CESifo WORKING PAPERS

8761
2020

December 2020

Effort Comparisons for a Class of Four-Player Tournaments

Deren Cağlayan, Emin Karagözoğlu, Kerim Keskin, Çağrı Sağlam

CESifo

Impressum:

CESifo Working Papers
ISSN 2364-1428 (electronic version)
Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH
The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute
Poschingerstr. 5, 81679 Munich, Germany
Telephone +49 (0)89 2180-2740, Telefax +49 (0)89 2180-17845, email office@cesifo.de
Editor: Clemens Fuest
https://www.cesifo.org/en/wp
An electronic version of the paper may be downloaded
· from the SSRN website: www.SSRN.com
· from the RePEc website: www.RePEc.org
· from the CESifo website: https://www.cesifo.org/en/wp

CESifo *Working Paper No. 8761*

Effort Comparisons for a Class of Four-Player Tournaments

Abstract

We propose a novel tournament design that incorporates the main properties of a round-robin tournament, a Swiss tournament, and a race. Following an equilibrium analysis, we compare 36 tournament structures inherent in our model and several well-known tournament models from the literature, on the basis of expected total equilibrium effort. We show that two of the tournament structures we introduce outperform all the other tournament structures considered.

JEL-Codes: C720.

Keywords: contest, multi-player contest, race, round-robin tournament, Swiss tournament, tournament.

Deren Çağlayan
Department of Economics
Bilkent University
Turkey - 06800 Bilkent, Ankara
caglayanderen@gmail.com

*Emin Karagözoğlu**
Department of Economics
Bilkent University
Turkey - 06800 Bilkent, Ankara
karagozoglu@bilkent.edu.tr

Kerim Keskin
ADA University
Azerbaijan - Baku, 1008
kkeskin@ada.edu.az

Çağrı Sağlam
Department of Economics
Bilkent University
Turkey - 06800 Bilkent, Ankara
csaglam@bilkent.edu.tr

*corresponding author

December 2020

1 Introduction

A tournament is a type of multi-battle contest game, which is commonly used in sports, labor markets, politics, and so on (see Lazear and Rosen, 1981; Rosen, 1986; Prendergast, 1999; Szymanski, 2003; Harbaugh and Klumpp, 2005). Two well-known examples are round-robin tournaments and Swiss tournaments. In a round-robin tournament, players are pairwise matched in each round to compete in two-player component battles. It is an *all-play-all* tournament in that each player competes with all other players in turn. The schedule (i.e., who is matched with whom in which round) is exogenously given and common knowledge among all players. In a Swiss tournament, as in a round-robin tournament, players are pairwise matched in each round to compete in two-player component battles. However, a player does not necessarily compete with all other players, but competes with a selection of them, which is determined based on their performances in previous rounds. The latter property can be referred to as *endogenous scheduling*. Finally, although it cannot be classified as a tournament model, a property of another well-known multi-battle contest is worth mentioning here. In a race, multiple players compete in a component battle in each round and a player who reaches a certain number of battle victories (i.e., *winning threshold*) wins the race (see Klumpp and Polborn, 2006; Konrad and Kovenock, 2009, and Doğan et al., 2018).

In this paper we propose a novel tournament design with four symmetric players, which incorporates the main properties of a round-robin tournament, a Swiss tournament, and a race. In each round, players are pairwise matched to compete in two-player component battles. The winner of each battle is determined by a Tullock contest success function. The aim is to win a total of three component battles, before another player achieves the same (similar to a race). A player competes against each of the other players, and in the first-three rounds, no player competes against the same opponent twice (similar to a round-robin tournament). Moreover, we implement endogenous scheduling, which means that the order of games depends on the outcomes of the earlier component battles (similar to a Swiss tournament). With endogenous scheduling, considering different possibilities for second-round and fourth-round match-ups, we obtain 36 different tournament structures.

