive equilibrium (BRE) with entry types* $x \in X^E$ is a recursive equilibrium where v, $\hat{a}$ and $\hat{p}$ do not depend on the distribution measure μ and where $m(x, z, \mu) > 0$ for $x \in X^E$, $m(x, z, \mu) = 0$ for $x \notin X^E$, for all $z \in Z$ and $\mu \in \mathcal{M}$.

In the next subsection, I establish conditions under which BRE with aggregate shocks exist in the neighborhood of a stationary equilibrium. To begin with, suppose that a stationary equilibrium with I entry types exists:

Assumption 1: *There exists a stationary equilibrium with aggregate state* $(\bar{z}, \bar{\mu}, \bar{\mu}_X)$ *where* $\bar{z} \in \text{int}(\bar{Z})$, *stationary equilibrium price* $\bar{p} \in \text{int}(P)$ *and a subset* X^E *with cardinality* I *such that:*

(a) $m(x, \bar{z}, \bar{\mu}) > 0$ *for all* $x \in X^E$,

[13] The same logic applies if the entry cost further depends on the price vector (for example, firm entry requires labor input whose cost depends on wages). However, it may not vary in other dimensions across agents or depend on the mass of entrants, as is the case in some of the papers listed in footnote 4.

(b) $\mathbb{E}_x v(0, x_0, \bar{z}, \bar{\mu}) < \bar{v}(x, \bar{z}) + c(x, \bar{z})$ *for all* $x \notin X^E$.

The two conditions say that there are exactly I entry types in the stationary equilibrium whereas all other agents strictly prefer not to enter. Under plausible independence conditions specified in the next subsection, this is a generic feature: A small deviation of the stationary aggregate state $\bar{z}$ induces a small change of stationary commodity prices $\bar{p} = \hat{p}(\bar{z}) \in P \subset \mathbb{R}^I$ consistent with exactly I entry conditions for agents $x \in X^E$ while all other agents strictly prefer not to enter. By a similar reasoning, a stationary equilibrium with *more than* I entrant types cannot be generic: a small deviation of the stationary aggregate state $\bar{z}$ typically cannot induce a reaction of I commodity prices such that entry remains optimal for more than I agent types. On the other hand, stationary equilibria *without entry* or with entry of *less than* I agent types may easily exist (and they do exist in various classes of models). In these situations, however, one cannot expect existence of a BRE: there are not enough entrants to price all commodities in all aggregate states. Then, commodity prices must depend on the cross-sectional distribution to guarantee market clearing in all states of the world.[14]

2.3 Existence of Block-Recursive Equilibria

This section establishes two existence theorems. The first theorem builds on continuity arguments for small deviations of the aggregate states z_n from the stationary value $\bar{z}$, obtaining BRE in a weak topology neighborhood of the stationary distribution measure $\bar{\mu}$. The second result starts out from a given solution of equations (2) and (3) and characterizes conditions for the existence of a BRE where the set of distribution measures $\mathcal{M}$ is bounded in the strong topology.

Establishing the existence of BRE in the topological neighborhood of a stationary equilibrium requires two steps. First, equations (2) and (3) must have a solution $p = \hat{p}(z)$ for all aggregate states when $z \in Z$ is close to $\bar{z}$ and varies over time. Throughout this section, I fix the transition probabilities of the Markov process for z and consider variation of the aggregate states z_n, $n = 1, \ldots, N$ around $\bar{z}$. This step is relatively straightforward and requires a few regularity conditions specified below. One of these conditions also guarantees the genericity of a stationary equilibrium.

Second, the dynamics of distribution measures μ and μ_X must remain close to the stationary distribution measures, which is required to ensure that commodity markets clear with positive entry in all states of the world. Otherwise the block-recursive prices obtained from equations (2) and (3) cannot be equilibrium prices. Obtaining such a stability property is far from trivial.

[14]Proving the existence of a stationary equilibrium with the properties of Assumption 1 is feasible in specific applications of this framework invoking fixed-point or continuity arguments; see e.g., Hopenhayn (1992) and Hopenhayn and Rogerson (1993) for existence of an equilibrium with entry in heterogeneous-firm models with one input commodity ($I = 1$). See Acemoglu and Jensen (2015) for existence results for a broad class of models.

Beginning with the first step, write the value function of a *candidate* BRE as a function of the current aggregate state index $n = 1, \ldots, N$, parameterized by the vector of aggregate states, denoted $\vec{z} = (z_1, \ldots, z_N) \in \bar{Z}^N$, and by the vector of commodity prices over all aggregate states, denoted $\vec{p} = (p_1, \ldots, p_N) \in P^N$. The Bellman equation (2) is then written as

$$v(a, x, n; \vec{z}, \vec{p}) = \max_{a' \in \mathcal{A}(a, p_n, x, z_n)} u(a', a, p_n, x, z_n) + \beta_0(x, z)\mathbb{E}_{x,n} v(a', x', n'; \vec{z}, \vec{p}) \ . \tag{4}$$

Using standard arguments, Lemma 1 in the Appendix shows that value and policy functions $\hat{a}(a, x, n; \vec{z}, \vec{p})$ for this problem exist which are continuous in $(a, \vec{z}, \vec{p})$. Further, the value function of inactive agents $\bar{v}(x, n; \vec{z})$ exists and is continuous in $\vec{z}$.

Then rewrite the entry conditions as follows:

$$w(x, n; \vec{z}, \vec{p}) \equiv \mathbb{E}_x v(0, x_0, n; \vec{z}, \vec{p}) - \bar{v}(x, z_n) - c(x, z_n) = 0 \ , \ n = 1, \ldots, N, \ x \in X^E \ . \tag{5}$$

Given a vector of aggregate states $\vec{z}$, these are $N \cdot I$ equations in $N \cdot I$ unknowns $\vec{p}$. Let $\vec{\bar{z}} = (\bar{z}, \ldots, \bar{z}) \in \bar{Z}^N$ and $\vec{\bar{p}} = (\bar{p}, \ldots, \bar{p}) \in P^N$ be the vectors of stationary aggregate states and prices from Assumption 1. Further write a generic price vector as $\vec{p} = (p_o^j)$ where $j = 1, \ldots, I$ are commodities and $o = 1, \ldots, N$ are aggregate states. Write the set of entrants $X^E = \{x_i \in X : \ i = 1, \ldots, I\}$ and impose the following differentiability and independence assumptions:

Assumption 2: *The expected net benefit of an entrant $w(.)$ is differentiable in $(\vec{z}, \vec{p})$ in a neighborhood of $(\vec{\bar{z}}, \vec{\bar{p}})$. Furthermore:*

(a) The $I \times I$ matrix

$$\left(\sum_{o=1}^{N} \frac{\partial w(x_i, n, \vec{\bar{z}}, \vec{\bar{p}})}{\partial p_o^j} \right)_{i,j}$$

is invertible for some aggregate state n.

(b) For every entry type $x_i \in X^E$ there exists some commodity j such that the $N \times N$ matrix

$$W_{ij} = \left(\frac{\partial w(x_i, n, \vec{\bar{z}}, \vec{\bar{p}})}{\partial p_o^j} \right)_{n,o}$$

is invertible.

Assumptions 2(a) and 2(b) are distinct independence conditions. 2(a) says that entry types benefit differently from changes in commodity prices, when these are varied uniformly across aggregate states. This assumption is needed to ensure that all I commodities can be priced by heterogeneous market entrants. In fact, the matrix in (a) is independent of the aggregate state n since a change of commodity j's price, uniformly across aggregate states o, takes the same impact on agent i's value in every aggregate state n, given that $z_n = \bar{z}$ (see the proof of

Proposition 1 for a formal argument). It follows from Assumption 2(a) (invoking the implicit function theorem) that the stationary equilibrium is generic in the sense that a small deviation of the *stationary* aggregate state permits another *stationary* commodity price vector consistent with entry conditions (5) (which are I equations in I unknowns in stationary equilibrium).

Assumption 2(b) says that every entrant agent benefits differently from price changes *across* aggregate states, at least for one of the commodities. This assumption is needed to make sure that entrants determine market prices when the aggregate state varies.

Both elements of Assumption 2, together with the implicit function theorem, imply that the entry equations (5) can be inverted for $\vec{p}$ when the aggregate state vector $\vec{z}$ varies around the stationary vector $\bar{\vec{z}}$.

Proposition 1: *Under Assumptions 1 and 2, the entry conditions (5) have a unique solution $\vec{p} = \tilde{p}(\vec{z})$ for all aggregate state vectors $\vec{z}$ in an open neighborhood of $\bar{\vec{z}}$. Function $\tilde{p}$ is continuous.*

When all commodity prices are pinned down by market entrants, are commodity markets in equilibrium? The answer to this question requires to deal with the second step described above. Write the commodity absorption of an agent (a, x) in aggregate state n more compactly as

$$a_n(a, x; \vec{z}) \equiv \hat{A}\left(\hat{a}(a, x, n; \vec{z}, \tilde{p}(\vec{z})), a, \tilde{p}(\vec{z})\right) \ ,$$

where $\vec{z}$ is in the neighborhood of $\bar{\vec{z}}$. Proposition 1 and continuity of the policy function (see Lemma 1 in the Appendix) ensure that a_n are continuous functions of a and $\vec{z}$.

For any measurable function $\phi : A \times X \to \mathbb{R}^I$ and any bounded Borel measure μ on $A \times X$, define

$$< \phi, \mu > \equiv \int \phi d\mu \in \mathbb{R}^I \ .$$

In a candidate BRE, suppose that $m_n(x) > 0$ agents of types $x \in X^E$ enter in aggregate state z_n, while entry of all other types $x \notin X$ is zero. The distribution of active agents after entry is then $\tilde{\mu} = \mu + \sum_{x \in X^E} m_n(x)\mu_x^0$. Commodity market clearing (Definition 1.c) can then be written

$$\begin{aligned} S(\tilde{p}_n(\vec{z}), z_n) &= < a_n(.; \vec{z}), \tilde{\mu} > \\ &= < a_n(.; \vec{z}), \mu > + \sum_{x \in X^E} < a_n(.; \vec{z}), \mu_x^0 > m_n(x) \\ &= < a_n(.; \vec{z}), \mu > + A_n^0(\vec{z}) m_n \ , \end{aligned} \tag{6}$$

where $A_n^0(\vec{z}) \equiv (< a_n(.; \vec{z}), \mu_x^0 >)_{x \in X^E}$ is an $I \times I$ matrix defining the absorption of I commodities in rows by $x \in X^E$ entry types in columns. $m_n \equiv (m_n(x))_{x \in X^E} \in \mathbb{R}^I$ is the column vector of entrant measures of types $x \in X^E$. Without aggregate uncertainty ($\vec{z} = \bar{\vec{z}}$), matrix $A_n^0(\vec{z})$

is independent of n and denoted $\bar{A}^0$. To make sure that equation (6) can be inverted, impose the following independence assumption:

Assumption 3: *The $I \times I$ matrix $\bar{A}^0$ is invertible.*

Assumption 3 says that the I entry types trade the I commodities to a linearly independent extent in stationary equilibrium. For example, different entrant firms hire heterogeneous labor inputs differently. Because of continuity (cf. Lemma 3 in the Appendix), matrix $A_n^0(\vec{z})$ is invertible for $\vec{z}$ close to $\vec{\bar{z}}$ and all n. This permits inversion of (6) (with m_n, S and a_n written as column vectors):

$$m_n = A_n^0(\vec{z})^{-1}\Big[S(\tilde{p}_n(\vec{z}), z_n) - < a_n(.;\vec{z}), \mu > \Big] \ . \tag{7}$$

At the stationary equilibrium ($\vec{z} = \vec{\bar{z}}$ and $\mu = \bar{\mu}$), Assumption 1 says that the vector of entry measures is strictly positive, $\bar{m} \in \mathbb{R}^I_{++}$. Since all vectors and matrices in (7) are continuous functions of $\vec{z}$, one may expect that entry remains positive (and hence, commodity markets are in equilibrium at block-recursive prices) *if* the distribution measure of active agents μ remains sufficiently close to the stationary measure *and if* the right-hand side of (7) is continuous in μ in the corresponding topology.

To derive such results, consider the vector space of bounded, signed Borel measures on $A \times X$, denoted $\mathcal{M}(A \times X)$.[15] Consider the Kantorovich-Rubinstein metric (cf. Bogachev (2007b))

$$d(\mu, \nu) \equiv \sup \left\{ \int f d\mu - \int f d\nu : f \in \mathrm{Lip}_{1,1} \right\} \ , \ \mu, \nu \in \mathcal{M}(A \times X) \ ,$$

where $\mathrm{Lip}_{1,1}$ is the set of Lipschitz continuous real-valued functions on $A \times X$ satisfying $|f(a_0, x) - f(a_1, x)| \le |a_0 - a_1|$ for all $a_0, a_1 \in A$ and $x \in X$ and $|f|_s \le 1$ where $|.|_s$ is the sup norm. This metric generates the weak topology on the subset of non-negative measures; see Lemma 2 in the Appendix on this result and further properties of this metric. In Lemma 3, I show that the mapping $(\mu, \vec{z}) \mapsto < a_n(.;\vec{z}), \mu > \in \mathbb{R}^I$ is continuous under the following assumption:

Assumption 4: *Policy function $\hat{a}$ is Lipschitz continuous in the state variable a.*

It follows that the entry vectors m_n as defined in (7) are strictly positive provided that $\vec{z}$ is sufficiently close to $\vec{\bar{z}}$ and that $d(\mu, \bar{\mu})$ is sufficiently small.

It remains to establish that distribution measures stay sufficiently close to the stationary measure as they evolve over time. The dynamics of distribution measures in (d1) of Definition 1 can be written as

$$\mu' = T_n^*(\vec{z})\tilde{\mu} \ , \tag{8}$$

[15] It is necessary to start from the space of *signed* measures since some of the operators defined below cannot a priori be restricted to the set of non-negative measures.

where $T_n^*(\vec{z})$ is a linear operator on $\mathcal{M}(A \times X)$, defined by

$$T_n^*(\vec{z})\tilde{\mu}(A_0 \times X_0) = \int \sum_{x' \in X_0} [1 - \xi(x, z_n)]\pi_{xx'}(z_n)\mathbb{I}(a_n(a, x; \vec{z}) \in A_0)d\tilde{\mu}(a, x) \ ,$$

for all Borel sets $A_0 \times X_0$. Using (7), the measure of active agents after entry is

$$\begin{aligned}\tilde{\mu} &= \mu + \sum_{x \in X^E} \mu_x^0 m_n(x) = \mu + \mu^0 m_n \\ &= \left[\mathbb{1} - \mu^0 A_n^0(\vec{z})^{-1} < a_n(.; \vec{z}), . >\right] \mu + \mu^0 A_n^0(\vec{z})^{-1} S(\tilde{p}_n(\vec{z}), z_n) \ . \end{aligned} \tag{9}$$

Here $\mu^0 : \mathbb{R}^I \to \mathcal{M}(A \times X)$ is a linear operator defined by $(y_i) \mapsto \sum_i \mu_{x_i}^0 y_i$, and $\mathbb{1}$ is the identity operator. Combining (8) and (9) shows that the distribution measure μ in aggregate state n adjusts according to the affine-linear mapping Ψ_n defined by

$$\mu' = \Psi_n \mu \equiv S_n^*(\vec{z})\mu + \mu_n^*(\vec{z}) \ , \tag{10}$$

where $S_n^*(\vec{z})$ is the linear operator on $\mathcal{M}(A \times X)$ defined by[16]

$$S_n^*(\vec{z}) \equiv T_n^*(\vec{z}) \circ \left[\mathbb{1} - \mu^0 A_n^0(\vec{z})^{-1} < a_n(.; \vec{z}), . >\right] \ ,$$

and $\mu_n^*(\vec{z})$ is the (signed) measure on $A \times X$ defined by

$$\mu_n^*(\vec{z}) \equiv T_n^*(\vec{z})\mu^0 A_n^0(\vec{z})^{-1} S(\tilde{p}_n(\vec{z}), z_n) \ .$$

At the stationary equilibrium vector $\bar{\vec{z}}$, all mappings $\Psi_n = \bar{\Psi}$ are identical so that distribution measures adjust according to

$$\mu' = \bar{\Psi}\mu \equiv \bar{S}^* \mu + \bar{\mu}^* \ , \tag{11}$$

where $\bar{S}^* = S_n^*(\bar{\vec{z}})$ and $\bar{\mu}^* = \mu_n^*(\bar{\vec{z}})$ for all n. By definition of the stationary equilibrium, $\bar{\mu}$ is a steady state of (11).

As discussed above, without stability of the stationary measure in the absence of aggregate shocks, as defined by the deterministic dynamical system (11), the stochastic dynamical system (10) can hardly permit locally bounded solutions. Therefore, the existence of a BRE requires stability of system (11). I impose a somewhat stronger contraction property, namely:

Assumption 5: *Operator $\bar{S}^*$ is a contraction with modulus $\lambda < 1$, i.e.,*

$$d(\bar{S}^*\mu, \bar{S}^*\nu) \le \lambda d(\mu, \nu) \qquad \textit{for all} \qquad \mu, \nu \in \mathcal{M}(A \times X) \ .$$

[16]While operator $T_n^*(\vec{z})$ maps non-negative measures $\mu \in \mathcal{M}_+(A \times X)$ into non-negative measures, operator $S_n^*(\vec{z})$ *does not* have this property, precisely because positive entry cannot a priori be guaranteed for all $\mu \in \mathcal{M}_+(A \times X)$. For this reason, $S_n^*(\vec{z})$ must be defined on the vector space of signed measures. Similarly, $\mu_n^*(\vec{z})$ may not be a non-negative measure because $A_n^0(\vec{z})^{-1} S(\tilde{p}_n(\vec{z}), z_n) \in \mathbb{R}^I$ may have negative entries.

In the Appendix, I prove the following:

Proposition 2: *Suppose that a continuous price function $\vec{p} = \tilde{p}(\vec{z})$ exists in a neighborhood of $\vec{\bar{z}}$ (such as derived from Proposition 1). Further suppose that Assumptions 1, 3, 4 and 5 are fulfilled. Then for every sufficiently small $\varepsilon_\mu > 0$ there exists $\varepsilon_z > 0$ such that for every $\vec{z}$ with $|\vec{z} - \vec{\bar{z}}| < \varepsilon_z$, all affine linear functions $\Psi_n : \mu \mapsto S_n^*(\vec{z})\mu + \mu_n^*(\vec{z})$ map the open ball of non-negative measures around $\bar{\mu}$ of radius ε_μ into itself, i.e., $\Psi_n(B(\bar{\mu}, \varepsilon_\mu)) \subset B(\bar{\mu}, \varepsilon_\mu)$ where $B(\bar{\mu}, \varepsilon_\mu) \equiv \{\mu \in \mathcal{M}_+(A \times X) \ : \ d(\mu, \bar{\mu}) < \varepsilon_\mu\}$.*

Proposition 2 implies that distribution measure μ remains contained in a neighborhood of $\bar{\mu}$ when z undergoes a Markov process close enough to $\bar{z}$. From the considerations above it follows that commodity markets are then in equilibrium at strictly positive entry vectors m_n defined in (7). The following theorem states the first main result about the existence of BRE in the presence of aggregate shocks. To close the remaining requirements, it must be guaranteed that none of the agents $x \notin X^E$ wishes to enter and to ensure that there are enough inactive agents to enter in every state of the world. The last requirement needs another regularity assumption when the total measure of agents is finite:

Assumption 6: *If $\bar{M} < \infty$, then the transition matrix of idiosyncratic states at the stationary equilibrium $(\pi_{xx'}(\bar{z}))$ is a contraction on the space of probability distributions on X endowed with the ℓ_1-metric.*

Theorem 1: *Under the assumptions of Proposition 2 together with Assumption 6, if $\varepsilon_z > 0$ is sufficiently small, there exists a block-recursive equilibrium for every $\vec{z}$ with $|\vec{z} - \vec{\bar{z}}| < \varepsilon_z$.*

Theorem 1 builds on continuity arguments to establish the existence of a BRE for aggregate shocks in a neighborhood around the stationary aggregate state. It does not specify the size of this neighborhood. For practical purposes, it may be useful to know whether a BRE exists for a *given* vector of aggregate states $\vec{z} = (z_1, \ldots, z_N) \in \bar{Z}^N$. Such a statement, formally shown in Theorem 2 below, necessarily requires stronger assumptions. To begin with, assume directly that there is a vector of prices solving the entry conditions for I entry types:

Assumption 7: *For the given vector of aggregate states $\vec{z} = (z_1, \ldots, z_N)$, there is a price vector $\vec{p}$ and value function v satisfying the Bellman equation (4), the entry conditions (5), and $\mathbb{E}_x v(0, x_0, n; \vec{z}, \vec{p}) - \bar{v}(x, n, \vec{z}) - c(x, z_n) \leq 0$ for $x \notin X^E$.*

To make sure that commodity markets are in equilibrium in all aggregate states, assume that all $I \times I$ matrices A_n^0 as defined above are invertible:

Assumption 8: *For all n, matrix $A_n^0 = (< a_n(.), \mu_x^0 >)_{x \in X^E}$ is invertible.*

It follows that distribution measures in any *candidate* BRE follow the stochastic system of affine-linear mappings

$$\mu' = \Psi_n \mu = S_n^* \mu + \mu_n^* \ , \tag{12}$$

where S_n^* are the linear operators on $\mathcal{M}(A \times X)$ and $\mu_n^* \in \mathcal{M}(A \times X)$ are the (signed) measures, as defined above. In a given aggregate state (n, μ), the I-dimensional vector of entrant measures follows from the market-clearing condition (cf. equation (7)),

$$m_n = (A_n^0)^{-1} \left\{ S(p_n, z_n) - < a_n(.), \mu > \right\} \ . \tag{13}$$

To guarantee that the system (12) is stable and has positive entry in all aggregate states, use the strong topology on $\mathcal{M}(A \times X)$, induced by the total variation norm $\|.\|_{TV}$, and the following contraction property:

Assumption 9: *Operators S_n^* are contractions in the total variation norm with modulus $\lambda < 1$.*

Since $(\mathcal{M}(A \times X), \|.\|_{TV})$ is a Banach space (see Theorem 4.6.1 in Bogachev (2007a)), every mapping Ψ_n has a unique fixed point,[17] denoted $\bar{\mu}_n$. Assume that these are non-negative measures with associated strictly positive entry vectors:

Assumption 10: *For all n, $\bar{\mu}_n \in \mathcal{M}_+(A \times X)$ and $\bar{m}_n \equiv (A_n^0)^{-1} \left\{ S(p_n, z_n) - < a_n(.), \bar{\mu}_n > \right\} \in \mathbb{R}^I_{++}$.*

With these assumptions at hand, and imposing a further parameter restriction, there exists an invariant set of non-negative measures:

Proposition 3: *Suppose that Assumptions 7–10 are fulfilled. Define*

$$\eta \equiv \min_{n,i} \frac{\bar{m}_{n,i}}{\sup_{A \times X} |H_{n,i}(.)|} > 0 \quad , \quad \delta \equiv \max_{n,o} \|\bar{\mu}_n - \bar{\mu}_o\|_{TV} \geq 0 \ ,$$

where $\bar{m}_{n,i}$ is the ith component of vector $\bar{m}_n$, and $H_{n,i}$ is the ith component of the continuous function $(A_n^0)^{-1} a_n(.) : A \times X \to \mathbb{R}^I$. If $\delta < \eta(1 - \lambda)$, then the set

$$\mathcal{M} \equiv \left\{ \mu \in \mathcal{M}_+(A \times X) \ : \ \|\mu - \bar{\mu}_n\|_{TV} < \eta \text{ for all } n \right\}$$

is invariant under all mappings Ψ_n. Further, m_n as defined in (13) is strictly positive for all n and $\mu \in \mathcal{M}$.

[17] The vector space of *signed* measures $\mathcal{M}(A \times X)$ endowed with the Kantorovich-Rubinstein metric d is not complete (cf. Bogachev (2007b), p. 192) which is why the contraction mapping theorem cannot be applied for this metric. This is the reason why Theorem 2 uses the total variation norm. On the other hand, the space of *non-negative* measures $\mathcal{M}_+(A \times X)$ is complete with metric d (see Lemma 2(a)). This property is used in the next section on asymptotic behavior.

Proposition 3 directly implies the existence of a BRE if the measure of agents is infinite. Otherwise another condition is required to ensure that sufficiently many inactive agents of every entrant type exist.

Assumption 11: *If $\bar{M} < \infty$, then*

(i) $\pi_0(x|x) = 1$ *(i.e., the idiosyncratic state does not change upon entry).*

(ii) The transition matrices $\Pi_n \equiv (\pi_{xx'}(z_n))$ are contractions with modulus $\rho < 1$ (in ℓ_1-metric), and the unique invariant measures $\bar{\mu}_{X,n}$ of Π_n are strictly positive such that

$$|\bar{\mu}_{X,n_1} - \bar{\mu}_{X,n_2}|_1 \leq (1-\rho) \min_{x \in X} \bar{\mu}_{X,n_3}(x) \ , \ \textit{for all } n_1, n_2, n_3 = 1, \ldots, N \ .$$

Theorem 2: *For given $(z_1, \ldots, z_N) \in \bar{Z}^N$ a block-recursive equilibrium exists provided that Assumptions 7–11 are satisfied, $\bar{M}$ is sufficiently large, and $\delta < \eta(1-\lambda)$ with δ and η defined in Proposition 3.*

2.4 Asymptotic Behavior and Computation

Theorems 1 and 2 specify conditions under which BRE with aggregate shocks exist on a bounded, complete metric space $(\mathcal{M}, d)$ where d is either the Kantorovich-Rubinstein metric[18] or the total variation metric, thus inducing either the weak or the strong topology on the measure space. The stochastic dynamics on this metric space is described by a finite number of affine-linear mappings $\Psi_n : \mathcal{M} \to \mathcal{M}$ which are all contractions of modulus $\lambda < 1$.[19] Over time, the aggregate state index n undergoes a Markov process with given transition probabilities $\psi_{nn'}$. If this exogenous Markov process defines a contraction on the space of probability measures on $\mathcal{N} = \{1, \ldots, N\}$, then the aggregate state (μ, n) converges in probability to an invariant probability measure on $\mathcal{M} \times \mathcal{N}$. Theorem 3 makes this ergodic statement precise. First of all, impose a standard condition that makes the Markov process for n contractive (cf. Lemma 11.3 in Stokey et al. (1989)):

Assumption 12: *There exists $\varepsilon > 0$ and aggregate state index $\bar{n} \in \mathcal{N}$ such that $\psi_{n\bar{n}} \geq \varepsilon$ for all $n \in \mathcal{N}$.*

Next, define the metric D on $\Omega \equiv \mathcal{M} \times \mathcal{N}$ by

$$D((\mu_1, n_1), (\mu_2, n_2)) \equiv d(\mu_1, \mu_2) + \alpha\rho(n_1, n_2) \ ,$$

[18] Note that $\mathcal{M} \subset \mathcal{M}_+(A \times X)$ and footnote 17 for completeness of $(\mathcal{M}, d)$.

[19] More precisely, Theorem 1 requires only that the steady-state mapping $\bar{\Psi}$ is a λ-contraction. But then, under the conditions of Proposition 2, all mappings Ψ_n are contractions, possibly with a larger modulus $\lambda' \in [\lambda, 1)$. This follows from similar continuity arguments as those used in the proof of Proposition 2.

where ρ is the discrete metric on $\mathcal{N}$, and

$$\alpha \equiv \frac{2}{\varepsilon} \cdot \sup \{d(\mu_1, \mu_2) \ : \ \mu_1, \mu_2 \in \mathcal{M}\} \ .$$

Evidently, (Ω, D) is a complete metric space. Consider the Borel σ-algebra on Ω, and let $\mathbb{P}(\Omega)$ denote the set of probability measures on Ω. Define the Kantorovich-Rubinstein metric on $\mathbb{P}(\Omega)$ by

$$d_K(Q_1, Q_2) \equiv \sup \left\{ \int f d(Q_1 - Q_2) \ : \ f \in \mathrm{Lip}_1(\Omega) \right\} \ ,$$

where $\mathrm{Lip}_1 \equiv \{f : \Omega \to \mathbb{R} \ : \ |f(\omega_1) - f(\omega_2)| \le D(\omega_1, \omega_2) \text{ for all } \omega_1, \omega_2 \in \Omega\}$ is the space of real-valued Lipschitz continuous functions with unit Lipschitz constant on Ω.[20]

The joint stochastic dynamics of $(\mu, n) \in \Omega$ can be defined by an *iterated function system* (see Barnsley and Demko (1985) and Kunze et al. (2012)):

$$\begin{aligned} &\Phi : \Omega \times [0,1] \to \Omega \ , \\ &(\omega, y) = ((\mu, n), y) \mapsto \omega' = (\mu' = \Psi_n \mu, n' = \hat{N}(n, y)) \ , \end{aligned}$$

where y is a uniformly distributed random variable and the mapping $\hat{N}$ induces the given Markov process on $\mathcal{N}$ (see the proof of Theorem 3 for the construction of $\hat{N}$). For any given initial aggregate state $\omega_0 \in \Omega$, the mapping Φ defines a sequence of random variables $\omega_t(\omega_0)$ recursively via $\omega_{t+1}(\omega_0) = \Phi(\omega_t(\omega_0), y_t)$, where y_t are stochastically independent and uniformly distributed on $[0,1]$. Let $Q_t^{\omega_0}$ denote the probability distribution of $\omega_t(\omega_0)$, i.e., $Q_t^{\omega_0}(\tilde{\Omega}) \equiv \mathrm{Prob}(\omega_t(\omega_0) \in \tilde{\Omega})$ for all Borel sets $\tilde{\Omega} \subset \Omega$. The following theorem, which makes use of a result on the ergodic dynamics of iterated function systems by Stenflo (2001), shows that for any initial state ω_0, the probability distributions $Q_t^{\omega_0}$ converge in metric d_K, at an exponential rate, to a unique invariant distribution $\bar{Q}$ on $\mathbb{P}(\Omega)$.

Theorem 3: *Let the assumptions of either Theorem 1 or Theorem 2 be fulfilled and let $(\mathcal{M}, d)$ be the invariant metric space implied by either of these theorems. Furthermore, let Assumption 12 be fulfilled. Then there exists a unique probability measure $\bar{Q} \in \mathbb{P}(\Omega)$ and a constant γ, such that for $\bar{\lambda} \equiv \max(\lambda, 1 - \frac{\varepsilon}{2}) < 1$,*

$$d_K(Q_t^{\omega_0}, \bar{Q}) \le \gamma \bar{\lambda}^t \ , \quad \textit{for all } \omega_0 \in \Omega \textit{ and } t \ge 0 \ .$$

Because Ω is the product of two sets, one of which is a space of measures on a subset of an Euclidean space, the invariant distribution $\bar{Q}$ is generally a high-dimensional object. Fortunately, due to Theorem 3, $\bar{Q}$ can be approximated by simulations of the underlying iterated function system from an arbitrary starting point ω_0.

[20]Different from the Kantorovich-Rubinstein metric defined above on the *measure space* $\mathcal{M}(A \times X)$, the Lipschitz functions in the definition of d_K are not restricted to be uniformly bounded. This is possible because this metric is defined on a space of *probability measures*.

The numeric computation of BRE and its invariant distribution can then be implemented as follows. First, discretize the state space A (cardinality $r \in \mathbb{N}$), and find (and calibrate) a stationary equilibrium with exogenous aggregate state $\bar{z}$ and commodity price vector $\bar{p}$ such that there are exactly I entry types. To solve the model with aggregate risk, use the following procedure:

1. For a given vector of exogenous aggregate states $\vec{z} = (z_1, \ldots, z_N)$ close to $\bar{z}$, guess a price vector $\vec{p} = (p_1, \ldots, p_N)$.

2. Solve Bellman equation (4) via value function iteration.

3. Verify the entry conditions (5) and update $\vec{p}$ if these conditions are not fulfilled with tolerable accuracy. In this case, go back to the previous step. Otherwise, proceed to the next step.

4. Solve for matrices A_n^0 and check invertibility. Stack elements of the *finite* state space $A \times X$ in vectors μ of dimension $Y = r \cdot |X|$ and compute the $Y \times Y$-matrices S_n^* and vectors $\mu_n^* \in \mathbb{R}^Y$, both defining the affine-linear mappings Ψ_n on $\mathbb{R}^Y$. If matrices S_n^* satisfy $\|S_n^*\|_1 < 1$, mappings Ψ_n are contractions in the ℓ_1-norm (i.e., the total variation norm). This verifies Assumptions 8 and 9 above.

5. Calculate the invariant measures $\bar{\mu}_n$ (fixed points of Ψ_n) and verify Assumption 10 (i.e. $\bar{\mu}_n \in \mathbb{R}_+^Y$ and $m_n \in \mathbb{R}_{++}^I$). Check if the parameter condition of Proposition 3 is fulfilled. If so, this defines invariant set $\mathcal{M} \subset \mathbb{R}_+^Y$.[21]

6. Take any vector $\mu \in \mathcal{M}$ and $n \in \mathcal{N}$. Iterating over the Markov process for n and mappings Ψ_n asymptotically approximates the unique ergodic distribution according to Theorem 3.

3 Applications

This section shows how the theoretical results of the previous section can be usefully applied to different economic models with heterogeneous firms or heterogeneous households. It largely builds on illustrative examples which highlight the requirements for the existence of BRE and their properties.

[21] Note that Proposition 3, and the assumptions and derivations preceding it, apply equally for the finite (discretized) commodity state space A.

3.1 Firm Dynamics

Consider an equilibrium model with heterogeneous firms which hire workers of two types, production workers and managers, in competitive labor markets. Firms face idiosyncratic and aggregate productivity risk and operate production technologies with decreasing returns in both labor inputs. This extends the classic Hopenhayn (1992) and Hopenhayn and Rogerson (1993) model to two types of labor and aggregate uncertainty.

Specifically, a firm with ℓ^p production workers and ℓ^m managers produces $f(\ell^p, \ell^m, x, z)$ units of output where x is idiosyncratic productivity and z is aggregate productivity. The production function f is increasing and concave in both labor inputs with jointly decreasing returns in (ℓ^p, ℓ^m). x and z follow Markov processes on finite sets with transition probabilities $\pi_{xx'}$ and $\psi_{zz'}$, respectively. If a firm adjusts labor inputs from (ℓ^p_-, ℓ^m_-) to (ℓ^p, ℓ^m) it pays adjustment cost $h(\ell^p, \ell^m, \ell^p_-, \ell^m_-)$. Firms exit at the end of a period with exogenous probability ξ. There is an unbounded mass of two types of potential entrants, $i = 1, 2$: If entrant firm i pays cost $c^i(z)$, it draws initial idiosyncratic productivity x from distribution $\pi^i_0(x)$.

Labor inputs are supplied by a representative household whose expected utility over streams of consumption C_t and labor supply (L^p_t, L^m_t) is given by

$$\mathbb{E} \sum_{t \geq 0} \beta^t \left[C_t - v_p(L^p_t) - v_m(L^m_t) \right] ,$$

where v_p and v_m are increasing and convex functions describing the disutility of labor of production workers and managers. The household owns all firms, receiving aggregate profit income Π_t in period t. The budget constraint of the household in period t is $C_t \leq \Pi_t + w^p_t L^p_t + w^m_t L^m_t$, where w^p_t and w^m_t denote the real wages of the two labor types in period t. The household optimally supplies $L^p_t = (v'_p)^{-1}(w^p_t)$ units of production labor and $L^m_t = (v'_m)^{-1}(w^m_t)$ units of managerial labor.

Note that quasi-linear utility of the representative household is a necessary requirement for this model to have a BRE.[22] The reason is that firms discount future profits with the household's discount factor which is constant and equal to β when utility is quasi-linear, but stochastic and dependent on the aggregate state (including the cross-sectional firm distribution) otherwise.

In a candidate BRE, wages for both labor types depend only on the (exogenous) aggregate state, to be denoted $w^p(z)$ and $w^m(z)$. The recursive formulation of the firm's problem is

[22] The utility function of the representative household in Hopenhayn (1992) and Hopenhayn and Rogerson (1993) is strictly concave in consumption and linear in the single labor input. In generalizations with two (or more) labor inputs, it is required to have linearity either in consumption or in one of the multiple labor inputs (for instance, production labor in the present model).

then[23]

$$v(\ell^p_-, \ell^m_-, x, z) = \max_{\ell^p, \ell^m} f(\ell^p, \ell^m, x, z) - w^p(z)\ell^p - w^m(z)\ell^m - h(\ell^p, \ell^m, \ell^p_-, \ell^m_-) + \beta_0 \mathbb{E}_{x,z} v(\ell^p, \ell^m, x', z') ,$$

with $\beta_0 = \beta(1-\xi)$. Entry conditions for both firm types $i = 1, 2$ are

$$\int v(0, 0, x, z) d\pi^i_0(x) = c^i(z) ,$$

for all aggregate states z. In a BRE, these equations pin down value functions, policy functions, and the two wages $w^p(z)$ and $w^p(z)$ across aggregate states. Entry rates of both firm types $i = 1, 2$ must be consistent with market clearing in labor markets for production and managerial labor.

In the following I consider two versions of this model without adjustment costs such that value functions do not depend on previous labor inputs. First I discuss existence properties of BRE in a closed-form example with one labor input. Then I describe a calibrated model where the composition of startups changes over the business cycle. In both examples, aggregate productivity alternates between two levels $z_L < z_H$ with (symmetric) transition probability $\psi_{z_L z_H} = \psi_{z_H z_L} \equiv \psi$.

3.1.1 Closed-Form Solutions with Homogeneous Labor Input

Consider the special case of this model with one labor input ℓ and Cobb-Douglas production function $\frac{xz}{1-\alpha}\ell^{1-\alpha}$. In the absence of adjustment costs, a firm with productivity x employs $\ell^*(w, xz) \equiv (xz/w)^{1/\alpha}$ workers so that profit income is $\pi^*(w, xz) \equiv \frac{\alpha}{1-\alpha}(xz)^{1/\alpha} w^{-\frac{1-\alpha}{\alpha}}$. The features of BRE critically depend on the properties of the idiosyncratic productivity process. If entrants draw initial productivity from the stationary productivity distribution of incumbent firms, the distribution of firms is independent of the history of aggregate shocks. In this case, which is described in detail in Appendix B, a BRE exists whenever aggregate shocks are not too large.

Since young firms are smaller than older firms, it is often assumed that startups draw initial productivity from a different distribution than incumbent firms (cf. Hopenhayn and Rogerson (1993)). In such situations the firm distribution depends on the history of aggregate shocks. Provided that the dynamics of the firm distribution is stable (as required in Assumption 5 or Assumption 9 of Section 2), BRE exist when aggregate shocks are not too large (as they do in the calibrated model of Lee and Mukoyama (2018)). However, the following example demonstrates that the (unique) steady state of this model can also be *unstable* in which case the stationary equilibrium is not a valid long-run description of this model. Instead the dynamics

[23] Different from the notation in the general framework, the initial state (ℓ^p_-, ℓ^m_-) has subscript "$-$" while labor input choices in the current period have no index. This modification is for notational convenience since firms produce in the current period with (ℓ^p, ℓ^m) at productivity (x, z).

converges to an endogenous limit cycle alternating between periods with and without entry. Then no BRE exists, no matter how small aggregate shocks are.

To illustrate these results, consider a particularly simple process of firm productivity where a new ("young") firm enters the market with productivity x^y. At the end of every period with probability λ, a young firm becomes an established ("old") firm in which case productivity increases to $x^o > x^y$. Otherwise the firm keeps low productivity x^y.[24] The exit rate ξ is identical for all firms.

It is straightforward to show that the entry conditions in both aggregate states can be satisfied with equality with wages $w_H, w_L > 0$, provided that entry costs are not too cyclical (see Appendix B for details). For example, if entry costs are constant $c_L = c_H = c$, then wages are procyclical satisfying $(w_H/w_L)^{1-\alpha} = z_H/z_L$.

Now consider the evolution of the firm distribution over time. Write L_n for labor supply in aggregate state $n = H, L$, and write $\ell_n^y = \ell^*(w_n, x^y z_n)$ and $\ell_n^o = \ell^*(w_n, x^o z_n)$ for employment of young and old firms in aggregate state n. Write $\mu^y, \mu^o \in \mathbb{R}_+$ for the measures of young and old firms at the beginning of a period (prior to entry). When m firms enter in aggregate state n, the labor market clears if

$$(\mu^y + m)\ell_n^y + \mu^o \ell_n^o = L_n \ . \tag{14}$$

To the next period, the measures of young and old firms adjust to

$$(\mu^y)' = (1-\xi)(1-\lambda)(\mu^y + m) \ , \tag{15}$$

$$(\mu^o)' = (1-\xi)\left(\mu^o + \lambda(\mu^y + m)\right) \ . \tag{16}$$

Substituting (14) into (16) shows that the measure of old firms adjusts according to

$$(\mu^o)' = R_n - S\mu^o \ , \tag{17}$$

where

$$R_n \equiv (1-\xi)\lambda \frac{L_n}{\ell_n^y} \quad \text{and} \quad S \equiv (1-\xi)\left[\lambda \left(\frac{x^o}{x^y}\right)^{1/\alpha} - 1\right] \ .$$

The measure of young firms in the next period is obtained by substituting (14) into (15). It depends on the measure of old firms in the current period and on the current aggregate state n:

$$\mu^{y'} = (1-\xi)(1-\lambda)\left[\frac{L_n}{\ell_n^y} - \left(\frac{x^o}{x^y}\right)^{1/\alpha} \mu^o\right] \ . \tag{18}$$

[24] In this example, firm size and firm age are positively correlated due to the exogenous productivity process. Another feature giving rise to a positive firm age-size relation are upward adjustment costs (e.g. hiring or training costs). In fact, it is possible to construct examples of unstable dynamics where productivity is constant and firms are initially small due to the presence of adjustment costs.

The measure of entrants is

$$m = \frac{L_n}{\ell_n^y} - \mu^y - \left(\frac{x^o}{x^y}\right)^{1/\alpha} \mu^o \ . \tag{19}$$

A BRE exists if, for any stochastic realization of aggregate states $n = H, L$, the dynamics of distribution measures, defined by (17) and (18), is such that $(\mu^o, \mu^y) \gg 0$ and such that the entry measure m defined by (19) remains positive. This necessitates that the dynamics of μ^o remains bounded. Inspection of (17) shows that the critical parameter S is not necessarily contained in the interval $(-1, 1)$ which is the stability condition of this dynamic system. In particular, $S > 1$ if $x^o > 2^\alpha x^y$ (i.e., if the productivity gap between established firms and startups is large enough), if λ is close to one (i.e., firms grow large fast enough) and if the exit rate is sufficiently low. For such parameter constellations, even in the absence of aggregate shocks, the unique steady state is unstable so that this economy follows an endogenous equilibrium cycle. In Appendix B, I describe the dynamics of the deterministic model in a special case and show that it converges to equilibrium cycles (which may be periodic or non-periodic) alternating between states of positive and zero entry. Because of this instability, no BRE exists in the presence of (arbitrarily small) aggregate shocks when $S > 1$.