Our tournament model is inspired by the following observations: In round-robin tournaments, the winning players are *relatively* more successful than the losing players. However, this does not necessarily mean that the winning players are sufficiently successful in *absolute* terms. For instance, in a four-player tournament, three players can share the trophy by collecting two battle victories each. The contest designer may not be satisfied with such a result. In order to make sure that the champion is also successful in absolute terms, the designer may introduce a winning threshold to be achieved by the champion, as in a model of race.

Our model can be thought of as an alternative to a standard round-robin tournament in the sense that it implements modifications to a round-robin tournament in two dimensions: (i) endogenous scheduling and (ii) additional tie-braking games.[1] We structure our paper around this observation. We start with the equilibrium analyses of round-robin tournaments both with exogenously-given and endogenously-determined schedules. Afterward, we formally introduce the alternative tournament model, analyze its equilibrium for all possible versions, and compare all these tournament structures in terms of expected total equilibrium effort.[2]

We show that two of the tournament structures we introduce outperform all the other tournament structures considered in this paper. We refrain from going into the technical details here, but a brief intuition is in order: In the second round, a winner of the first round competes against the non-played loser of the first round. Normally, this would have resulted in a discouragement effect for the latter side, but then the fourth round match-ups are optimally selected such that if the player who lost in the first round wins in the second round, he/she will be incentivized to exert even more effort in the third round. Furthermore, independent of the outcomes of the second-round battles, there will always be at least two players who are very

[1] The latter is a direct consequence of introducing a winning threshold into a standard round-robin tournament.

[2] The maximization of expected total effort is, arguably, the most frequently-used objective criterion in contest theory. It is especially relevant in sport contests where higher total effort is related to higher attendance and greater revenue. See Dasgupta and Nti (1998), Moldovanu and Sela (2001), Borland and MacDonald (2003), Szymanski (2003), and Nti (2004) among others.

motivated to exert high efforts in the third round, and all players would do that in case each player collects one battle victory in the first two rounds. As a result, the maximum expected total effort is achieved in equilibrium. Further details are provided later in the paper.

Although round-robin tournaments are extensively studied in the operations research literature (see Fleurent and Ferland, 1993; Russel and Leung, 1994; Nemhauser and Trick, 1998; Henz et al., 2004; Rasmussen and Trick, 2008 among others), those studies mostly assume non-strategic players, and thus they ignore the effects of players' strategic effort choices on the tournament outcome. On the other hand, possibly due to the complexity of the respective equilibrium analysis (see Krumer et al., 2017a, pg. 634), there are only a few papers in the game theory literature that contributes a new perspective to the discussion by considering strategic players.

Among those papers, Krumer et al. (2017a) examine round-robin tournaments with three or four symmetric players. In case of four players, they assume that two battles in a round are scheduled one of the the after. Using an all-pay contest success function, they show that a player who plays in the first game of each of the first two rounds ends up with a higher equilibrium winning probability, indicating a first-mover advantage. Krumer et al. (2017b) analyze a three-player round-robin tournament with one strong player and two equally-weak players. They find that in order to maximize expected total equilibrium effort, the designer should use a round-robin tournament rather than a one-shot contest if the asymmetry between the strong player and the weak players is sufficiently high. Later, Sahm (2019) investigates fairness in a model similar to the one studied by Krumer et al. (2017a). Differently, however, he utilizes a Tullock contest success function and shows that the discrimination is weaker in Tullock contests compared to all-pay contests. Following this result, the author also analyzes endogenous scheduling in a three-player round-robin tournament. Finally, Krumer et al. (2020) study an optimal tournament design problem for a round-robin tournament with three symmetric players with either one prize or two prizes. They find that the designer should allocate only one prize if he/she aims to maximize expected total equilibrium effort.[3]

[3]We are also aware of two recent working papers. Similarly to the other papers mentioned above,

The paper is organized as follows. In Section 2, we provide equilibrium analyses of four-player round-robin tournaments with exogenous and endogenous schedules. In Section 3, we formulate a new tournament design. Section 4 presents our main results that compare all these tournament structures. Section 5 concludes.