Consider next the case where the deterministic steady state is stable with $S \in (0, 1)$ (i.e. Assumption 5 of Section 2 is fulfilled). Then BRE exist in the presence of (small) aggregate shocks. Depending on the cyclicality of the wage and on the labor supply elasticity, it is possible that R_H is larger or smaller than R_L. If the wage is relatively stable and/or labor supply is rather inelastic, $R_H < R_L$ applies. In this case, Figure 1 illustrates the dynamics of μ^o over time. It is evident from this graph that the invariant ergodic distribution of this stochastic system is contained in the interval $[\underline{\mu}^o, \overline{\mu}^o]$ with

$$\underline{\mu}^o \equiv \frac{R_H - SR_L}{1 - S^2} < \overline{\mu}^o \equiv \frac{R_L - SR_H}{1 - S^2} \ .$$

Positive distribution measures require the condition $\underline{\mu}^o > 0$, i.e. $R_H > SR_L$ (which is trivially satisfied when there are no aggregate shocks, $R_H = R_L$). Moreover, another parameter restriction is required to ensure that entry is positive in all aggregate states. In Appendix B, I prove that $m > 0$ at the invariant ergodic distribution of (μ^o, μ^y) if

$$\underline{\mu}^o \geq \max_{n,n'} \frac{R_n[(S + (1 - \xi)(2 - \lambda))] - R_{n'}}{(S + 1 - \xi)[S + (1 - \xi)(1 - \lambda)]} \ ,$$

which is also satisfied if aggregate shocks are small enough. Due to $R_H < R_L$ and because a larger measure of established firms μ^o implies a lower entry rate, entry is procyclical in the BRE.

3.1.2 A Calibrated Example with Two Worker Types

Now suppose that a firm's production function is $f(\ell^p, \ell^m, x, z) = xz(a + \ell^m)^{\alpha^m}(\ell^p)^{\alpha^p}$ where $a > 0$ stands for the managerial work of the firm's owner, ℓ^m and ℓ^p are managers and

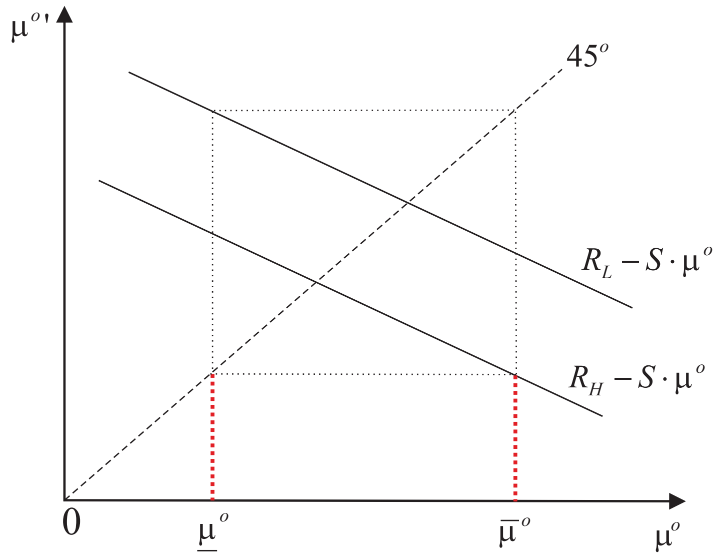

Figure 1: Dynamics of the firm distribution for $S < 1$: the BRE is contained in the interval $[\underline{\mu}^o, \overline{\mu}^o]$.

production workers hired in the labor market, and $\alpha^p + \alpha^m < 1$ so that overall returns to labor are decreasing. Write $w_n^p = w^p(z_n)$ and $w_n^m = w^m(z_n)$ for the wages of production workers and managers in aggregate state $n \in \{L, H\}$ in a BRE. Idiosyncratic productivity attains one of four levels $x_1 < x_2 < x_3 < x_4$ corresponding to four firm size classes.

Two types of firms $\tau = 1, 2$ enter the economy at cost $c^\tau(z)$. Firms of type 1 are small businesses with no growth potential. They enter at the lowest productivity state x_1 which stays constant over time. In the calibrated model, these firms do not find it optimal to hire managers, i.e. they choose $\ell^m = 0$ in all aggregate states. Firms of type 2 have potential to grow large. With probability $\pi_{0,i}$ such a firm draws initial productivity x_i. For an existing firm of this type with productivity x_i, $i \leq 3$, productivity increases to the next highest level x_{i+1} with probability π_i at the end of every period. Firms of type 2 and productivity $x \geq x_2$ find it optimal to hire managers at the calibrated parameters.

The firm distribution is a five-dimensional vector defining the masses of firms of both firm types, where firms of type 2 can be in one of four productivity states. By construction, the value function of type 1 firms depends negatively on the wages of production workers but is independent of the wages of managers, whereas the value function of type 2 firms decreases in wages of both worker types. Hence small business entrants pin down the wages of production workers, while wages of managers are pinned down by entrants with growth potential.

The model is calibrated at annual frequency to match empirical facts about the firm distribution from the Business Dynamics Statistics (BDS) of the U.S. Census Bureau and about wages and employment of "production workers" (production and nonsupervisory employees) and "managers" (all other employees) from the Current Employment Statistics of the U.S. Bureau of Labor Statistics (CES); see Appendix B for details. Set $\beta = 0.95$ and $\xi = 0.1$ to reflect a five-percent interest rate and a ten percent annual firm exit rate. With labor inputs ℓ^m and ℓ^p measured in the numbers of employees, $a = 1$ captures the managerial labor input of a small firm's (single) owner. Production elasticities are set to $\alpha^p = 0.587$ and $\alpha^m = 0.263$ which ensure that overall returns to scale are $\alpha^p + \alpha^m = 0.85$ (a standard value) and that the managers' share in earnings is 31 percent of total earnings.

With average aggregate productivity normalized at $z = 1$, the four idiosyncratic productivity levels x_i are set to match average firm size in the four size classes 1-9, 10-99, 100-999, and 1000+. The entry shares $\pi_{0,i}$ and the transition rates π_i for firms of type 2 are set to match the shares of firms in these size classes, both for entrants and for all firms in the BDS. Entry costs for both firm types in steady state ($z = 1$) are set such that the equilibrium wage of managers is 63% above the wage of production workers.

Cyclical parameters are calibrated as follows: Aggregate labor productivity fluctuates between $z_L = 1 - \varepsilon$ and $z_H = 1 + \varepsilon$ with transition probability ψ where ε and ψ are set so that the model matches the standard deviation and annual autocorrelation of detrended real GDP. Entry costs for both types of firms, $c^\tau(z_n)$, $\tau = 1, 2$ and $n = L, H$, fluctuate procyclically such that the standard deviation of the aggregate entry rate (0.6%) and its correlation with aggregate output (0.43) match the data targets. This implies that entry costs of both firms vary by about half as much as aggregate labor productivity, while wages for production workers and managers are procyclical and are 67% (110%) as volatile as aggregate labor productivity. The representative household's disutility of production and managerial labor is $v^p(L^p)^{\frac{1}{\gamma^p}+1} + v^m(L^m)^{\frac{1}{\gamma^m}-1}$. The scale parameters v^p and v^m and the Frisch elasticities γ^p and γ^m are set such that average employment for workers and managers, and their cyclical variations (relative to labor productivity) are in line with the data.

The calibrated model has a BRE with positive entry of both types of firms. This is illustrated in Figure 2 which shows a projection of the asymptotic equilibrium, i.e. the limit of the iterated function system on the six-dimensional aggregate state space (μ, n) where $n = H, L$ is the exogenous aggregate state and $\mu \in \mathbb{R}^5_+$ is the distribution vector of firm types. The entry rate of type 2 firms (firms with growth potential) is more volatile than the entry rate of type 1 firms (small businesses). Both entry rates are also procyclical. Specifically, the correlation coefficient between the entry rate of type 2 (type 1) firms and output is 0.46 (0.22), and the entry rate of type 2 firms is more than twice as volatile as the entry rate of type 1 firms.

This feature implies that the growth potential of startups varies over the business cycle which is in line with the facts documented by Sedláček and Sterk (2017). Firms that enter in booms are larger on average than firms that enter in recessions, and this difference in startup size propagates into subsequent years. Figure 3 shows the average firm size of startups in different periods of the cycle against the size of the same cohort of firms in the following three years. In periods of high aggregate output, firms start larger on average, and these firms also grow larger in subsequent periods. That is, firm size is highly positively correlated within the

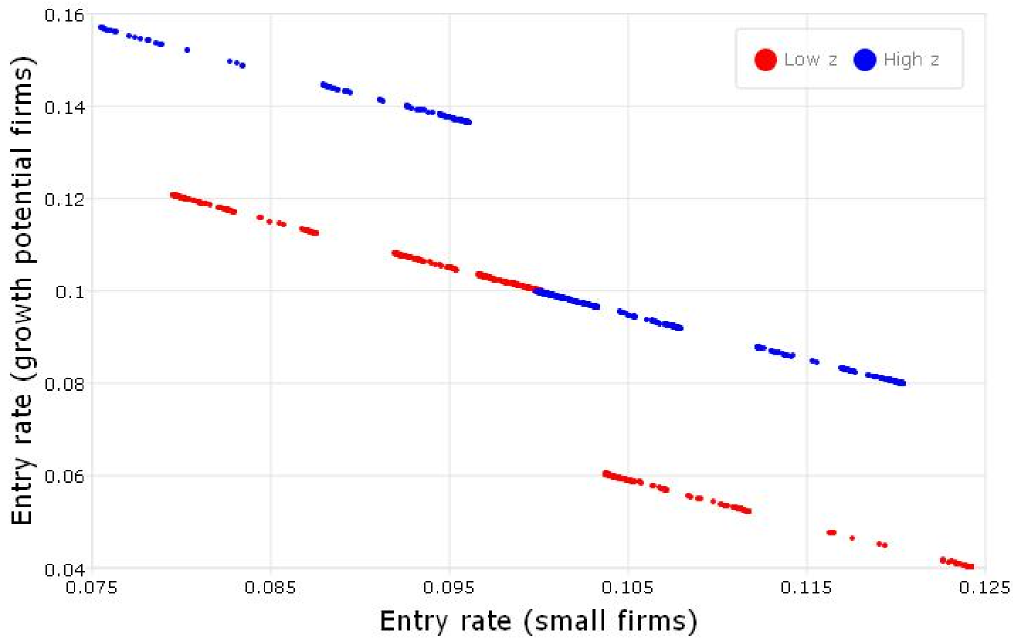

Figure 2: Entry rates for both firm types in the BRE.

Note: The dots show model outcomes of a simulation for 2000 periods where the first 500 periods are discarded. Red (blue) points are for low (high) realizations of aggregate productivity z.

same cohort of firms.

3.2 Firm Dynamics with Search Frictions

Consider a model with homogeneous labor input and matching frictions in the labor market. As in the previous section, heterogeneous firms operate production functions $f(\ell, x, z)$, increasing and strictly concave in labor input ℓ, where x and z are idiosyncratic and aggregate productivity. New firms enter at cost $c(z)$ in which case they draw initial productivity x with probability $\pi_0(x)$. Idiosyncratic productivity adjusts from x to x' with probability $\pi_{xx'}$. Posting V vacancies entails recruitment cost $\kappa(V)$ where κ is an increasing and (weakly) convex function. Each vacancy is filled with probability $q(\theta)$ where θ is market tightness (vacancies per unemployed worker) which generally depends on the aggregate state of the economy. The job-filling rate q is a decreasing function of θ.

On the other side of the labor market is a unit mass of risk-neutral workers who can be either employed or unemployed. Employed workers do not search for jobs but quit into unemployment with exogenous probability s_0 at the beginning of each period. An unemployed worker finds a job with probability $p(\theta) = \theta q(\theta)$; otherwise this worker receives unemployment income b and searches again next period. Firms and workers discount future income with

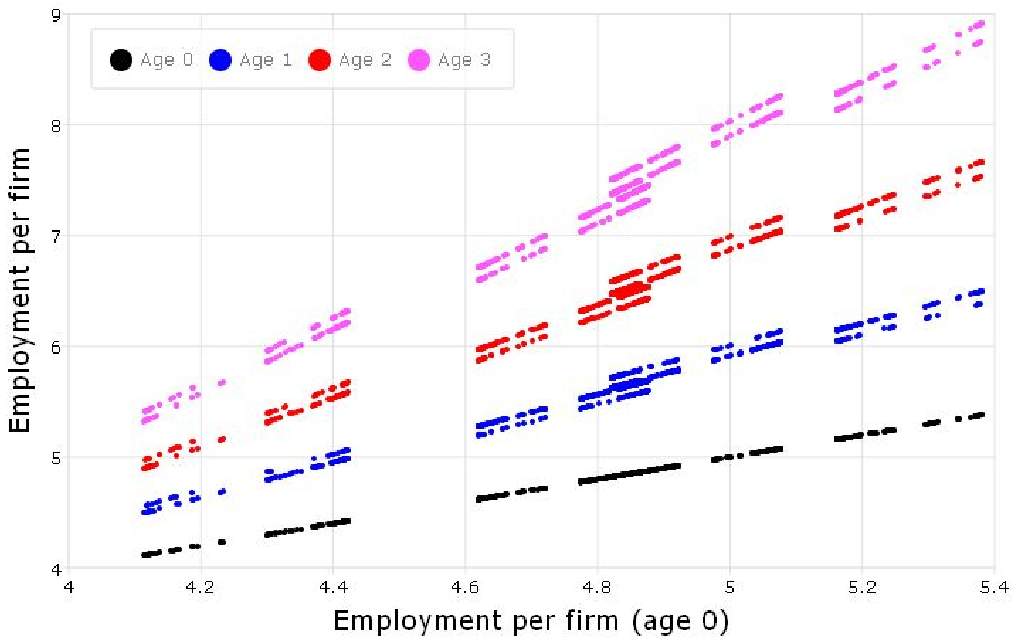

Figure 3: Average firm size at entry against firm size of the same cohort in the following three years.

Note: The graph is based on a simulation for 2000 periods with the first 500 periods discarded.

factor $\beta < 1$.

The timing within each period is as follows: First, idiosyncratic and aggregate productivities are realized and new firms enter. Second, each firm decides to separate from a fraction $s \geq s_0$ of its workers and to post $V \geq 0$ vacancies. Third, vacancies and unemployed workers are matched, where newly separated workers can search for employment within the same period. Fourth, output is produced and wages are paid. Fifth, a fraction ξ of firms exits the market in which case all its workers enter the unemployment pool. Again, $\beta_0 = \beta(1 - \xi)$ denotes the effective discount factor.

Consider two versions of this model: First, with *random search*, market tightness θ varies only with the aggregate state so that the job-filling rate is identical for all hiring firms and the job-finding rate is identical for all unemployed workers. Wages are negotiated between a firm and each of its workers in every period, taking into account the impact of every additional worker on the bargaining outcome with all other workers (cf. Stole and Zwiebel (1996)). This model is nearly identical to the one of Elsby and Michaels (2013) with free entry of firms. Building on Hawkins (2011) who shows that the Elsby-Michaels model with linear vacancy costs permits block-recursive solutions, I establish that this model is indeed a special case of the general framework developed in Section 2. Second, I consider the same model with *competitive search* where firms compete for workers by offering contingent long-term contracts.

In this model, market tightness (and job-filling rates) vary across firms which are matched with workers in different submarkets. Kaas and Kircher (2015) demonstrate that this model is block-recursive, and it also turns out to be a special case of the general framework of the previous section.

Write $\mu(\ell_-, x)$ for the distribution measure of existing firms, entering the period with ℓ_- workers and drawing idiosyncratic productivity x. After entry of $m \geq 0$ firms, the distribution measure of firms is $\tilde{\mu} = \mu + m \sum_{x \in X} \mu_x^0$ where μ_x^0 has mass $\pi_0(x)$ at $(0, x)$ and zero mass elsewhere. Consider a candidate BRE in which the employment policy of a given firm (ℓ_-, x) depends only on the exogenous aggregate state z, written as $\ell = \hat{\ell}(\ell_-, x, z)$. The job-filling rate of this firm is $q(\theta(\ell_-, x, z))$ where θ is independent of (ℓ_-, x) under random search. After separations and before hiring, firm (ℓ_-, x) employs $\min[\ell_-(1 - s_0), \ell]$ workers and it posts $V = \frac{1}{q(\theta(\ell_-,x,z))} \max[0, \ell - \ell_-(1 - s_0)]$ vacancies. Consistent with market tightness $\theta(\ell_-, x, z)$, these vacancies require $\frac{V}{\theta(\ell_-,x,z)}$ unemployed workers. Therefore, the employment policy of firm (ℓ_-, x) absorbs

$$\min[\ell_-(1 - s_0), \ell] + \frac{1}{p(\theta(\ell_-, x, z))} \max[0, \ell - \ell_-(1 - s_0)] \tag{20}$$

workers. Since total labor supply is normalized to unity, the aggregate resource constraint in any given period (after separations and before hires) is written

$$1 = \int \min\left[\ell_-(1 - s_0), \hat{\ell}(\ell_-, x, z)\right] + \frac{1}{p(\theta(\ell_-, x, z))} \max\left[0, \hat{\ell}(\ell_-, x, z) - \ell_-(1 - s_0)\right] d\tilde{\mu}(\ell_-, x) . \tag{21}$$

3.2.1 Random Search

In a candidate BRE with random search, market tightness is a function of the exogenous aggregate state but independent of firm characteristics, $\theta(z)$. Let $w(\ell, x, z)$ denote the bargained wage in a firm with ℓ workers when idiosyncratic (aggregate) productivity is equal to x (z). Let $\eta \in (0, 1)$ denote the bargaining power of workers, and let vacancy costs be linear, $\kappa(V) = \kappa_0 \cdot V$. With intrafirm bargaining the wage solves the differential equation

$$w(\ell, x, z) = (1 - \eta)b + \eta \left[f_\ell(\ell, x, z) - w_\ell(\ell, x, z)\ell + \frac{p(\theta(z))\kappa_0}{(1 - p(\theta(z)))q(\theta(z))}\right]$$

and it further satisfies the limiting condition that the wage bill in a firm with no workers is zero, $\lim_{\ell \to 0} w(\ell, x, z)\ell = 0$. The proof of this result follows Elsby and Michaels (2013) and is contained in Appendix B.[25]

[25] Their model has no quits and a slightly different timing assumption (separated workers do not search in the same period). In the special case $f(\ell, x, z) = xz\ell^\alpha$, the wage is $w(\ell, x, z) = (1 - \eta)b + \eta \left[\frac{\alpha zx\ell^{\alpha-1}}{1-\eta(1-\alpha)} + \frac{p(\theta(z))\kappa_0}{(1-p(\theta(z)))q(\theta(z))}\right]$.

The Bellman equation describing the employment adjustment of firms is

$$v(\ell_-, x, z) = \max_{\ell} f(\ell, x, z) - w(\ell, x, z)\ell - \frac{\kappa_0}{q(\theta(z))} \max[\ell - \ell_-(1-s_0), 0] + \beta_0 \mathbb{E}_{x,z} v(\ell, x', z') \ . \quad (22)$$

With positive entry in all aggregate states, the free-entry condition is

$$c(z) = \sum_x \pi_0(x) v(0, x, z) \ . \quad (23)$$

Equations (22) and (23) define policy and value function as well as market tightness $\theta(z)$ in all aggregate states in a BRE. The aggregate resource constraint (21), with $\theta(z)$ independent of (ℓ_-, x), takes the form of the market-clearing condition in Definition 1(c) of the general model framework with appropriately defined absorption function[26] $\hat{A}(\ell, \ell_-, \theta) > \ell$.

3.2.2 Competitive Search

Kaas and Kircher (2015) show that the competitive search equilibrium is constrained efficient in the sense that it maximizes the discounted value of aggregate surplus subject to the matching of workers and firms in different submarkets. At the beginning of each period, the planner takes the distribution of incumbent firms μ and the exogenous aggregate state z as given. The recursive formulation of the planner's problem is

$$\mathcal{V}(\mu, z) = \max \int f(\ell, x, z) - b\ell - \kappa(V) d\tilde{\mu}(\ell_-, x) - c(z)m + \beta \mathbb{E}_z \mathcal{V}(\mu', z') \ , \quad (24)$$

subject to $\tilde{\mu} = \mu + m \sum_{x \in X} \mu_x^0$ (the measure of active firms after entry), $\ell = \ell_-(1-s) + q(\theta)V$ (employment in firm (ℓ_-, x) after separations and hires), next period's firm distribution μ' given by

$$\mu'(A_0 \times X_0) = (1 - \xi) \int \sum_{x' \in X_0} \pi_{xx'} \mathbb{I}\big(\hat{\ell}(\ell_-, x, z) \in A_0\big) d\tilde{\mu}(\ell_-, x) \ ,$$

for all Borel sets $A_0 \times X_0$, and subject to the aggregate resource constraint (21). Maximization in (24) is over the entry measure $m \geq 0$ and over separation rates s, vacancies V and market tightness θ, specific for each firm (ℓ_-, x) in the support of the firm distribution $\tilde{\mu}$.

Consider a candidate solution of the planner's problem with positive entry in all aggregate states, and write $\rho(z)$ for the multiplier on the aggregate resource constraint (21). Kaas and Kircher (2015) show that the solution of the planner's problem is isomorphic to (i) the maximization of the social surplus value of each firm (ℓ_-, x) and (ii) optimal entry.[27] The

[26] See equation (20). In the absence of search frictions with $p(\theta) = 1$, the absorption function is simply $\hat{A}(\ell, \ell_-, \theta) = \ell$, as in a model with competitive labor markets.

[27] The equivalence of (25) and (26) to the original problem (24) follows by writing the planner's value function in the form $\mathcal{V}(\mu, z) = \int v(\ell_-, x, z) d\mu(\ell_-, x)$.

surplus value of firm (ℓ_-, x) solves the recursive problem

$$v(\ell_-, x, z) = \max_{s,V,\theta,\ell} f(\ell, x, z) - b\ell - \kappa(V) - \rho(z) \left[\ell_-(1-s) + \frac{V}{q(\theta)}\right] + \beta_0 \mathbb{E}_{x,z} v(\ell, x', z') \quad (25)$$

subject to $\ell = \ell_-(1-s) + q(\theta)V$ and $s \geq s_0$. Optimal entry requires

$$c(z) = \sum_x \pi_0(x) v(0, x, z) \ . \quad (26)$$

In problem (25), the planner takes into account the shadow price $\rho(z)$ of the firm's workers after separations, $\ell_-(1-s)$, and of $V/q(\theta)$ unemployed workers searching for jobs at this firm. Using first-order conditions, the separation rate s, vacancies V and market tightness θ can be expressed as functions of past and current employment (ℓ_-, ℓ) and of the shadow price $\rho(z)$. Therefore (25) and (26) have the same structure as (2) and (3) in the general model framework. Moreover, the aggregate resource constraint (21) with $\theta(\ell_-, x, z)$ rewritten as $\theta = \Theta(\ell_-, \ell, \rho)$ has the same form as the market-clearing condition in Definition 1 with suitably defined absorption function $\hat{A}(\ell_-, \ell, \rho)$. Therefore, the existence results for the BRE apply to the planner's solution (and hence for the competitive search equilibrium) in this example.

3.3 Incomplete Markets with Asset Market Participation

Consider a Huggett (1993) model in which households decide about asset market participation. There is a mass $\bar{M}$ of households with discount factor β and period utility $u(c)$ where c is consumption. The economy can be in low-income aggregate state $z = z_L$ or in high income aggregate state $z = z_H$ with transition probabilities $\psi_{zz'}$, $z, z' \in \{z_L, z_H\}$. A household can be in one of two idiosyncratic states: employment $(x = E)$ or unemployment $(x = U)$ with transition probabilities $\pi_{xx'}(z)$. Income of a household in state (x, z) is denoted $y(x, z)$. Further, households are either inactive (hand-to-mouth) or active (traders).

Households die exogenously with probability ξ at the end of a period so that $\beta_0 = \beta(1-\xi)$ is the effective discount factor of a household. A dying household is replaced by a newborn household who enters the economy in the same income state as the exiting household. Newborn households enter the economy as hand-to-mouth consumers. Each period, a hand-to-mouth consumer decides whether to acquire the ability to become a trader household at utility cost $\tau(z)$.

The only asset in this example is a Lucas tree in unit supply which pays dividend income $d(z)$.[28] In a candidate BRE, the (ex-dividend) price of this asset depends only on the aggregate state and is denoted $q(z)$. Trader households can buy arbitrary shares of the asset, while

[28] Extensions with multiple assets are possible. A BRE then requires hand-to-mouth households who differ in some characteristics (income or preferences) and who become traders in all aggregate states.

short-selling is not permitted. If a household dies at the end of the period, its asset shares are redistributed to the other shareholders in proportion to their asset holdings. Hence, if a household buys s_t shares in period t and survives to the next period, the household enters period $t+1$ with $\frac{s_t}{1-\xi}$ shares. This implies that the ex-post asset return, conditional on survival, is $\frac{q(z_{t+1})+d(z_{t+1})}{q(z_t)(1-\xi)}$.

The Bellman equation of a trader household in income state (x, z) who held s shares of the asset in the previous period is

$$v(s, x, z) = \max_{s' \geq 0} u(c) + \beta_0 \mathbb{E}_{x,z} v(s', x', z')$$

subject to

$$c + q(z)s' \leq y(x, z) + \frac{q(z) + d(z)}{1 - \xi} s \ . \tag{27}$$

Let $s' = \hat{s}(s, x, z)$ denote the policy function of this problem. Suppose that parameters and asset prices are such that $\hat{s}(.)$ maps the compact interval $A \equiv [0, \bar{s}]$ into itself for a large enough upper bound $\bar{s} > 0$ and every $(x, z) \in \{E, U\} \times \{z_H, z_L\}$. The lifetime utility of a hand-to-mouth household satisfies

$$\bar{v}(x, z) = u(y(x, z)) + \beta_0 \mathbb{E}_{x,z} \bar{v}(x', z') \ , \text{ for all } (x, z) \ .$$

Consider a BRE in which a positive measure of employed hand-to-mouth households become traders in every aggregate state. Unemployed hand-to-mouth households do not become traders. The corresponding asset-market participation conditions are

$$v(0, E, z) - \tau(z) = \bar{v}(E, z) \ , \tag{28}$$

$$v(0, U, z) - \tau(z) < \bar{v}(U, z) \ , \tag{29}$$

for $z \in \{z_H, z_L\}$. In the BRE, the participation conditions (28) pin down the asset price in the two aggregate states $z = z_H, z_L$.

Let $\mu(s, x)$ denote the distribution of traders over idiosyncratic states (s, x) at the beginning of a period. When the exogenous aggregate state is z, the asset market clears if measure $m(z, \mu)$ of (employed) hand-to-mouth households decides to become traders such that

$$\int \hat{s}(s, x, z) d\mu(s, x) + \hat{s}(0, E, z) m(z, \mu) = 1 \ ,$$

which implies that the measure of new traders is

$$m(z, \mu) = \frac{1}{\hat{s}(0, E, z)} \left(1 - \int \hat{s}(s, x, z) d\mu(s, x) \right) \ . \tag{30}$$

Let $\tilde{\mu} = \mu + m(z,\mu)\delta_{(0,E)}$ denote the distribution measure of traders in aggregate state (z,μ) (with δ denoting the Dirac measure). The distribution measure of traders at the beginning of the next period is denoted μ' and satisfies

$$\mu'(A_0 \times X_0) = (1-\xi) \int \mathbb{I}(\hat{s}(s,x,z) \in A_0) \cdot \left(\sum_{x' \in X_0} \pi_{xx'}(z) \right) d\tilde{\mu}(s,x) \ , \tag{31}$$

for all Borel sets $A_0 \subset A$ and $X_0 \subset \{E, U\}$. As described in Section 2, equations (30) and (31) can be expressed by two affine-linear functions, denoted Ψ_H and Ψ_L, mapping the space of distribution measures on $A \times X$ into itself.

Consider a numeric example to illustrate the existence (or non-existence) of a BRE. The period length is a year and $\xi = 0.025$ is set to reflect a 40-year working life. The utility function is $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with relative risk aversion $\gamma = 2$ and the discount factor is set to $\beta = 0.93$. Labor incomes of employed households are $1 \pm \varepsilon$ in aggregate states $z = z_H, z_L$ where $\varepsilon > 0$ controls cyclicality of aggregate labor income. Transitions between aggregate states occur with probability 0.5. Unemployment income is constant at 0.4 in both aggregate states. Transitions from E to U (U to E) occur with probability 0.1 (0.9, resp.) so that the unemployment rate is constant at 10 percent.

The dividend is constant at $d = 0.02$ and the asset price fluctuates around $\bar{q} = 0.4$ so that the average asset return is around 5%. The utility cost to become a trader is calibrated such that the asset price in the high (low) aggregate state is $q(z_H) = \bar{q}(1+\alpha)$ ($q(z_L) = \bar{q}(1-\alpha)$, resp.). Parameter α, which may be positive, zero, or negative, controls the cyclicality of the endogenous asset price. Solving the model for a steady state ($\varepsilon = \alpha = 0$) yields an equilibrium stationary mass of traders μ which averages to $\bar{\mu} \approx 0.82$. The mass of households is set to $\bar{M} = 2\bar{\mu}$, reflecting that half of all households are stock owners.

To explore the existence of BRE in the presence of aggregate shocks, set $\varepsilon = 0.005$, i.e. aggregate labor income has a standard deviation of about 0.5%. Assume first that $\alpha = 0.01$ which implies that the asset return fluctuates procyclically between 3 and 7 percent across aggregate states. This choice of α implies that the utility cost to become a trader must be set to $\tau(z_H) = 0.5697$ and $\tau(z_L) = 0.5700$ and hence is almost constant over time. The utility gain to become a trader is 0.5114 for unemployed households (the same in both aggregate states); therefore, these households decide not to become traders in equilibrium. This is intuitive since these households would start trading only in the first period after they find a job.

For this parameterization, the smaller (red) points in Figure 4 show the share of traders and new traders (among all households) in a simulation of this model for 20,000 periods where the first 500 periods are discarded. Hence these points illustrate a projection of the stationary ergodic distribution in the numeric approximation of a BRE.[29] The BRE exists in this example

[29] The simulation uses a grid of 100 equally-spaced points for assets, which implies that the measure of traders in the approximate equilibrium has dimension 200 (two idiosyncratic income states and 100 asset levels). The approximate invariant ergodic measure is a probability distribution on $\mathbb{R}^{200} \times \{L, H\}$ (cf. Section 2.4).

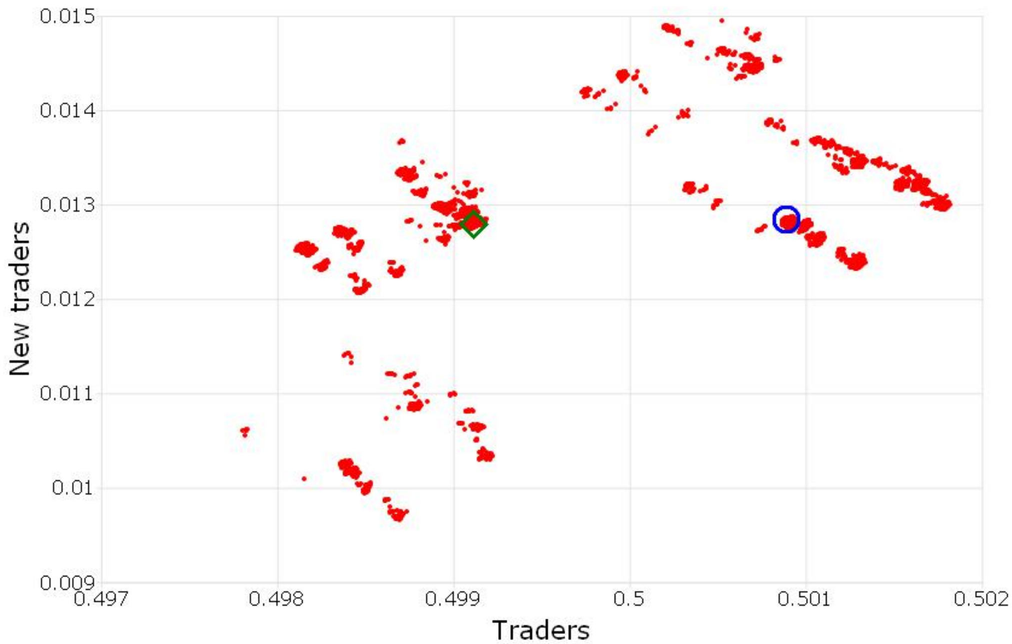

Figure 4: Shares of traders and new traders in the BRE with procyclical asset price ($\alpha = 0.01$).
Note: The red dots show model outcomes for a simulation for 20,000 periods where the first 500 periods are discarded. The blue circle (green diamond) shows the stationary state if this model realizes the high-income (low-income) state in all periods.

because the measure of new traders is strictly positive in this asymptotic equilibrium.

The two larger points in this figure show the hypothetical outcomes if this economy would realize either the high-income state (blue circle) or the low-income state (green diamond) in all periods. In fact, the red points in the upper right (lower left) half of this figure are realizations for high (low) aggregate productivity. Therefore, asset market participation and entry into asset market trade are procyclical.

The existence of a BRE requires a particular configuration of participation costs (and hence, equilibrium prices). If the model is calibrated such that the asset price is either constant across aggregate states ($\alpha = 0$) or twice as cyclical (e.g., $\alpha = 0.02$), then no BRE exists.[30] This is illustrated in the two graphs of Figure 5. In both cases, the dynamics of the distribution measure μ converges to an ergodic distribution (because the affine-linear maps Ψ_H and Ψ_L are contractions). However, entry into asset market trade is not positive in all aggregate states of the invariant ergodic distribution: Both graphs reveal occasionally a negative number of new traders which is inconsistent with equilibrium.

[30]The first case ($\alpha = 0$) requires procyclical participation costs, $\tau(z_H) = 0.5717 > \tau(z_L) = 0.5679$, whereas a more volatile asset price ($\alpha = 0.02$) requires countercyclical participation costs, $\tau(z_H) = 0.5678 < \tau(z_L) = 0.5721$.

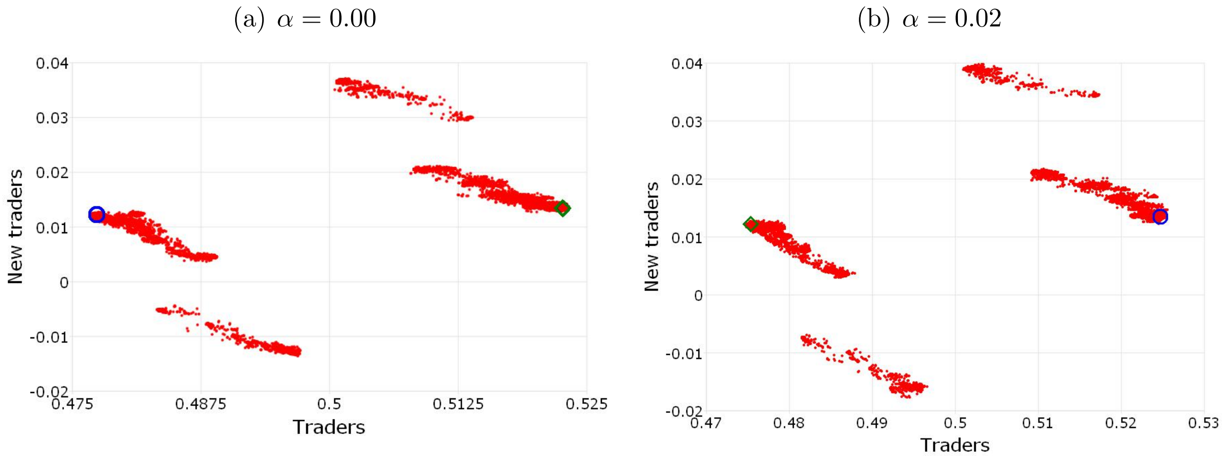

Figure 5: Non-existence of BRE for $\alpha = 0$ (constant asset price) and $\alpha = 0.02$ (strongly procyclical asset price): The hypothetical shares of traders and new traders in a candidate equilibrium are shown by the small (red) points. Equilibrium does not exist because the share of new traders occasionally becomes negative.

Note: See the notes for Figure 4.

4 Conclusions and Extensions

This paper presents a general model framework and derives existence and ergodicity results for recursive equilibria which have a particularly simple structure: Value and policy functions of heterogeneous agents vary with the exogenous aggregate state but are independent of the high-dimensional cross-sectional distribution. This simplification is feasible due to block recursivity. Market prices and value functions are solved via participation conditions of market entrants (the first block), while entry is consistent with market clearing (the second block which remains high-dimensional).

Using several dynamic equilibrium models with heterogeneous firms and heterogeneous households, I demonstrate how this framework can be applied and under which conditions the existence of block-recursive equilibria can be guaranteed and when it fails. The crucial requirements are first, the stability of the stationary equilibrium (which is itself difficult to analyze in infinite-dimensional state spaces), and second, that aggregate uncertainty is not too large so that entry stays positive in all states of the world. The examples are deliberately simple and illustrative; they can certainly be extended to include, for example, firm dynamics with more than two factor inputs or incomplete-market economies in which households trade multiple assets.

Although this framework covers some standard dynamic models, it does not include models with endogenous aggregate state variables. However, block-recursive equilibria may also exist in such broader classes of models, as I will explain in the remaining few paragraphs. Exploring such extensions in detail should be an interesting avenue for future research.

If there are endogenous aggregate state variables which do not enter the agents' payoff functions, extending the previous results is relatively straightforward. As an example, consider again the model of Section 3.3 and suppose that the Lucas tree is not in fixed unit supply. Instead the tree depreciates at rate δ at the end of every period, whereas a production sector produces new trees with strictly increasing and convex cost function C. Tree producers in period t maximize $q_t I_t - C(I_t)$ which leads to $C'(I_t) = q_t$. The stock of new trees adjusts to $K_{t+1} = (1-\delta)K_t + I_t$. This model permits a block-recursive equilibrium in which the price $q(z)$ is determined via the asset-market participation conditions (28). The Bellman equation is the same as in the model with a fixed asset supply, but the asset return in the budget constraint (27) is diminished to $\frac{(1-\delta)q(z)+d(z)}{1-\xi}$ because of the depreciation of the asset.

If the aggregate state is (z_n, K, μ), the capital market clears if the measure of employed households starting to trade assets is

$$m(z_n, K, \mu) = \frac{1}{\hat{s}(0, E, z_n)} \left(K - \int \hat{s}(s, x, z_n) d\mu(s, x) \right) .$$

Therefore, $\tilde{\mu} = \mu + m(z_n, K, \mu)\delta_{(0,E)}$ is the distribution measure of traders in aggregate state (z_n, K, μ). The distribution measure of traders at the beginning of the next period is then

$$\mu' = T_n^* \tilde{\mu} = S_n^* \mu + \mu_n^* K \ , \tag{32}$$

where linear operator S_n^* and $\mu_n^* \in \mathcal{M}_+(A, X)$ as in the model without depreciation and production of trees. The stock of Lucas trees in the next period is

$$K' = (1 - \delta)K + (C')^{-1}(q(z_n)) \ . \tag{33}$$

This shows that the joint dynamics of (K, μ) is described by N affine linear mappings $\Psi_n : \mathbb{R}_+ \times \mathcal{M} \to \mathbb{R}_+ \times \mathcal{M}$.[31] It is straightforward to extend Theorem 2 (Existence) and Theorem 3 (Ergodicity) under appropriate modifications of the assumptions. In particular, all mappings Ψ_n must be contractions and their unique fixed points must be sufficiently close as specified in Proposition 3.

Matters are more complex if there are endogenous state variables which (directly or indirectly) enter the agents' payoff functions. Consider a modified version of the model of Aiyagari (1994). There is a unit mass of households that differ in their stochastic labor efficiency x. Similar to the example of Section 3.3, households enjoy utility $u(c)$ of consumption and they discount future utility with factor $\beta_0 = \beta(1 - \xi)$ where ξ is the death rate. Dying households

[31] Another example with a similar structure is an extension of the Hopenhayn (1992) model with capital accumulation.

are replaced by newborns who enter as hand-to-mouth consumers. A hand-to-mouth household with labor efficiency x may become a trader household at utility cost $\tau(x,z)$. Trader households can buy shares of capital $a' \geq 0$ which earn gross return $R(z',K')$ in the next period. The real wage per efficiency unit $w(z,K)$ and the real gross return $R(z,K)$ are determined from the marginal products of firms operating the constant returns production function $F(z,K,L)$, i.e. $w(z,K) = F_L(z,K,1)$ and $R(z,K) = 1-\delta+F_K(z,K,1)$ where δ is the capital depreciation rate and where average labor efficiency is normalized to one.

Consider a candidate block-recursive equilibrium in which value functions depend only on (z,K) but not on the cross-sectional distribution. The Bellman equation of a trader household with capital holdings a and labor efficiency x is

$$v(a,x,z,K) = \max_{a'\geq 0} u\left(xw(z,K) + R(z,K)a - a'\right) + \beta_0 \mathbb{E}_{x,z} v(a',x',z',K') \,. \tag{34}$$

The utility value of hand-to-mouth households solves

$$\bar{v}(x,z,K) = u\left(xw(z,K)\right) + \beta_0 \mathbb{E}_{x,z} \bar{v}(x',z',K') \,. \tag{35}$$

Both Bellman equations include a forecast of the capital stock in the next period K'. In a block-recursive equilibrium, this capital stock must be independent of the cross-sectional distribution. In other words, the current aggregate state (z,K) is sufficient to forecast next period's capital stock $K' = \hat{K}(z,K)$.[32] Such a solution is compatible with equilibrium, if there is a unique hand-to-mouth household type x^e who decides to participate in asset market trade in every aggregate state. That is,

$$v(0,x^e,z,K) = \tau(x^e,z) + \bar{v}(x^e,z,K) \tag{36}$$

holds in all relevant aggregate states (z,K) of this model (while all other hand-to-mouth households do not participate). Proving analytically that equations (34), (35) and (36) have a solution $(v,\bar{v},\hat{K})$ is a more intricate problem than before. Given a guess for function $\hat{K}$: $Z\times[0,K_0] \to [0,K_0]$ with appropriate upper bound for capital K_0, the Bellman equations (34) and (35) determine v and $\bar{v}$ uniquely, both of which depend implicitly on function $\hat{K}$. Therefore, the participation condition (36) can be understood as an implicit equation which maps function $\hat{K}$ into another function $\Phi(z,K) \equiv v(0,x^e,z,K) - \tau(x^e,z) - \bar{v}(x^e,z,K) : Z\times[0,K_0] \to \mathbb{R}$. If the functional equation $\Phi(.) = 0$ can be inverted for a class of functions defined in the neighborhood of a steady-state equilibrium $(\bar{z},\bar{K})$, a candidate for a block-recursive solution to equations (34)–(36) would exist.