2 A Round-robin Tournament

Consider four symmetric players in a player set $N = \{1, 2, 3, 4\}$. They compete in a round-robin tournament. The procedure is as follows. In each round, players are pairwise matched. For each pair of players, there is a component battle in which the players choose how much effort to exert, a Tullock contest success function determines who wins the battle, and the winning player collects one point from this round. For example, assuming that players $i, j \in N$ are matched in a given round, player i wins the battle with probability.

$$\frac{e_i}{e_i + e_j}$$

where e_i and e_j denote the exerted effort levels. If both players exert zero effort, then a tie-breaker rule applies: the player with more battle victories wins with probability 1, but if both players have the same number of battle victories, then each player would have a winning probability of 1/2. We also assume that the marginal cost of effort is one for every player and in each round.

A player competes against each of the other players and no player competes against the same opponent twice. The tournament always ends in three rounds. At the end, whichever player has more battle victories becomes the tournament champion and collects a winning prize of $V > 0$. If there are multiple such players, they become co-champions and share the winning prize equally.

The standard round-robin tournament as studied in the literature considers an exogenously-given schedule. This means that who will be matched with whom in each round is known before the tournament starts. A different specification may

Laica et al. (2017) examine round-robin tournaments with three or four symmetric players and multiple prizes, whereas Sela et al. (2020) analyze a round-robin tournament with four symmetric players and two prizes.

consider a tournament with an endogenously-determined schedule. This means that, depending on the results of the first-round battles, it is possible for player $i \in N$ to compete against either of the other two players in the second round. Notice that once the second-round battles are set, there is only one possible match-up for the third-round battles. In this specification, there are two versions to be considered: (a) winners of the first round compete against each other, or (b) a winner of the first round competes with the non-played loser of the first round.[4]

Our first result analyzes the unique symmetric subgame perfect Nash equilibrium (SPNE) in each version with an endogenously-determined schedule. As it will be revealed later, this result makes it easier to analyze the unique symmetric SPNE in the standard round-robin tournament with an exogenously-given schedule.

Proposition 1. *There exists a unique symmetric SPNE in either version of a round-robin tournament with an endogenously-determined schedule. The expected total effort that will be exerted in equilibrium is 0.7407V if the winners of the first round compete against each other in the second round and 0.7585V if each winner of the first round competes with the non-played loser of the first round in the second round.*

Proof. We analyze SPNE via backward induction. There are three possible cases at the beginning of the third round.[5]

Case 1. Assume that there are two players with two battle victories each (without loss of generality, say players 1 and 2), so that the other two players could not win any battle in the first two rounds. In the third round, the laggards compete against each other. Since there is no possibility that they will become a champion, both players exert zero effort in equilibrium.

In the battle between players 1 and 2, player $i \in \{1, 2\}$ maximizes

$$\frac{e_i}{e_1 + e_2} V - e_i$$

[4]Notice that such a round-robin tournament with an endogenously-determined schedule is a natural combination of a round-robin tournament and a Swiss tournament.

[5]In the following, by an abuse of notation, we omit the current round or state when denoting players' effort choices. Furthermore, in all utility maximization problems considered below, the respective second-order conditions hold.

In the equilibrium, we find $e_1 = e_2 = V/4$. This yields an expected payoff of $V/4$ to each player $i \in \{1, 2\}$.

Case 2. Assume that all players have one battle victory each. A player $i \in N$ maximizes

$$\frac{e_i}{e_1 + e_2} \frac{V}{2} - e_i$$

in her respective battle. In the equilibrium, we find $e_i = V/8$ for each player $i \in N$. This yields an expected payoff of $V/8$ to each player $i \in N$.

Case 3. Assume that there is one player with two battle victories (without loss of generality, say player 1), there are two players with one battle victory each (without loss of generality, say players 2 and 3), and the remaining player could not win any battle in the first two rounds.