Given such a solution, the dynamics of distribution measures is complicated by the fact that the aggregate capital stock enters policy functions. Let μ denote the cross-sectional

[32] This is reminiscent of the perceived law of motion in approximations of recursive equilibria (cf. Krusell and Smith (1998)) where households forecast next period's capital stock on the basis of today's aggregate state z and today's capital stock. If equilibrium is block-recursive, such a law of motion is exact.

distribution of trader households over idiosyncratic states (a, x) at the beginning of a period. If the current aggregate state is (z_n, K), next period's distribution takes the form

$$\mu' = S_n^*(K)\mu + \mu_n^*(K) \ , \tag{37}$$

where linear operators $S_n^*(.)$ and distribution measures $\mu_n^*(.)$ depend nonlinearly on K.[33] If the N mappings $(\mu, K) \mapsto (\mu', K')$ defined by (37) and $K' = \hat{K}(K, z_n)$ are contractive, existence and ergodicity of a block-recursive equilibrium can potentially be established with similar methods as in the proofs of Theorems 2 and 3.

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[33] From capital market equilibrium, the measure of new traders is

$$m = \frac{1}{a'(0, x^e, z_n, K)} \left(\hat{K}(z_n, K) - \int a'(a, x, z_n, K) d\mu(a, x) \right) \ ,$$

where $a'(.)$ is the policy function of problem (34). Then the measure of trader households in state (z, K) is $\tilde{\mu} = \mu + \delta_{(0,x^e)} m$. Next period's distribution measure is $\mu' = T_n^*(K)\tilde{\mu}$ where operator $T_n^*(K)$ depends on policy functions $a'(.)$, on the Markov process for x and on the death rate ξ. This gives rise to an equation of the form (37).

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Appendix A: Proofs

Lemma 1: *The Bellman equation*

$$v(a,x,n;\vec{z},\vec{p}) = \max_{a'\in\mathcal{A}(a,p_n,x,z_n)} u(a',a,p_n,x,z_n) + \beta_0(x,z_n)\mathbb{E}_{x,n}v(a',x',n';\vec{z},\vec{p}) \tag{38}$$

has a unique solution v with corresponding policy function $\hat{a}$. Both are continuous functions of $(a,\vec{z},\vec{p})$. Furthermore, the value function of inactive agents

$$\bar{v}(x,n;\vec{z}) = \bar{u}(x,z_n) + \beta_0(x,z_n)\mathbb{E}_{x,n}\bar{v}(x',n';\vec{z})$$

exists and is continuous in $\vec{z}$.

Proof of Lemma 1: Write $v_0 \equiv v - \bar{v}$ which satisfies the Bellman equation

$$v_0(a,x,n;\vec{z},\vec{p}) = \max_{a'\in\mathcal{A}(a,p_n,x,z_n)} \underbrace{u(a',a,p_n,x,z_n) - \bar{u}(x,z_n)}_{\equiv\, u_0(a',a,p_n,x,z_n)} + \beta_0(x,z_n)\mathbb{E}_{x,n}v_0(a',x',n';\vec{z},\vec{p}) \ . \tag{39}$$

Let $\mathcal{N} \equiv \{1,\ldots,N\}$ and let $\mathcal{C}$ be the set of real-valued continuous functions on the compact set $A \times X \times \mathcal{N} \times \bar{Z}^N \times P^N$. For any $v_0 \in \mathcal{C}$ and $n \in \mathcal{N}$, the objective function in (39) is continuous in $(a',a,\vec{z},\vec{p})$ because u, $\bar{u}$, $\beta_0(x,.)$ and transition probabilities $\pi_{xx'}(.)$ are continuous functions. Since $\mathcal{A}(a,p_n,x,z_n)$ is non-empty and compact valued, the maximum exists. Therefore, the right-hand side in (39) defines a function $Tv_0 : A \times X \times \mathcal{N} \times \bar{Z}^N \times P^N \to \mathbb{R}$, and the Theorem of the Maximum (Stokey et al., 1989, Theorem 3.6) with continuity of $\mathcal{A}$ imply that $Tv_0 \in \mathcal{C}$. Furthermore, when $\mathcal{C}$ is endowed with the sup norm, T is a contraction with modulus $\bar{\beta}_0 \equiv \beta \max_{x,z}[1 - \xi(x,z)] < 1$ because Blackwell's sufficient conditions apply, cf. Stokey et al. (1989, Theorems 3.3 and 9.6). By the contraction mapping theorem, T has a unique fixed point $v_0 \in \mathcal{C}$.

Using similar arguments, the simple Bellman equation for inactive agents has a unique solution $\bar{v}$ which is continuous in $\vec{z}$. Therefore, $v = v_0 + \bar{v}$ is the unique and continuous solution to the original Bellman equation (38). Both problems (38) and (39) have the same policy correspondence $G : A \times X \times \mathcal{N} \times \bar{Z}^N \times P^N \hookrightarrow A$ which is non-empty, compact-valued and upper hemi-continuous, cf. Stokey et al. (1989, Theorem 3.6).

The existence of a continuous policy function also uses standard arguments. Consider the subspace $\mathcal{C}' \subset \mathcal{C}$ which contains continuous functions which are concave in a. Take any arbitrary $v_0 \in \mathcal{C}'$. Because u is strictly concave in a', the right-hand side in (39) is strictly concave in a'. Therefore, the maximum is unique. Furthermore, $Tv_0 \in \mathcal{C}'$: Take any $a_0 \neq a_1$ and $(x,n,\vec{z},\vec{p})$, and let a_0' and a_1' denote the unique maxima of (39) for $a = a_0$ and $a = a_1$, respectively. By assumption on $\mathcal{A}(.)$, $a_\lambda' \equiv \lambda a_1' + (1-\lambda)a_0' \in \mathcal{A}(a_\lambda,p_n,x,z_n)$ with $a_\lambda \equiv$

$\lambda a_1 + (1-\lambda)a_0$ for all $\lambda \in [0,1]$. Therefore,

$$\begin{aligned} Tv_0(a_\lambda, x, n; \vec{z}, \vec{p}) &= \sup_{a'} u_0(a', a_\lambda, p_n, x, z_n) + \beta_0(x, z_n)\mathbb{E}_{x,n} v_0(a', x', n'; \vec{z}, \vec{p}) \\ &\geq u_0(a'_\lambda, a_\lambda, p_n, x, z_n) + \beta_0(x, z_n)\mathbb{E}_{x,n} v_0(a'_\lambda, x', n'; \vec{z}, \vec{p}) \\ &\geq \lambda \Big[u_0(a'_1, a_1, p_n, x, z_n) + \beta_0(x, z_n)\mathbb{E}_{x,n} v_0(a'_1, x', n'; \vec{z}, \vec{p}) \Big] \\ &\quad + (1-\lambda) \Big[u_0(a'_0, a_0, p_n, x, z_n) + \beta_0(x, z_n)\mathbb{E}_{x,n} v_0(a'_0, x', n'; \vec{z}, \vec{p}) \Big] \\ &= \lambda T v_0(a_1, x, n; \vec{z}, \vec{p}) + (1-\lambda) T v_0(a_0, x, n; \vec{z}, \vec{p}) \ . \end{aligned}$$

Here the second inequality follows since u is concave in (a', a) and since v_0 is concave in a'. This proves that $Tv_0 \in \mathcal{C}'$. Since $\mathcal{C}'$ is a closed subset of $\mathcal{C}$, it follows that the unique fixed point of T is in $\mathcal{C}'$. This implies that the (upper hemi-continuous) policy correspondence is a continuous policy function which is denoted $\hat{a}$. $\square$

Proposition 1: *Under Assumptions 1 and 2, the entry conditions (5) have a unique solution $\vec{p} = \tilde{p}(\vec{z})$ for all aggregate state vectors $\vec{z}$ in an open neighborhood of $\vec{\bar{z}}$. Function $\tilde{p}$ is continuous.*

Proof of Proposition 1: In order to apply the implicit function theorem, differentiate the $N \cdot I$ equations (for $x_i \in X^E$, $i = 1, \ldots I$, and $n = 1, \ldots, N$) in (5) with respect to the $N \cdot I$ prices (p^j_o). This gives rise to the Jacobian matrix

$$J = \left(w^{ij}_{no}\right) \qquad \text{with} \qquad w^{ij}_{no} \equiv \frac{\partial w(x_i, n; \vec{\bar{z}}, \vec{\bar{p}})}{\partial p^j_o} \ .$$

Here (j, o) are columns (commodity prices in different states) and (i, n) are rows (entrant types in different states). We can write this matrix in the form

$$J = \begin{pmatrix} W_{11} & \ldots & W_{1I} \\ \vdots & \ddots & \vdots \\ W_{I1} & \ldots & W_{II} \end{pmatrix} \ ,$$

with $N \times N$ submatrices $W_{ij} = (w^{ij}_{no})_{n,o}$. Assumption 2(b) says that for every i there exists some j such that W_{ij} is invertible.

For each (i, j, n), define $\bar{w}^{ij}_n \equiv \sum_{o=1}^N w^{ij}_{no} = \sum_{o=1}^N \frac{\partial w}{\partial p^j_o}(x_i, n, \vec{\bar{z}}, \vec{\bar{p}})$ which is the marginal change in the value of entrant i in aggregate state n if price j is marginally increased, uniformly in all aggregate states $o = 1, \ldots, N$. Note that the value function v (and therefore w) does not depend on the aggregate state index n at any constant aggregate state vector $\vec{z} = (z, \ldots, z)$ and constant aggregate price vector $\vec{p} = (p, \ldots, p)$. This is because Bellman equations (38) are identical for all values of n in a stationary environment. Hence, the unique fixed point of these Bellman equations cannot depend on n either. Therefore, one can write $\bar{w}^{ij} = \bar{w}^{ij}_n$ for all n. Assumption 2(a) says that the $N \times N$ matrix $(\bar{w}^{ij})_{i,j}$ is invertible.

Now suppose that the Jacobian matrix J is not invertible. Then there are K different rows (i_1, n_1), …, (i_K, n_K), and $\lambda_1, \dots, \lambda_K \in \mathbb{R}\setminus\{0\}$ such that

$$\sum_{k=1}^{K} \lambda_k w_{n_k o}^{i_k j} = 0 \quad \text{ for all } (j,o) \ . \tag{40}$$

It follows that

$$0 = \sum_{o=1}^{N} \sum_{k=1}^{K} \lambda_k w_{n_k o}^{i_k j} = \sum_{k=1}^{K} \lambda_k \bar{w}^{i_k j} \qquad \text{for all } j \ .$$

Then either $i_k \neq i_{k'}$ for some k and k', contradicting invertibility of $(\bar{w}^{ij})$ (Assumption 2(a)). Alternatively, if $i_k = i$ for all k, Assumption 2(b) implies that there exists some j such that $W_{ij} = (w_{no}^{ij})_{n,o}$ is invertible. This contradicts (40) for $i_k = i$.

This demonstrates that the Jacobian matrix J is invertible. The implicit function theorem yields a solution $\vec{p} = \tilde{p}(\vec{z})$ for $\vec{z}$ close to $\vec{\bar{z}}$, with differentiable function $\tilde{p}$. □

Lemma 2: *Let $\mathcal{M}(A \times X)$ be the vector space of bounded, signed Borel measures on the compact set $A \times X$. Let $\mathcal{M}_+(A \times X)$ be the subset of bounded, nonnegative measures. Define the Kantorovich-Rubinstein metric by*

$$d_{p,q}(\mu,\nu) = \sup\left\{ \int f d\mu - \int f d\nu : f \in \mathrm{Lip}_{p,q} \right\} \ , \ \mu, \nu \in \mathcal{M}(A \times X) \ ,$$

where $\mathrm{Lip}_{p,q}$ are real-valued functions on $A \times X$ satisfying $|f(a_0,x) - f(a_1,x)| \le p|a_0 - a_1|$ and $|f|_s \le q$ with sup norm $|.|_s$ and $p, q > 0$. Write $d = d_{1,1}$. The following properties hold:

(a) *$(\mathcal{M}_+(A \times X), d)$ is a complete and separable metric space and d induces the weak topology on this measure space.*

(b) *$d(\mu, 0) = \mu(A \times X)$ for $\mu \in \mathcal{M}_+(A \times X)$*

(c) *$d(\mu,\nu) \ge |\mu(A \times X) - \nu(A \times X)|$*

(d) *$d(\mu,\nu) \le \|\mu - \nu\|$ where $\|.\|$ is the total variation norm.*

(e) *$d(\lambda\mu, \lambda\nu) = \lambda d(\mu,\nu)$ for $\lambda > 0$.*

(f) *$d(\mu + \zeta, \nu) = d(\mu, \nu - \zeta)$.*

(g) $\min(p,q) d_{1,1}(\mu,\nu) \le d_{p,q}(\mu,\nu) \le \max(p,q) d_{1,1}(\mu,\nu)$.

Proof of Lemma 2:

(a) follows from Theorems 8.3.2 and 8.9.4. in Bogachev (2007b). (b) and (c) follow by setting $f \triangleq 1$ or $f \triangleq -1$. (d) follows because the total variation norm satisfies

$$\|\mu\| = \sup\left\{ \int f d\mu \ : \ f \text{ measurable and } |f|_s \le 1 \right\} \ .$$

(e) and (f) follow because $d(.,.)$ is induced by a norm on $\mathcal{M}(A \times X)$. (g) follows because for every $f \in \mathrm{Lip}_{1,1}$, $\min(p,q) \cdot f \in \mathrm{Lip}_{p,q}$, and for every $f \in \mathrm{Lip}_{p,q}$, $\max(p,q) \cdot f \in \mathrm{Lip}_{1,1}$. $\square$

Lemma 3: *The function $(\mu, \vec{z}) \mapsto < a_n(.;\vec{z}), \mu >$ (from $\mathcal{M}(A \times X) \times \bar{Z}^N$ into $\mathbb{R}^I$) is continuous.*

Proof of Lemma 3:

(a) Continuity in $\vec{z}$: Take any μ, and $\vec{z}^{\,'} \neq \vec{z}$. Then

$$< a_n(.;\vec{z}^{\,'}), \mu > = \int a_n(a,x;\vec{z}^{\,'}) d\mu(a,x) \to \int a_n(a,x;\vec{z}) d\mu(a,x)$$

when $\vec{z}^{\,'} \to \vec{z}$ because a_n is continuous in a and in $\vec{z}$ and because $A \times X$ is compact.

(b) Continuity in μ: Take and $\vec{z}$ and $\mu' \neq \mu$. By Assumption 4 and the properties of the absorption function $\hat{A}(.)$, each of the I elements of function a_n is Lipschitz continuous with some Lipschitz parameter p. Because of continuity and since $A \times X$ is compact, each element of a_n is bounded in absolute value by some constant q. Therefore, for each element a_n^i, $i = 1, \ldots, I$:

$$\begin{aligned} \left| \int a_n^i(.;\vec{z}) d(\mu - \mu') \right| &\leq \sup \left\{ \int g d(\mu - \mu') \ : \ g \in \mathrm{Lip}_{p,q} \right\} \\ &\leq \max(p,q) \sup \left\{ \int g d(\mu - \mu') \ : \ g \in \mathrm{Lip}_{1,1} \right\} = \max(p,q) d(\mu, \mu') \ . \end{aligned}$$

The second inequality follows from Lemma 2(g). This proves continuity of $< a_n^i(.,\vec{z}), \mu >$ (and hence $< a_n(.;\vec{z}), \mu >$) in μ. $\square$

Proposition 2: *Suppose that a continuous price function $\vec{p} = \tilde{p}(\vec{z})$ exists in a neighborhood of $\bar{\vec{z}}$ (such as the one derived from Proposition 1). Further suppose that Assumptions 1, 3, 4 and 5 are fulfilled. Then for every sufficiently small $\varepsilon_\mu > 0$ there exists $\varepsilon_z > 0$ such that for every $\vec{z}$ with $|\vec{z} - \bar{\vec{z}}| < \varepsilon_z$, all affine linear functions $\Psi_n : \mu \mapsto S_n^*(\vec{z})\mu + \mu_n^*(\vec{z})$ map the open ball of non-negative measures around $\bar{\mu}$ of radius ε_μ into itself, i.e., $\Psi_n(B(\bar{\mu}, \varepsilon_\mu)) \subset B(\bar{\mu}, \varepsilon_\mu)$ where $B(\bar{\mu}, \varepsilon_\mu) \equiv \{\mu \in \mathcal{M}_+(A \times X) \ : \ d(\mu, \bar{\mu}) < \varepsilon_\mu\}$.*

Proof of Proposition 2: Let

$$\begin{aligned} B(\bar{\vec{z}}, \varepsilon_z) &= \left\{ \vec{z} \in \bar{Z}^N \ : \ |\vec{z} - \bar{\vec{z}}| < \varepsilon_z \right\} \\ B(\bar{\mu}, \varepsilon_\mu) &= \{ \mu \in \mathcal{M}_+(A \times X) \ : \ d(\mu, \bar{\mu}) < \varepsilon_\mu \} \end{aligned}$$

be open balls around the stationary values $\bar{\vec{z}}$ and $\bar{\mu}$. Note that $B(\bar{\mu}, \varepsilon_\mu)$ includes non-negative measures only. Lemma 4 below shows that for ε_z and ε_μ small enough, every Ψ_n maps $B(\bar{\mu}, \varepsilon_\mu)$ into $\mathcal{M}_+(A \times X)$ when $\vec{z} \in B(\bar{\vec{z}}, \varepsilon_z)$.

Now take any small enough ε_μ and ε_z with the property $\Psi_n(B(\bar{\mu}, \varepsilon_\mu)) \subset \mathcal{M}_+(A \times X)$ for $|\vec{z} - \bar{\vec{z}}| < \varepsilon_z$. To prove the assertion of this proposition requires to set ε_z sufficiently small such that $\Psi_n(B(\bar{\mu}, \varepsilon_\mu)) \subset B(\bar{\mu}, \varepsilon_\mu)$ for $|\vec{z} - \bar{\vec{z}}| < \varepsilon_z$.

Write $\bar{\Psi}$ for the map $\mu \mapsto \bar{S}^*\mu + \bar{\mu}^*$ which has fixed point $\bar{\mu}$. By Assumption 5, $\bar{\Psi}$ is a λ-contraction. Define a metric on the set of continuous linear operators $T^* : \mathcal{M}(A \times X) \rightarrow \mathcal{M}(A \times X)$ by

$$d^O(T^*, T^{*'}) \equiv \sup \left\{ d(T^*\mu, T^{*'}\mu) \ : \ \mu \in \mathcal{M}_+(A \times X) \text{ s.t. } \mu(A \times X) = 1 \right\} \ . \tag{41}$$

Since the subset of probability measures on a compact set is compact in the weak topology (see Theorem 8.9.3 in Bogachev (2007b)) and T^*, $T^{*'}$ are continuous, the supremum is finite, so that d^O is a well-defined metric. It is straightforward to verify $d(T^*\mu, T^{*'}\mu) \leq d^O(T^*, T^{*'})\mu(A \times X)$ for every $\mu \in \mathcal{M}_+(A \times X)$ and continuous linear operators T^*, $T^{*'}$. In Lemma 5 below I prove that $S_n^*(\vec{z})$ are continuous operators on $\mathcal{M}(A \times X)$ so that the metric d^O can be applied to these operators.

For any $\mu \in B(\bar{\mu}, \varepsilon_\mu)$,

$$\begin{aligned} d(\Psi_n\mu, \bar{\mu}) \leq \ & d(\Psi_n\mu, \bar{\Psi}\mu) + d(\bar{\Psi}\mu, \bar{\mu}) \\ = \ & d(S_n^*(\vec{z})\mu + \mu_n^*(\vec{z}), \bar{S}^*\mu + \bar{\mu}^*) + d(\bar{\Psi}\mu, \bar{\Psi}\bar{\mu}) \\ \leq \ & d(S_n^*(\vec{z})\mu, \bar{S}^*\mu + \bar{\mu}^* - \mu_n^*(\vec{z})) + \lambda d(\mu, \bar{\mu}) \\ \leq \ & d(S_n^*(\vec{z})\mu, \bar{S}^*\mu) + d(\bar{S}^*\mu, \bar{S}^*\mu + \bar{\mu}^* - \mu_n^*(\vec{z})) + \lambda\varepsilon_\mu \\ \leq \ & d^O(S_n^*(\vec{z}), \bar{S}^*)\mu(A \times X) + d(\mu_n^*(\vec{z}), \bar{\mu}^*) + \lambda\varepsilon_\mu \ . \end{aligned} \tag{42}$$

Here, the inequalities in the third and fifth line make use of Lemma 2(f). Note that $\mu(A \times X) = d(\mu, 0) \leq d(\mu, \bar{\mu}) + d(\bar{\mu}, 0) < \varepsilon_\mu + \bar{\mu}(A \times X)$, so that $\mu(A \times X)$ is bounded above. Lemma 6 below shows that $\vec{z} \mapsto S_n^*(\vec{z})$ is a continuous function from $\bar{Z}^N$ into the operator space on $\mathcal{M}(A \times X)$ with the d^O metric. Lemma 7 shows that $\vec{z} \mapsto \mu_n^*(\vec{z})$ from $\bar{Z}^N$ into $(\mathcal{M}(A \times X), d)$ is continuous as well. Therefore, the first two terms in the last line of (42) are arbitrarily small for small enough ε_z (because of $S_n^*(\vec{\bar{z}}) = \bar{S}^*$ and $\mu_n^*(\vec{\bar{z}}) = \bar{\mu}^*$). Since $\lambda < 1$, all three terms together, and therefore $d(\Psi_n\mu, \bar{\mu})$, are smaller than ε_μ for small enough ε_z. This shows that Ψ_n maps $B(\bar{\mu}, \varepsilon_\mu)$ into itself for small enough ε_z. □

Lemma 4: *For ε_z and ε_μ small enough and $\vec{z}$ in an ε_z-neighborhood of $\vec{\bar{z}}$, all functions Ψ_n map measures $\mu \in B(\bar{\mu}, \varepsilon_\mu)$ into a non-negative measure.*

Proof of Lemma 4: Recall the entry vectors

$$m_n = A_n^0(\vec{z})^{-1} \Big[S(\tilde{p}_n(\vec{z}), z_n) - < a_n(.; \vec{z}), \mu > \Big] \in \mathbb{R}^I \ .$$

By Lemma 3, $(\mu, \vec{z}) \mapsto < a_n(.; \vec{z}), \mu >$ depends continuously on μ and on $\vec{z}$. By assumption on S, by continuity of $\tilde{p}$, the policy function and the absorption function $\hat{A}$, the matrix $A_n^0(.)^{-1}$ and the vector $S(.)$ also depend continuously on $\vec{z}$. Since $m_n = \bar{m} \in \mathbb{R}_{++}^I$ at $(\mu, \vec{z}) = (\bar{\mu}, \vec{\bar{z}})$, m_n is a strictly positive vector for $\vec{z}$ and μ in ε_z (ε_μ, resp.) neighborhoods of $\vec{\bar{z}}$ ($\bar{\mu}$, resp.) with ε_z and ε_μ small.

For any $\mu \in B(\bar{\mu}, \varepsilon_\mu)$ (which by definition includes non-negative measures), and $m_n \in \mathbb{R}^I_{++}$,

$$\tilde{\mu} = \mu + \sum_{x \in X^E} \mu^0_x m_n(x)$$

is a non-negative measure. Since operator $T^*_n(\vec{z})$ maps any non-negative measure into a non-negative measure, $\mu' = T^*_n(\vec{z})\tilde{\mu} = \Psi_n \mu$ is a non-negative measure. This proves the assertion of the lemma. $\square$

Lemma 5: *The linear operators $S^*_n(\vec{z})$ are continuous mappings from the metric space $(\mathcal{M}(A \times X), d)$ into itself.*

Proof of Lemma 5: Recall the definition

$$S^*_n(\vec{z}) \equiv T^*_n(\vec{z}) \circ [\mathbb{1} - \mu^0 A^0_n(\vec{z})^{-1} < a_n(.; \vec{z}), . >] \ .$$

For any real-valued measurable function f on $A \times X$ and any $\mu \in \mathcal{M}(A \times X)$,

$$\int f d(T^*_n(\vec{z})\mu) = \int [1 - \xi(x, z_n)] \mathbb{E}_{x,n} f(a_n(a, x; \vec{z}), x') d\mu(a, x) \ .$$

Here the expectations operator is over the realization of x' with probability $\pi_{xx'}(z_n)$. Therefore,

$$\begin{aligned}
\int f d(S^*_n(\vec{z})\mu) = & \int [1 - \xi(x, z_n)] \mathbb{E}_{x,n} f(a_n(a, x; \vec{z}), x') d\left(\left[\mathbb{1} - \mu^0 A^0_n(\vec{z})^{-1} < a_n(.; \vec{z}), . >\right] \mu\right)(a, x) \\
= & \int [1 - \xi(x, z_n)] \mathbb{E}_{x,n} f(a_n(a, x; \vec{z}), x') d\mu(a, x) \\
& - \sum_{x_i \in X^E} \left\{ \int [1 - \xi(x_0, z_n)] \mathbb{E}_{x_0,n} f(a_n(a_0, x_0; \vec{z}), x') d\mu^0_{x_i}(a_0, x_0) \sum_j a_{ij}(\vec{z}) \int a^j_n(.; \vec{z}) d\mu \right\} \ ,
\end{aligned}$$

where $a_{ij}(\vec{z})$ are the elements of $I \times I$ matrix $A^0_n(\vec{z})^{-1}$. Rearranging the integral terms in the last line, this can be written more compactly:

$$\int f d(S^*_n(\vec{z})\mu) = \int g_n((a, x), f, \vec{z}) d\mu(a, x) \ ,$$

where $g_n(., f, \vec{z})$ is the measurable function defined by

$$\begin{aligned}
g_n((a, x), f, \vec{z}) \equiv & \ [1 - \xi(x, z_n)] \mathbb{E}_{x,n} f(a_n(a, x; \vec{z}), x') \qquad (43) \\
& - \sum_{x_i \in X^E} \left[\int [1 - \xi(x_0, z_n)] \mathbb{E}_{x_0,n} f(a_n(a_0, x_0; \vec{z}), x') d\mu^0_{x_i}(a_0, x_0) \sum_j a_{ij}(\vec{z}) a^j_n(a, x; \vec{z}) \right] \ .
\end{aligned}$$

By Assumption 4, a_n is Lipschitz continuous in a. If $f \in \mathrm{Lip}_{1,1}$, then $g_n(., f, \vec{z})$, which is the sum, product and composite of bounded Lipschitz continuous functions, is also an element of $\mathrm{Lip}_{p,q}$ for some $p, q > 0$ which are independent of f.

Take any $\mu, \nu \in \mathcal{M}(A \times X)$. Then

$$\begin{aligned} d(S_n^*(\vec{z})\mu, S_n^*(\vec{z})\nu) = & \sup \left\{ \int g_n(., f, \vec{z}) d(\mu - \nu) \ : \ f \in \mathrm{Lip}_{1,1} \right\} \\ \leq & \sup \left\{ \int h d(\mu - \nu) \ : \ h \in \mathrm{Lip}_{p,q} \right\} \\ \leq & \max(p,q) d(\mu, \nu) \ , \end{aligned}$$

where the last inequality follows from Lemma 2(g). This proves that $S_n^*(\vec{z})$ is continuous. □

Lemma 6: *The map S_n^* from $\bar{Z}^N$ into the space of continuous, linear operators on $(\mathcal{M}(A \times X), d)$ endowed with the d^O metric defined in (41) is continuous.*

Proof of Lemma 6: Take any $\vec{z}, \vec{z}^{'} \in \bar{Z}^N$. Then

$$\begin{aligned} d^O(S_n^*(\vec{z}), S_n^*(\vec{z}^{'})) = & \sup \left\{ d(S_n^*(\vec{z})\mu, S_n^*(\vec{z}^{'})\mu) \ : \ \mu \geq 0, \mu(A \times X) = 1 \right\} \\ = & \sup \left\{ \int f d\left(S_n^*(\vec{z})\mu - S_n^*(\vec{z}^{'})\mu \right) \ : \ \mu \geq 0, \ \mu(A \times X) = 1, \ f \in \mathrm{Lip}_{1,1} \right\} \\ = & \sup \left\{ \int g_n(., f, \vec{z}) - g_n(., f, \vec{z}^{'}) d\mu \ : \ \mu \geq 0, \ \mu(A \times X) = 1, \ f \in \mathrm{Lip}_{1,1} \right\} \\ \leq & \sup \left\{ \sup_{(a,x)} \left| g_n(a, x, f, \vec{z}) - g_n(a, x, f, \vec{z}^{'}) \right| \ : \ f \in \mathrm{Lip}_{1,1} \right\} \ , \end{aligned}$$

with $g_n(., f, \vec{z})$ defined in (43). Now

$$\begin{aligned} & \left| g_n(a, x, f, \vec{z}) - g_n(a, x, f, \vec{z}^{'}) \right| \qquad (44) \\ \leq & \ \left| [1 - \xi(x, z_n)] \mathbb{E}_{x,n} f(a_n(a, x; \vec{z}), x') - [1 - \xi(x, z_n')] \mathbb{E}_{x,n} f(a_n(a, x; \vec{z}^{'}), x') \right| \\ & + \sum_{x_i \in X^E} \left| \int [1 - \xi(x_0, z_n)] \mathbb{E}_{x_0,n} f(a_n(a_0, x_0; \vec{z}), x') d\mu_{x_i}^0(a_0, x_0) \sum_j a_{ij}(\vec{z}) a_n^j(a, x; \vec{z}) \right. \\ & \left. - \int [1 - \xi(x_0, z_n')] \mathbb{E}_{x_0,n} f(a_n(a_0, x_0; \vec{z}^{'}), x') d\mu_{x_i}^0(a_0, x_0) \sum_j a_{ij}(\vec{z}^{'}) a_n^j(a, x; \vec{z}^{'}) \right| \ . \end{aligned}$$

Because $f \in \mathrm{Lip}_{1,1}$, the terms in this expression can be estimated above by the sum of several terms of the form,

$$\left| [1 - \xi(x, z_n)] \pi_{xx'}(\vec{z}) - [1 - \xi(x, z_n')] \pi_{xx'}(\vec{z}^{'}) \right| \ , \ \left| a_n(a, x; \vec{z}) - a_n(a, x; \vec{z}^{'}) \right| \ , \qquad (45)$$

$$\left| \int a_n(a_0, x_0; \vec{z}) - a_n(a_0, x_0; \vec{z}^{'}) d\mu_{x_i}^0 \right| \ , \ \left| a_{ij}(\vec{z}) - a_{ij}(\vec{z}^{'}) \right|$$

(multiplied by some constants which are independent of f). For instance, the first term on the right-hand side of (44) can be estimated above by

$$\begin{aligned}
&\Big|[1-\xi(x,z_n)]\mathbb{E}_{x,n}f(a_n(a,x;\vec{z}),x') - [1-\xi(x,z_n')]\mathbb{E}_{x,n}f(a_n(a,x;\vec{z}^{\,'}),x')\Big| \\
= &\ \Big|\sum_{x'}[1-\xi(x,z_n)]\pi_{xx'}(z_n)f(a_n(a,x;\vec{z}),x') - \sum_{x'}[1-\xi(x,z_n')]\pi_{xx'}(z_n')f(a_n(a,x;\vec{z}^{\,'}),x')\Big| \\
\leq &\ \left|\sum_{x'}\left([1-\xi(x,z_n)]\pi_{xx'}(z_n) - [1-\xi(x,z_n')]\pi_{xx'}(z_n')\right)f(a_n(a,x;\vec{z}),x')\right| \\
&+ \left|\sum_{x'}[1-\xi(x,z_n')]\pi_{xx'}(z_n')(f(a_n(a,x;\vec{z}),x') - f(a_n(a,x,\vec{z}^{\,'}),x'))\right| \\
\leq &\ \sum_{x'}|[1-\xi(x,z_n)]\pi_{xx'}(z_n) - [1-\xi(x,z_n')]\pi_{xx'}(z_n')| + \left|f(a_n(a,x;\vec{z}),x') - f(a_n(a,x,\vec{z}^{\,'}),x')\right| \ .
\end{aligned}$$

The other terms can be estimated similarly, making use of the triangle inequality and the fact that any continuous function on the compact domains $A\times X$ or $\bar{Z}^N$ is bounded above. Because of continuity, all terms in (45) go to zero, uniformly over $(a,x)\in A\times X$ (compact), when $\vec{z}^{\,'}\to\vec{z}$. Hence the left-hand side in (44) goes to zero, uniformly in (a,x) and independently of $f\in \mathrm{Lip}_{1,1}$. This shows that $S_n^*(\vec{z}^{\,'})\to S_n^*(\vec{z})$ in the d^O-topology. □

Lemma 7: *The map μ_n^* from $\bar{Z}^N$ into the measure space $(\mathcal{M}(A\times X),d)$ is continuous.*

Proof of Lemma 7: Take any $\vec{z},\vec{z}^{\,'}\in\bar{Z}^N$ and recall that

$$\mu_n^*(\vec{z}) \equiv T_n^*(\vec{z})\mu^0 A_n^0(\vec{z})^{-1}S(\tilde{p}_n(\vec{z}),z_n) \ .$$

Write $\mu_n^2(\vec{z})\equiv\mu^0 A_n^0(\vec{z})^{-1}S(\tilde{p}_n(\vec{z}),z_n)\in\mathcal{M}(A\times X)$. Because of continuity of $A_n^0(.)^{-1}S(\tilde{p}_n(.),.):\bar{Z}^N\to\mathbb{R}^I$, $\mu_n^2(.)$: is also continuous. Then,

$$\begin{aligned}
&d\left(T_n^*(\vec{z})\mu_n^2(\vec{z}),T_n^*(\vec{z}^{\,'})\mu_n^2(\vec{z}^{\,'})\right) = \sup\left\{\int fd(T_n^*(\vec{z})\mu_n^2(\vec{z})) - \int fd(T_n^*(\vec{z}^{\,'})\mu_n^2(\vec{z}^{\,'})) \ : \ f\in\mathrm{Lip}_{1,1}\right\} \\
&= \sup\left\{\int\sum_{x'}[1-\xi(x,z_n)]\pi_{xx'}(z_n)f(a_n(a,x;\vec{z}),x')d\mu_n^2(\vec{z})\right. \\
&\quad \left. - \int\sum_{x'}[1-\xi(x,z_n')]\pi_{xx'}(z_n')f(a_n(a,x;\vec{z}^{\,'}),x')d\mu_n^2(\vec{z}^{\,'}) \ : \ f\in\mathrm{Lip}_{1,1}\right\} \\
&\quad = \sup\left\{\int\Big[\sum_{x'}[1-\xi(x,z_n)]\pi_{xx'}(z_n)f(a_n(a,x;\vec{z}),x')\right. \qquad (46) \\
&\qquad - [1-\xi(x,z_n')]\pi_{xx'}(z_n')f(a_n(a,x;\vec{z}^{\,'}),x')\Big]d\mu_n^2(\vec{z}) \\
&\qquad \left. + \int\sum_{x'}[1-\xi(x,z_n')]\pi_{xx'}(z_n')f(a_n(a,x;\vec{z}^{\,'}),x')d(\mu_n^2(\vec{z})-\mu_n^2(\vec{z}^{\,'})) \ : \ f\in\mathrm{Lip}_{1,1}\right\}
\end{aligned}$$

For arbitrary $\varepsilon > 0$,

$$\left| \sum_{x'} [1-\xi(x,z_n)]\pi_{xx'}(z_n) f(a_n(a,x;\vec{z}),x') - [1-\xi(x,z_n')]\pi_{xx'}(z_n') f(a_n(a,x;\vec{z}^{\,'}),x') \right| < \varepsilon \ ,$$

for all $f \in \mathrm{Lip}_{1,1}$ and all $(a,x) \in A \times X$ when $\vec{z}^{\,'}$ is sufficiently close to $\vec{z}$. This follows because a_n, $\pi_{xx'}(.)$ and $\xi(x,.)$ are continuous, $A \times X$ is compact, and $f \in \mathrm{Lip}_{1,1}$. Hence, the first term in (46) goes to zero when $\vec{z}^{\,'} \to \vec{z}$. Regarding the second term, note that $(a,x) \mapsto [1-\xi(x,z_n')]\pi_{xx'}(z_n') f(a_n(a,x;\vec{z}^{\,'}),x')$ is a function $h \in \mathrm{Lip}_{p,q}$ for some $p,q > 0$ (independent of f) because a_n and $f \in \mathrm{Lip}_{1,1}$ are Lipschitz continuous functions. Since $\mu_n^2(\vec{z}^{\,'}) \to \mu_n^2(\vec{z})$ in the d-metric when $\vec{z}^{\,'} \to \vec{z}$ (see the beginning of this proof), this term goes to zero (because of Lemma 2(g)). This proves that $d(T_n^*(\vec{z})\mu_n^2(\vec{z}), T_n^*(\vec{z}^{\,'})\mu_n^2(\vec{z}^{\,'})) \to 0$ when $\vec{z}^{\,'} \to \vec{z}$ so that the assertion of the lemma follows. □

Theorem 1: *Under the assumptions of Proposition 2 together with Assumption 6, if $\varepsilon_z > 0$ is sufficiently small, there exists a block-recursive equilibrium for every $\vec{z}$ with $|\vec{z} - \vec{\bar{z}}| < \varepsilon_z$.*

Proof of Theorem 1: We know from the proof of Lemma 4 that the entry vector

$$m_n = A_n^0(\vec{z})^{-1} \Big[S(\tilde{p}_n(\vec{z}), z_n) - < a_n(.;\vec{z}), \mu > \Big] \tag{47}$$

is strictly positive for $(\vec{z},\mu) \in B(\vec{\bar{z}}, \varepsilon_z) \times B(\bar{\mu}, \varepsilon_\mu)$. By Assumption 1 and continuity of v, $\bar{v}$, c and $\tilde{p}$,

$$\mathbb{E}_x v(0,x_0,n;\vec{z},\tilde{p}(\vec{z})) < \bar{v}(x,z_n) + c(x,z_n) \qquad \text{for all} \qquad x \in X \setminus X^E \ ,$$

for $|\vec{z} - \vec{\bar{z}}| < \varepsilon_z$. Hence no inactive agent $x \notin X^E$ wishes to enter. It remains to prove the last feature of Definition 1 (d2) saying that the number of entrant and incumbent agents is not constrained by the total number of agents. This is obvious if $\bar{M} = \infty$. Otherwise, write the dynamics of the distribution μ_X (Definition 1, d2) in matrix notation

$$\mu_X' = \Pi(z) \Big[\mu_X + (\Pi_0 - \mathbb{1}) m(z,\mu) \Big] \ , \tag{48}$$

where $\Pi(z)$ is the transpose of the Markov transition matrix $(\pi_{xx'}(z))_{xx'}$, Π_0 is the transpose of the matrix of conditional probabilities over entry states, $(\pi_0(x|\hat{x}))_{\hat{x}x}$, and $m(z,\mu) \in \mathbb{R}^{|X|}$ is the entry vector, defined by (47) for elements $x \in X^E$ and with zero entries for $x \notin X^E$. Assumption 1 says that for constant $z = \bar{z}$ a stationary equilibrium exists with constant entry vector $\bar{m}$ (which is positive at elements $x \in X^E$ and zero in all other elements). The stationary distribution measure $\bar{\mu}_X$ satisfies

$$\bar{\mu}_X = \Pi(\bar{z}) \Big[\bar{\mu}_X + (\Pi_0 - \mathbb{1}) \bar{m} \Big] \ .$$

The last feature of Definition 1 (d2) requires that

$$\bar{\mu}_X(x) > \bar{\mu}(A \times \{x\}) + \bar{m}(x) \quad \text{for all} \quad x \in X.$$

It must be shown that there is a set $\mathcal{M}_X$ of distribution measures μ_X which is invariant under (48) for $(\vec{z},\mu) \in B(\bar{\vec{z}},\varepsilon_z) \times B(\bar{\mu},\varepsilon_\mu)$ and $(\varepsilon_z,\varepsilon_\mu)$ small enough such that

$$\mu_X(x) > \mu(A \times \{x\}) + m(x) \tag{49}$$

for all $\mu_X \in \mathcal{M}_X$, entry vectors defined by (47), $\mu \in B(\bar{\mu},\varepsilon_\mu)$, and $(\vec{z},\mu) \in B(\bar{\vec{z}},\varepsilon_z) \times B(\bar{\mu},\varepsilon_\mu)$. By continuity of transition probabilities, we can set

$$\|\Pi(z) - \Pi(\bar{z})\|_1 < \varepsilon_0 \ ,$$

with matrix norm $\|.\|_1$ induced from the ℓ_1 vector norm, for arbitrary $\varepsilon_0 > 0$ for small enough $(\varepsilon_z,\varepsilon_\mu)$. Because of continuity of (47),

$$|m - \bar{m}| < \varepsilon_1 \ ,$$

for arbitrary $\varepsilon_1 > 0$ for small enough $(\varepsilon_z,\varepsilon_\mu)$. Consider an arbitrary $\varepsilon_2 > 0$ and the ε_2 open ball around $\bar{\mu}_X$ (in $|.|_1$ norm). For small ε_2, $B(\bar{\mu}_X,\varepsilon_2)$ is invariant under $\mu'_X = \Pi\Big[\mu_X + (\Pi_0 - \mathbb{1})m\Big]$ if $m \in B(\bar{m},\varepsilon_1)$ and $\Pi \in B(\Pi(\bar{z}),\varepsilon_0)$ for $(\varepsilon_0,\varepsilon_1)$ small enough. To prove this assertion, take $\mu_X \in B(\bar{\mu}_X,\varepsilon_2)$, $m \in B(\bar{m},\varepsilon_1)$ and $\Pi \in B(\Pi(\bar{z}),\varepsilon_0)$. By Assumption 6, $\|\Pi(\bar{z})\|_1 = \rho < 1$. Then,

$$\begin{aligned}
|\Pi(\mu_X + (\Pi_0 - \mathbb{1})m) - \bar{\mu}_X|_1 \leq\ & |\Pi(\mu_X + (\Pi_0 - \mathbb{1})m) - \Pi(\bar{z})(\mu_X + (\Pi_0 - \mathbb{1})m)|_1 \\
& + |\Pi(\bar{z})(\mu_X + (\Pi_0 - \mathbb{1})m) - \Pi(\bar{z})(\bar{\mu}_X + (\Pi_0 - \mathbb{1})\bar{m})|_1 \\
\leq\ & \|\Pi - \Pi(\vec{z})\|_1 \cdot |\mu_X + (\Pi_0 - \mathbb{1})m|_1 \\
& + \|\Pi(\bar{z})\|_1 \left\{|\mu_X - \bar{\mu}_X|_1 + \|\Pi_0 - \mathbb{1}\|_1 \cdot |m - \bar{m}|_1\right\} \\
\leq\ & \varepsilon_0 \left\{|\bar{\mu}_X + (\Pi_0 - \mathbb{1})\bar{m}|_1 + \varepsilon_2 + \|\Pi_0 - 1\|_1 \varepsilon_1\right\} \\
& + \rho \left\{\varepsilon_2 + \|\Pi_0 - 1\|_1 \varepsilon_1\right\} \\
<\ & \varepsilon_2 \ .
\end{aligned}$$

Because of $\rho < 1$, the last inequality holds if ε_0, ε_1 and ε_2 are small enough.

For such values of ε_0, ε_1 and ε_2, $B(\bar{\mu}_X,\varepsilon_2)$ is invariant under $\mu'_X = \Pi\Big[\mu_X + (\Pi_0 - \mathbb{1})m\Big]$ if $m \in B(\bar{m},\varepsilon_1)$ and $\Pi \in B(\Pi(\bar{z}),\varepsilon_0)$. Therefore, $B(\bar{\mu}_X,\varepsilon_2)$ is invariant under (48) if $(\varepsilon_z,\varepsilon_\mu)$ is small enough. Then (49) holds when ε_2, ε_1 and ε_μ are set sufficiently small. This proves the last requirement of the theorem. $\square$

Proposition 3: *Suppose that Assumptions 7–10 are fulfilled. Define*

$$\eta \equiv \min_{n,i} \frac{\bar{m}_{n,i}}{\sup_{A\times X}|H_{n,i}(.)|} > 0 \qquad , \qquad \delta \equiv \max_{n,o} \|\bar{\mu}_n - \bar{\mu}_o\|_{TV} \geq 0 \ ,$$

where $\bar{m}_{n,i}$ is the ith component of vector $\bar{m}_n$, and $H_{n,i}$ is the ith component of the continuous function $(A_n^0)^{-1}a_n(.) : A \times X \to \mathbb{R}^I$. If $\delta < \eta(1-\lambda)$, then the set

$$\mathcal{M} \equiv \Big\{\mu \in \mathcal{M}_+(A \times X) \ : \ \|\mu - \bar{\mu}_n\|_{TV} < \eta \ \textit{for all}\ n\Big\}$$

is invariant under all mappings Ψ_n. Further, m_n as defined in (13) is strictly positive for all n and $\mu \in \mathcal{M}$.

Proof of Proposition 3: Take $\mu \in \mathcal{M}$ and consider

$$m_n = (A_n^0)^{-1} S(p_n, z_n) - < H_n(.), \mu > = \bar{m}_n - < H_n(.), \bar{\mu}_n - \mu > ,$$

where $\bar{\mu}_n$ is the unique fixed point of Ψ_n (Assumption 9 and Banach's fixed point theorem) and $\bar{m}_n = (A_n^0)^{-1} S(p_n, z_n) - < H_n(.), \bar{\mu}_n > \in \mathrm{I\!R}^I_{++}$ (Assumption 10). m_n is a strictly positive vector iff

$$< H_n(.), \bar{\mu}_n - \mu > \ll \bar{m}_n .$$

For each component of $\bar{m}_n$, this follows if $< H_{n,i}(.), \bar{\mu}_n - \mu > = \int H_{n,i} d(\bar{\mu}_n - \mu) \ < \bar{m}_{n,i}$. This follows if

$$\sup_{A \times X} |H_{n,i}(.)| \cdot \|\mu - \bar{\mu}_n\|_{TV} < \bar{m}_{n,i} \ , \text{ for all } i \text{ and } n \ ,$$

which holds by definition of $\mathcal{M}$ and η.

Because $m_n \in \mathrm{I\!R}^I_{++}$ for every $\mu \in \mathcal{M}$ and n, $\Psi_n \mu$ is a non-negative measure. This uses a similar argument as in the proof of Lemma 4: the measure of active agents after entry $\tilde{\mu}$ is non-negative, and T_n^* maps non-negative measures into non-negative measures. It remains to show that $\Psi_n \mu \in \mathcal{M}$. For every $o = 1, \ldots, N$,

$$\begin{aligned} \|\Psi_n \mu - \bar{\mu}_o\|_{TV} \leq & \ \|\Psi_n \mu - \bar{\mu}_n\|_{TV} + \|\bar{\mu}_n - \bar{\mu}_o\|_{TV} \\ \leq & \ \lambda \eta + \delta < \eta \ , \end{aligned}$$

where the last inequality follows from the assumption $\delta < \eta(1 - \lambda)$. This completes the proof of Proposition 3. □

Theorem 2: *For given $(z_1, \ldots, z_N) \in \bar{Z}^N$ a block-recursive equilibrium exists provided that Assumptions 7–11 are satisfied, $\bar{M}$ is sufficiently large, and $\delta < \eta(1 - \lambda)$ with δ and η defined in Proposition 3.*

Proof of Theorem 2: Proposition 3 implies that with equilibrium distributions in invariant set $\mathcal{M}$, all requirements of Definition 1 are fulfilled except (d2). This last requirement is not needed if $\bar{M} = \infty$. Otherwise, the distribution measures of active and inactive agents, denoted μ_X, evolve over time according to

$$\mu'_X = \Pi_n \mu_X \ , \tag{50}$$

where Π_n is the transpose of the Markov transition matrix $(\pi_{xx'}(z_n))_{xx'}$ (because of Assumption 11(i), entrants do not draw a new idiosyncratic state). By Assumption 11(ii), all matrices Π_n are contractions of modulus ρ, so that invariant measure $\bar{\mu}_{X,n}$ exists. Then with $\bar{d} \equiv \max_{n_1,n_2} |\bar{\mu}_{X,n_1} - \bar{\mu}_{X,n_2}|_1$, it is straightforward to verify that the set

$$\mathcal{M}_X \equiv \left\{ \mu_X \in \mathcal{M}(X, \bar{M}) \ : \ |\mu_X - \bar{\mu}_{X,n}|_1 \leq \frac{\bar{d}}{1 - \rho} \text{ for all } n \right\}$$

is invariant under all mappings Π_n. Furthermore, $\mathcal{M}_X \subset \mathbb{R}^{|X|}_{++}$. To show this, take any $\mu_X \in \mathcal{M}_X$, $x \in X$ and arbitrary n. Assumption 11(ii) implies $\bar{\mu}_{X,n} \in \mathbb{R}^{|X|}_{++}$. Then

$$\mu_X(x) \geq \bar{\mu}_{X,n}(x) - |\bar{\mu}_{X,n}(x) - \mu_X(x)| > 0 ,$$

if $|\bar{\mu}_{X,n}(x) - \mu_X(x)| < \bar{\mu}_{X,n}(x)$. The first inequality follows from

$$|\bar{\mu}_{X,n}(x) - \mu_X(x)| \leq |\bar{\mu}_{X,n} - \mu_X|_1 \leq \frac{\bar{d}}{1-\rho} \leq \min_{x \in X} \bar{\mu}_{X,n}(x) ,$$

where the last inequality follows from Assumption 11(ii). This proves that $\mathcal{M}_X$ contains strictly positive vectors. Since all these vectors sum up to $\bar{M}$, they can be made arbitrarily large if $\bar{M}$ is large enough. Because the invariant sets of distribution measures $\mathcal{M}$, $\mathcal{M}_X$ and entry vectors m are bounded (see Proposition 3), the last inequality in Definition 1, (d2) is satisfied for all $\mu \in \mathcal{M}$, entry vectors m defined by (13), and $\mu_X \in \mathcal{M}_X$, if $\bar{M}$ is sufficiently large. □

Theorem 3: *Let the assumptions of either Theorem 1 or Theorem 2 be fulfilled and let $(\mathcal{M}, d)$ be the invariant metric space implied by one of these theorems. Furthermore, let Assumption 12 be fulfilled. Then there exists a unique probability measure $\bar{Q} \in \mathbb{P}(\Omega)$ and a constant γ, such that for $\bar{\lambda} \equiv \max(\lambda, 1 - \frac{\varepsilon}{2}) < 1$,*

$$d_K(Q_t^{\omega_0}, \bar{Q}) \leq \gamma \bar{\lambda}^t \ , \quad \textit{for all } \omega_0 \in \Omega \textit{ and } t \geq 0 .$$

Proof of Theorem 3: To apply Theorem 2.1 of Stenflo (2001), the following two properties must be shown:

$$\mathbb{E}D(\Phi(\omega_1, y), \Phi(\omega_2, y)) \leq \bar{\lambda} D(\omega_1, \omega_2) \ \text{ for all } \ \omega_1, \omega_2 \in \Omega , \tag{51}$$

$$\mathbb{E}D(\omega_0, \Phi(\omega_0, y)) < \infty \ \text{ for some } \ \omega_0 \in \Omega , \tag{52}$$

where the expectations operators are over the realization of uniformly distributed $y \in [0, 1]$. Property (52) trivially follows because $\Omega = \mathcal{M} \times \mathcal{N}$ is bounded in the metric $D = d + \alpha\rho$.

Regarding property (51), define first the mapping $\hat{N} : \mathcal{N} \times [0, 1] \to \mathcal{N}$ inducing the Markov chain (ψ_{no}). Let $\{n_i \ : \ i = 1, \ldots, N\}$ be some ordering of $\mathcal{N}$ such that $n_1 = \bar{n}$ with $\psi_{n\bar{n}} \geq \varepsilon > 0$ (Assumption 12). For every $j = 1, \ldots, N$, let $\psi^0_{nn_j} = \sum_{i=1}^{j} \psi_{nn_i}$ denote the cumulative transition probability from n to next period's states $n' \in \{n_1, \ldots, n_j\}$. Then define

$$\hat{N}(n, y) = \left\{ n_i \ : \ i = \min\{j \ : \ y \leq \psi^0_{nn_j}\} \right\} .$$

When y is uniformly drawn from $[0, 1]$, then $\text{Prob}(\hat{N}(n, y) = n') = \psi_{nn'}$, hence the iterated function system defined by $\hat{N}$ induces the original Markov chain.

Next establish the following property:

$$\mathbb{E}\rho(\hat{N}(n,y),\hat{N}(o,y)) \leq (1-\varepsilon)\rho(n,o) \quad \text{for all} \quad n,o \in \mathcal{N} \ . \tag{53}$$

If $n = o$, (53) trivially holds since both sides are equal to zero. If $n \neq o$, then $\rho(n,o) = 1$. Let λ^* denote the Lesbesgue measure on the real numbers. Then

$$\begin{aligned}
\mathbb{E}\rho(\hat{N}(n,y),\hat{N}(o,y)) = & \ \text{Prob}\left(\hat{N}(n,y) \neq \hat{N}(o,y)\right) \\
= & \ 1 - \text{Prob}\left(\hat{N}(n,y) = \hat{N}(o,y)\right) \\
= & \ 1 - \lambda^* \left\{ \left([0,\psi^0_{nn_1}] \cap [0,\psi^0_{on_1}]\right) \cup \bigcup_{i=1}^{N-1} \left((\psi^0_{nn_i},\psi^0_{nn_{i+1}}] \cap (\psi^0_{on_i},\psi^0_{on_{i+1}}]\right) \right\} \\
\leq & \ 1 - \varepsilon \ ,
\end{aligned}$$

because $\psi^0_{nn_1} = \psi_{n\bar{n}} \geq \varepsilon$ for all $n \in \mathcal{N}$. This proves (53).

Finally, for any $\omega_1 = (\mu_1, n_1)$, $\omega_2 = (\mu_2, n_2) \in \Omega$,

$$\begin{aligned}
\mathbb{E}D(\Phi(\mu_1,n_1,y),\Phi(\mu_2,n_2,y)) = & \ d(\Psi_{n_1}\mu_1,\Psi_{n_2}\mu_2) + \alpha\mathbb{E}\rho(\hat{N}(n_1,y),\hat{N}(n_2,y)) \\
\leq & \ d(\Psi_{n_1}\mu_1,\Psi_{n_1}\mu_2) + d(\Psi_{n_1}\mu_2,\Psi_{n_2}\mu_2) + \alpha(1-\varepsilon)\rho(n_1,n_2) \\
\leq & \ \lambda d(\mu_1,\mu_2) + \bar{d}\cdot\rho(n_1,n_2) + \alpha(1-\varepsilon)\rho(n_1,n_2) \ ,
\end{aligned}$$

where $\bar{d} \equiv \sup\{d(\mu_1,\mu_2) \ : \ \mu_1,\mu_2 \in \mathcal{M}\}$. Here the second line uses property (53) and the third line uses that all mappings Ψ_n are λ-contractions. By definition of $\bar{\lambda} = \max(\lambda, 1-\frac{\varepsilon}{2})$ and $\alpha = \frac{2\bar{d}}{\varepsilon}$,

$$\lambda d(\mu_1,\mu_2) + (\bar{d} + \alpha(1-\varepsilon))\rho(n_1,n_2) \leq \bar{\lambda}\left(d(\mu_1,\mu_2) + \alpha\rho(n_1,n_2)\right) = \bar{\lambda}D(\omega_1,\omega_2) \ .$$

This proves property (51) and therefore the requirements of Theorem 2.1 of Stenflo (2001). □

Appendix B: Further Details

Firm Dynamics with Homogeneous Labor and History-Independent Firm Distribution

Suppose that the Markov process for idiosyncratic productivity has a unique ergodic distribution $\pi^*(x)$. Entrants draw initial productivity from this stationary distribution. Thus, the distribution over idiosyncratic productivities is independent of the dynamics of entry and aggregate shocks. In a BRE, write $w_n = w(z_n)$ for the wage and $v(x, n)$ for a firm's discounted profit value in aggregate state $n \in \{H, L\}$. Profit values satisfy

$$v(x, n) = \pi^*(w_n, xz_n) + \beta_0\psi\mathbb{E}_x v(x', n') + \beta_0(1-\psi)\mathbb{E}_x v(x', n) \ , \tag{54}$$

for $n' \neq n \in \{H, L\}$ and with π^* as defined in the main text. Entry conditions in both aggregate states are $c_n \equiv c(z_n) = \mathbb{E}v(x, n)$ where the expectations operator is over the initial realization of x, drawn from the stationary distribution π^*. Because of this feature and equation (54), the entry condition in aggregate state n is

$$c_n = A\zeta_n + \beta_0\psi c_{n'} + \beta_0(1-\psi)c_n \ ,$$

with $n' \neq n$ and $A \equiv \frac{\alpha}{1-\alpha}\mathbb{E}x^{1/\alpha}$, $\zeta_n \equiv z_n^{1/\alpha} w_n^{-(1-\alpha)/\alpha}$. These two equations can be solved for

$$\begin{aligned} \zeta_H &= \frac{1}{A}\left\{c_H\left(1-\beta_0(1-\psi)\right) - c_L\beta_0\psi\right\} \ , \\ \zeta_L &= \frac{1}{A}\left\{c_L\left(1-\beta_0(1-\psi)\right) - c_H\beta_0\psi\right\} \ . \end{aligned}$$

Hence, there are wages $w_n > 0$ consistent with entry in both aggregate states whenever $\zeta_n > 0$, $n \in \{H, L\}$. This is the case if

$$\max\left(\frac{c_H}{c_L}, \frac{c_L}{c_H}\right) < 1 + \frac{1-\beta_0}{\beta_0\psi} \ , \tag{55}$$

which requires that entry costs are not too cyclical. For instance, if $c_H = c_L$, then $\zeta_H = \zeta_L$ and wages are procyclical with $(w_H/w_L)^{1-\alpha} = z_H/z_L$. On the other hand, with moderately procyclical entry costs, wages are constant across aggregate states.

Suppose there are $\bar{\mu} > 0$ incumbent firms at the beginning of a period (whose idiosyncratic productivities are distributed with π^*) and that the aggregate state is $n \in \{H, L\}$. If there are $m > 0$ entrants, then aggregate labor demand is $\ell_n(\bar{\mu} + m)$ where $\ell_n \equiv \mathbb{E}(x^{1/\alpha})(z_n/w_n)^{1/\alpha}$ is average employment of a firm in state n. With labor supply $L_n \equiv (v')^{-1}(w_n)$, the labor market is in equilibrium if

$$L_n = \ell_n(\bar{\mu} + m) \ , \ n \in \{H, L\} \ .$$

Next period there are

$$\bar{\mu}' = (1-\xi)(\bar{\mu}+m) = (1-\xi)\frac{L_n}{\ell_n} \equiv \bar{\mu}_n^*$$

incumbent firms. Therefore, in this example, the affine-linear functions (10) mapping the current firm distribution into next period's distribution are constant functions (independent of μ), i.e. the firm distribution is independent of the history of aggregate shocks. The measures of incumbent firms alternate stochastically between two levels, $\bar{\mu}_H^*$ and $\bar{\mu}_L^*$. Entry is positive if, and only if, for all aggregate states $(n,n') \in \{H,L\}^2$,

$$\bar{\mu}_{n'}^* - (1-\xi)\bar{\mu}_n^* > 0 \ . \tag{56}$$

These considerations demonstrate the existence of a BRE when aggregate shocks are not too large: Then condition (55) ensures that entry conditions are fulfilled for wages $w_H, w_L > 0$, and (56) is satisfied (because $\bar{\mu}_n^*$ depend continuously on z_n).

Details about the Model in Section 3.1.1.

Here the following results mentioned in the main text are shown:

1. If entry costs c_n are sufficiently close across aggregate states $n = H, L$, there exist wages w_n such that the entry conditions are satisfied in both aggregate states.
2. A BRE exists if $S \in (0,1)$ and if the parameter conditions

$$\underline{\mu}^o \geq \max_{n,n'} \frac{R_n[(S+(1-\xi)(2-\lambda))] - R_{n'}}{(S+1-\xi)[S+(1-\xi)(1-\lambda)]} \ , \tag{57}$$

   and $R_H > SR_L$ are satisfied (if $R_H \leq R_L$; otherwise, if $R_L > R_H$, $R_L > SR_H$ must hold).

Regarding the first point, start with the recursive equations for profit values V_n^y, V_n^o for young and old firms in both aggregate states n, where $n' \neq n$:

$$\begin{aligned}
V_n^y &= \pi^*(w_n, x^y z_n) + \beta_0\psi\left[(1-\lambda)V_{n'}^y + \lambda V_{n'}^o\right] + \beta_0(1-\psi)\left[(1-\lambda)V_n^y + \lambda V_n^o\right] \ , \\
V_n^o &= \pi^*(w_n, x^o z_n) + \beta_0\psi V_{n'}^o + \beta_0(1-\psi)V_n^o \ .
\end{aligned}$$

With $\pi_n^i \equiv \pi^*(w_n, x^i z_n)$ for $i = y, o$, $n = H, L$, the profit values of young firms in both states are solved for

$$\begin{aligned}
V_L^y &= \frac{1}{1-\beta_0(1-\lambda)}\left\{\pi_L^y + \frac{\beta_0\lambda}{1-\beta_0}\pi_L^o + C^y(\pi_H^y - \pi_L^y) + C^o(\pi_H^o - \pi_L^o)\right\} \ , \\
V_H^y &= \frac{1}{1-\beta_0(1-\lambda)}\left\{\pi_H^y + \frac{\beta_0\lambda}{1-\beta_0}\pi_H^o - C^y(\pi_H^y - \pi_L^y) - C^o(\pi_H^o - \pi_L^o)\right\} \ ,
\end{aligned}$$

with positive constants

$$C^y \equiv \frac{\beta_0(1-\lambda)\psi}{1-\beta_0(1-\lambda)(1-2\psi)} \ , \ C^o \equiv \frac{\beta_0\lambda\psi}{1-\beta_0(1-2\psi)} \left\{ \frac{1}{1-\beta_0} + \frac{\beta_0(1-\lambda)(1-2\psi)}{1-\beta_0(1-\lambda)(1-2\psi)} \right\} \ .$$

Because $\pi_n^y = \zeta_n \pi^y$, $\pi_n^o = \zeta_n \pi^o$ with $\zeta_n \equiv z_n^{1/\alpha} w_n^{-(1-\alpha)/\alpha}$ and constants π^y, π^o, the entry conditions $c_n = V_n^y$ can be rewritten in the form

$$\begin{aligned} c_L = & \ A^0 \zeta_L + A^1(\zeta_H - \zeta_L) \ , \\ c_H = & \ A^0 \zeta_H - A^1(\zeta_H - \zeta_L) \ , \end{aligned}$$

with positive constants A^0, A^1. These equations can be solved uniquely for $\zeta_H > 0$ and $\zeta_L > 0$ (and hence for wages $w_H > 0$, $w_L > 0$), provided that entry costs c_H and c_L are sufficiently close.

Regarding the second statement, under the assumptions $S \in (0,1)$ and $R_H \leq R_L$, the iterated function system $(\mu^o)' = R_n - S\mu^o$, where n undergoes a Markov process on $\{H, L\}$ converges to an invariant ergodic distribution with support $(\mu^o, n) \in [\underline{\mu}^o, \overline{\mu}^o] \times \{H, L\}$ where

$$\underline{\mu}^o \equiv \frac{R_H - SR_L}{1-S^2} \leq \overline{\mu}^o \equiv \frac{R_L - SR_H}{1-S^2} \ ,$$

with equality if $R_H = R_L$. The lower bound $\underline{\mu}^o$ is strictly positive under the assumption $R_H > SR_L$. It remains to prove that entry is strictly positive in all aggregate states under the additional parameter condition (57). From (19),

$$m = \frac{L_n}{\ell_n^y} - \mu^y - \left(\frac{x^o}{x^y}\right)^{1/\alpha} \mu^o = \frac{R_n}{(1-\xi)\lambda} - \mu^y - \frac{S+1-\xi}{(1-\xi)\lambda}\mu^o \ .$$

From (17) and (18),

$$\begin{aligned} \mu^o = & \ R_{n_-} - S\mu^{o-} \ , \\ \mu^y = & \ \frac{1-\lambda}{\lambda} \left[R_{n_-} - (S+1-\xi)\mu^{o-} \right] \ , \end{aligned}$$

where μ^{o-} is the measure of old firms in the previous period and n_- is the aggregate state in the previous period. Hence, $m > 0$ iff

$$R_n > R_{n_-}\left[S + (1-\xi)(2-\lambda)\right] - (S+1-\xi)\left[S + (1-\xi)(1-\lambda)\right]\mu^o \ .$$

For $\mu^o \in [\underline{\mu}^o, \overline{\mu}^o]$, this inequality is fulfilled if $\underline{\mu}^o$ satisfies condition (57).

Endogenous Equilibrium Cycles in the Hopenhayn (1992) Model

This section describes equilibrium cycles in a special case of the model described in Section 3.1.1. To simplify, set $\lambda = 1$; that is, startups produce with low productivity x^y in the first period and then continue to operate with higher productivity $x^o > x^y$ in all subsequent periods. Suppose that the steady state of this model is unstable which happens when

$$S = (1-\xi)\left(\left(\frac{x^o}{x^y}\right)^{1/\alpha} - 1\right) > 1 \ ,$$

as has been shown in the main text.

Let labor supply be $L(w) = L_0 w^\gamma$ with scale parameter L_0 and Frisch elasticity γ. A competitive equilibrium is a sequence of wages w_t, distribution measures of incumbent firms μ_t^o, discounted profit values of incumbents v_t^o, and entry measures m_t such that

$$L_0 w_t^\gamma = \left[(x^o)^{\frac{1}{\alpha}}\mu_t^o + (x^y)^{\frac{1}{\alpha}} m_t\right] w_t^{-\frac{1}{\alpha}} \ , \tag{58}$$

$$\mu_{t+1}^o = (1-\xi)(\mu_t^o + m_t) \ , \tag{59}$$

$$v_t^o = \pi^o(w_t) + \beta_0 v_{t+1}^o \ , \tag{60}$$

$$m_t \geq 0 \ , \quad c \geq \pi^y(w_t) + \beta_0 v_{t+1}^o \quad \text{(c.s.)} \ , \tag{61}$$

where $\pi^i(w) = \frac{\alpha}{1-\alpha}(x^i)^{\frac{1}{\alpha}} w^{-\frac{1-\alpha}{\alpha}}$, $i = y, o$, are profits of young and old firms. Equation (58) is labor market equilibrium, (59) is the adjustment of incumbent firms over time, and (60) is the discounted profit value of incumbents. The last complementary-slackness condition says that when no firms enter, entry costs exceed the gains from entry, and they are equal otherwise.

This system of equations has a unique steady state with positive entry; indeed, this example is a special case of the classic Hopenhayn (1992) model. The steady state is locally unstable, however, when $S > 1$. The global dynamics of the system (58)–(61) are not straightforward to characterize, in particular because (61) may or may not bind and because (60) is a forward-looking equation which potentially gives rise to a large set of dynamic equilibria, including sunspot cycles. To simplify even further, shut down this forward-looking channel by setting the household discount factor to zero such that $\beta_0 = \beta(1-\xi) = 0$. In this case, condition (61) can be expressed more compactly as follows:

$$m_t \geq 0 \quad , \quad w_t \geq \bar{w} = \left(\frac{\alpha}{1-\alpha}\frac{1}{c}(x^y)^{1/\alpha}\right)^{\frac{\alpha}{1-\alpha}} \qquad \text{(c.s.)}.$$

Either entry is positive and the equilibrium wage equals $\bar{w}$ (i.e., the BRE wage consistent with entry), or entry is zero in which case the market-clearing wage exceeds $\bar{w}$. Solving this simple model shows that the dynamics reduces to one equation describing the evolution of the measure of incumbent firms:

$$\mu_{t+1}^o = \begin{cases} (1-\xi)\mu_t^0 & , \ \mu_t^0 \geq \hat{\mu}^o \ , \\ R - S\mu_t^o & , \ \mu_t^o < \hat{\mu}_t^o \ , \end{cases}$$

where $R = (1-\xi)(x^y)^{-1/\alpha} L_0 \bar{w}^{\gamma+\frac{1}{\alpha}}$ and S are defined as in the main text. If the measure of incumbent firms is below the threshold $\hat{\mu}^o = R/(S+1-\xi)$, entry is positive and the market-clearing wage equals $\bar{w}$. If the measure of incumbent firms exceeds $\hat{\mu}^o$, entry is zero, and the market-clearing wage satisfies $L_0 w_t^{\gamma+\frac{1}{\alpha}} = (x^o)^{\frac{1}{\alpha}} \mu_t^o$.

Figure 6 illustrates the dynamics of μ_t^o in the case $S > 1$. The unique steady state $\mu^{o*} < \hat{\mu}^o$ is unstable. However, the dynamics of μ_t^o does not diverge but rather settles down on equilibrium cycles alternating between periods with positive entry and periods with zero entry.[34] Such cycles can have arbitrary periodicity or may be non-periodic. In fact, cycles of period three exist in this model if the condition $S \geq \frac{2-\xi}{1-\xi}$ holds.[35] Such a cycle is illustrated in Figure 6. Because this one-dimensional system is continuous, a cycle of period three implies that cycles of any other periodicity as well as chaotic cycles exist.[36]

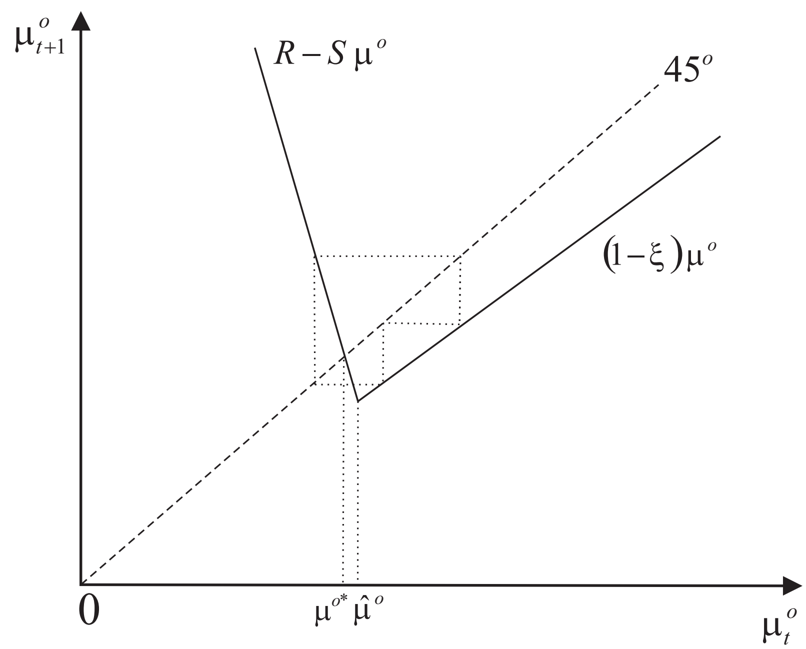

Figure 6: Endogenous cycles in the Hopenhayn (1992) model.

Calibration of the Model in Section 3.1.2

Table 1 shows the values of all model parameters and their calibration targets. The parameters for firm productivities x_i ($i = 1, 2, 3, 4$), the initial productivity distribution $\pi_{0,i}$ and the

[34] A variation of this model where entry costs are not constant but increasing in the measure of entrant firms (as in, e.g., Sedláček and Sterk, 2017) can also produce endogenous equilibrium cycles where entry is positive in all periods.

[35] The cycle is $(\mu_1^o, \mu_2^o, \mu_3^o)$ with $\mu_3^0 < \hat{\mu}^o \leq \mu_2^o < \mu_1^o$ where $\mu_1^o = R/[1 + S(1-\xi)^2]$.

[36] See Li, T.Y. and Yorke, J.A. (1975), "Period three implies chaos", *The American Mathematical Monthly*, Vol. 82 (10), 985–992.

Table 1: Calibrated parameters.

Parameter		Value	Explanation/Target
Discount factor	β	0.95	Annual interest rate
Exit probability	ξ	0.1	Annual firm exit rate (BDS)
Manager prod. elasticity	α^m	0.263	31% manager earnings share
Worker prod. elasticity	α^p	0.587	Returns to scale 0.85
Owner labor input	a	1	Single owner in small firms
Firm productivities	$(x_1, \ldots, x_4)$	(2.86,3.75,5.25,8.24)	Firm size (Table 2, row 2)
Entry shares type 2	$(\pi_{0,1}, \ldots, \pi_{0,4})$	(0.896,0.102,0.002,0)	Entrant size dist. (Table 2, row 3)
Transition prob. type 2	(π_1, π_2, π_3)	(0.066,0.009,0.014)	Firm size dist. (Table 2, row 4)
LS scale (managers)	v^m	0.063	17.1m managers (1970–2018)
LS scale (workers)	v^p	0.129	76.7m workers (1970–2018)
LS elasticity (managers)	γ^m	1.1	Cyclicality manager employment
LS elasticity (workers)	γ^p	2.5	Cyclicality worker employment
Aggregate productivity	(z_H, z_L)	(1.008,0.992)	Std. dev. real GDP
Transition prob	ψ	0.2	Autocorrelation real GDP
Entry costs (mean)	$(\bar{c}^1, \bar{c}^2)$	(10.71,18.99)	Mean wages $w^p = 1$, $w^m = 1.63$
Entry costs (cycle)	$(\frac{c_H^1 - c_L^1}{\bar{c}^1}, \frac{c_H^2 - c_L^2}{\bar{c}^2})$	(0.98%,0.81%)	See text

annual upward transition rates π_i for $i \leq 3$ are set to calibrate average firm size and the firm distribution of entrants (age 0 firms) and all firms over the four size classes 1-9, 10-99, 100-999 and 1000+. The corresponding data targets are obtained from the firm age×size tabulations of the Business Dynamics Statistics of the U.S. Census Bureau, averaged over the years 1977–2014. See Table 2 for these numbers which are exactly replicated by the model in steady state with aggregate productivity $z = 1$.

Table 2: Firm size distribution.

Size class	1–9	10–99	100–999	1000+
Mean employment	3.5	25.3	239.8	4832.4
Share entrants (%)	94.8	5.1	0.1	0.0
Share firms (%)	76.2	21.9	1.7	0.2

Employment and wages of production workers and managers are obtained from the Current Employment Statistics (CES) of the U.S. Bureau of Labor Statistics (1970–2018). Production workers are "Production and nonsupervisory employees", whereas managers are the difference between "All employees" and this number. On average during 1970 and 2018, there are 17.1 million managers and 76.7 million production workers. Given their respective wages and the Frisch elasticities (see below), these targets pin down the labor disutility scale parameters.

Regarding wages, average real hourly earnings of all employees (w^a) and of production and nonsupervisory employees (w^p) are available from the CES since 2006. With aggregate weekly hours of all employees (h^a) and of production and nonsupervisory employees (h^p), this allows to back out the implied average hourly wage of managers as a residual: $w^m = \frac{w^a h^a - w^p h^p}{h^a - h^p}$. The average wage gap between managers and production workers during 2006-2018 is $w^m/w^p = 1.63$. This target together with the normalization $w^p = 1$ implies entry costs for both firm types in steady state. The earnings share of managers $\frac{w^m h^m}{w^a h^a}$ is 31 percent on average. This target pins down the production function elasticity parameter α^m relative to α^p.

Aggregate labor productivity is measured by the ratio between real U.S. GDP and average weekly hours from the CES (1970-2018). All quarterly series are logged and HP-detrended with $\lambda = 1600$. The cyclical components have standard deviations 1.68% (production workers), 1.17% (managers) and 0.96% (labor productivity). Given the calibrated cyclicality of wages for both types of workers, the cyclicality of employment (relative to productivity) is used to set the Frisch elasticities of labor supply for both worker types.

Four parameters of the model are calibrated internally to match further business-cycle features. These are the cyclical variation of the labor productivity parameter z, the transition rate between high and low labor productivity, and the cyclical variation of both entry costs around their steady-state values. These are set to match (i) the standard deviation and annual autocorrelation of real GDP (logged and HP-detrended, 1.9% and 0.6 resp.), (ii) the standard deviation of the firm entry rate in the BDS (HP-detrended, 0.6%), (iii) the correlation between the the cyclical components of the entry rate and GDP (0.43).

Intrafirm Bargaining in the Random Search Model of Section 3.2.1

This section derives the equilibrium wage with Stole-Zwiebel intrafirm bargaining for the model in Section 3.2.1. It adapts the proof of Proposition 1 in Elsby and Michaels (2013) to the setup of Section 3.2. (which includes quits, firm exit, aggregate risk and a different timing specification). As Elsby and Michaels (2013), start with the presumption that the wage function $w(\ell, x, z)$ is such that the firm's flow profit is concave in ℓ and supermodular in (ℓ, x). The first of these features implies that the firm's employment policy is characterized by first-order conditions. In particular, the firm hires workers (i.e. $\ell > \ell_-(1 - s_0)$) if

$$f_\ell(\ell, x, z) - w(\ell, x, z) - w_\ell(\ell, x, z)\ell - \frac{\kappa_0}{q(\theta(z))} + \beta_0 D(\ell, x, z) = 0 \ ,$$

where

$$D(\ell, x, z) \equiv \mathbb{E}_{x,z} \frac{dv}{d\ell}(\ell, x', z') \ .$$

The firm fires workers ($\ell < \ell_-(1 - s_0)$) if

$$f_\ell(\ell, x, z) - w(\ell, x, z) - w_\ell(\ell, x, z)\ell + \beta_0 D(\ell, x, z) = 0 \ .$$

These two conditions determine employment levels for a hiring firm $\ell_h(x,z)$ and for a separating firm $\ell_s(x,z)$ such that $\ell_h(.) < \ell_s(.)$. That is, firms with employment $\ell_-(1-s_0)$ below $\ell_h(x,z)$ hire up to $\ell_h(x,z)$, firms with $\ell_-(1-s_0)$ above $\ell_s(x,z)$ separate from $\ell_-(1-s_0)-\ell_s(x,z)$ workers, and those in between choose $\ell = \ell_-(1-s_0)$. Supermodularity of the profit function implies that ℓ_s and ℓ_h are increasing functions of x: More productive firms choose larger employment. This also implies that there are cutoff levels for productivity $x_s(\ell_-,z) < x_h(\ell_-,z)$ such that (i) firms with $x < x_s(\ell_-,z)$ fire workers, (ii) firms with $x > x_h(\ell_-,z)$ hire workers, and (iii) firms with $x \in [x_s(\ell_-,z), x_h(\ell_-,z)]$ remain passive. Define the separation rate for firm (ℓ_-,x) in aggregate state z as $s(\ell_-,x,z) = s_0$ if $x \geq x_s(\ell_-,z)$ and $s(\ell_-,x,z) = \frac{\ell_- - \ell_s(x,z)}{\ell_-}$ otherwise.

Define the value of the marginal worker to a firm

$$J(\ell,x,z) \equiv f_\ell(\ell,x,z) - w(\ell,x,z) - w_\ell(\ell,x,z)\ell + \beta_0 D(\ell,x,z) \ .$$

Optimal employment adjustment implies that

$$J(\ell,x,z) = \begin{cases} 0 & , \ \ell = \ell_s(x,z) \ , \\ \frac{\kappa_0}{q(\theta(z))} & , \ \ell = \ell_h(x,z) \ . \end{cases} \tag{62}$$

Let $U(z)$ denote the value of unemployment and let $W(\ell,x,z)$ be the value of employment in a firm of size ℓ. Because $W-U$ (J) increases (decreases) linearly in the bargained wage (see below), Nash bargaining implies the sharing rule

$$(1-\eta)\left[W(\ell,x,z) - U(z)\right] = \eta J(\ell,x,z) \ , \tag{63}$$

where η is the workers' bargaining power parameter. The Bellman equation for an unemployed worker is

$$U(z) = \left[1-p(\theta(z))\right]\left[b + \beta\mathbb{E}_z U(z')\right] + p(\theta(z)) \int W(\ell_h(x,z),x,z) d\nu(\ell_-,x,z) \ , \tag{64}$$

where

$$d\nu(\ell_-,x,z) \equiv \frac{\max[0, \ell_h(x,z) - \ell_-(1-s_0)]}{H(z,\tilde{\mu})} d\tilde{\mu}(\ell_-,x)$$

is the probability of finding a job at firm (ℓ_-,x) conditional on being hired where $\tilde{\mu}$ is the firm distribution after entry and

$$H(z,\tilde{\mu}) \equiv \int \max[0, \ell_h(x,z) - \ell_-(1-s_0)] d\tilde{\mu}(\ell_-,x)$$

is the aggregate number of hires. For any worker in a firm that is currently hiring, (62) and (63) imply that

$$W(\ell_h(x,z),x,z) - U(z) = \frac{\eta}{1-\eta} J(\ell_h(x,z),x,z) = \frac{\eta}{1-\eta} \frac{\kappa_0}{q(\theta(z))}$$

is independent of the type of the hiring firm (ℓ_-, x). Therefore, the integral term in (64) simplifies to $U(z) + \frac{\eta}{1-\eta}\frac{\kappa_0}{q(\theta(z))}$ so that it follows

$$U(z) = b + \beta \mathbb{E}_z U(z') + \frac{p(\theta(z))}{1-p(\theta(z))}\frac{\eta}{1-\eta}\frac{\kappa_0}{q(\theta(z))}\ . \tag{65}$$

The Bellman equation for an employed worker is

$$\begin{aligned} W(\ell,x,z) = w(\ell,x,z) + \beta\xi\mathbb{E}_z U(z) + \beta(1-\xi)\mathbb{E}_z \Bigg\{ & \sum_{x'<x_s(\ell,z')} \pi_{xx'} \Big[s(\ell,x',z')U(z') \\ & + (1-s(\ell,x',z'))W(\ell_s(x',z'),x',z')\Big] \\ & + \sum_{x'\in[x_s(\ell,z'),x_h(\ell,z')]} \pi_{xx'}\Big[s_0 U(z') + (1-s_0)W(\ell(1-s_0),x',z')\Big] \\ & + \sum_{x'>x_h(\ell,z')} \pi_{xx'}\Big[s_0 U(z') + (1-s_0)W(\ell_h(x',z'),x',z')\Big]\Bigg\}\ . \end{aligned} \tag{66}$$

Because of (62) and (63)

$$\begin{aligned} W(\ell_s(x',z'),x',z') &= U(z') && \text{if } x' < x_s(\ell,z')\ , \\ W(\ell(1-s_0),x',z') &= U(z') + \frac{\eta}{1-\eta}J(\ell(1-s_0),x',z') && \text{if } x_s(\ell,z') \le x' \le x_h(\ell,z')\ , \\ W(\ell_h(x',z'),x',z') &= U(z') + \frac{\eta}{1-\eta}\frac{\kappa_0}{q(\theta(z'))} && \text{if } x' > x_h(\ell,z')\ . \end{aligned}$$

This can be substituted into the integral expressions of (66). Together with (65) follows that

$$\begin{aligned} (1-\eta)\left[W(\ell,x,z) - U(z)\right] = (1-\eta)(w(\ell,x,z)-b) - \eta\frac{p(\theta(z))}{1-p(\theta(z))}\frac{\kappa_0}{q(\theta(z))} \\ +\beta_0(1-s_0)\mathbb{E}_z\Big\{ \sum_{x'\in[x_s(\ell,z'),x_h(\ell,z')]} \pi_{xx'}\eta J(\ell(1-s_0),x',z') + \sum_{x'>x_h(\ell,z')} \pi_{xx'}\eta\frac{\kappa_0}{q(\theta(z'))}\Big\}\ . \end{aligned} \tag{67}$$

On the other hand, the expected marginal profit in a firm with ℓ workers is

$$\begin{aligned} D(\ell,x,z) &= \mathbb{E}_z \sum_{x'} \pi_{xx'}\frac{dv}{d\ell}(\ell,x',z') \\ &= \mathbb{E}_z\Big\{ \sum_{x'\in[x_s(\ell,z'),x_h(\ell,z')]} \pi_{xx'}(1-s_0)J(\ell(1-s_0),x',z') + \sum_{x'>x_h(\ell,z')} \pi_{xx'}(1-s_0)\frac{\kappa_0}{q(\theta(z'))}\Big\}\ , \end{aligned}$$

which uses the envelope condition for (22). Equating $(1-\eta)\left[W(\ell,x,z)-U(z)\right]$ in (67) with

$$\eta J(\ell,x) = \eta f_\ell(\ell,x,z) - \eta w(\ell,x,z) - \eta w_\ell(\ell,x,z)\ell + \eta\beta_0 D(\ell,x,z)$$

shows that all forward-looking terms cancel out so that it remains

$$(1-\eta)(w(\ell,x,z)-b)-\eta\frac{p(\theta(z))}{1-p(\theta(z))}\frac{\kappa_0}{q(\theta)}=\eta f_\ell(\ell,x,z)-\eta w(\ell,x,z)-\eta w_\ell(\ell,x,z)\ell\ ,$$

which is the differential equation for $w(\ell,x,z)$ reported in Section 3.2.1.