For a subcase, assume that player 1 competes against player 4 in the third round. The result is trivial: since player 4 is totally discouraged, she exerts zero effort. Thus, player 1 wins this battle for sure even when she exerts zero effort herself (due to the tie-breaking assumption). As a result, player 1 becomes a champion with an expected payoff of V. Anticipating the outcome of that battle, and knowing that there is no possibility that they will become a champion, players 2 and 3 exert zero effort in their own component battle.

For another subcase, assume that player 3 competes against player 4 in the third round. The result is again trivial: since player 4 is totally discouraged, she exerts zero effort. Thus, player 3 wins this battle for sure even when she exerts zero effort herself (due to the tie-breaking assumption). As a result, player 3 becomes one of the three co-champions in case player 2 wins against player 1 in their own battle. Anticipating this outcome, player 1 maximizes

$$\frac{e_1}{e_1 + e_2} V + \frac{e_2}{e_1 + e_2} \frac{V}{3} - e_1$$

whereas player 2 maximizes

$$\frac{e_2}{e_1 + e_2} \frac{V}{3} - e_2$$

We find that $e_1 = 4V/27$ and $e_2 = 2V/27$, which yields an expected payoff of $17V/27$

to player 1 and $V/27$ to player 2. Furthermore, player 3 has an expected payoff of $3V/27$.

This completes the equilibrium analysis of the third-round battles. There is only one possible case at the beginning of the second round. In that case, there are two players with one battle victory each (without loss of generality, say players 1 and 2) and the other two players could not win in the first round.

First, we consider the version in which the winners of the first round compete against each other. Notice that, in this version, only Case 3 can be observed in the third round. Assuming that player 1 played against player 3 in the first round, player 1 maximizes

$$\frac{e_3}{e_3+e_4}\left(\frac{e_1}{e_1+e_2}V+\frac{e_2}{e_1+e_2}\frac{3V}{27}\right)+\frac{e_4}{e_3+e_4}\frac{e_1}{e_1+e_2}\frac{17V}{27}-e_1$$

in this round. Player 2's maximization problem can be written symmetrically. Furthermore, player 3 maximizes

$$\frac{e_2}{e_1+e_2}\frac{e_3}{e_3+e_4}\frac{V}{27}-e_3$$

in this round. Player 4's maximization problem can be written symmetrically. Taking the respective first-order conditions and restricting our attention to symmetric equilibrium where $e_1 = e_2$ and $e_3 = e_4$, we find $e_1 = e_2 = 41V/216 \approx 0.1898V$ and $e_3 = e_4 = V/216 \approx 0.0046V$. This yields an expected payoff of $53V/216 \approx 0.2454V$ to players 1 and 2 and $V/216 \approx 0.0046V$ to players 3 and 4.

Second, we consider the version in which each winner of the first round competes with the non-played loser of the first round. Notice that, in this version, all three cases can be observed in the third round. Assuming that player 1 played against player 3 in the first round, player 1 maximizes

$$\frac{e_2}{e_2+e_3}\left(\frac{e_1}{e_1+e_4}\frac{V}{4}+\frac{e_4}{e_1+e_4}\frac{V}{27}\right)+\frac{e_3}{e_2+e_3}\left(\frac{e_1}{e_1+e_4}\frac{17V}{27}+\frac{e_4}{e_1+e_4}\frac{V}{8}\right)-e_1$$

in this round. Furthermore, player 3 maximizes

$$\frac{e_1}{e_1+e_4}\frac{e_3}{e_2+e_3}\frac{3V}{27}+\frac{e_4}{e_1+e_4}\frac{e_3}{e_2+e_3}\frac{V}{8}-e_3$$

in this round. The maximization problems for players 2 and 4 can be written symmetrically. Taking the respective first-order conditions and restricting our attention to symmetric equilibrium where $e_1 = e_2$ and $e_3 = e_4$, we find $e_1 = e_2 \approx 0.0595V$ and $e_3 = e_4 \approx 0.0232V$. This yields an expected payoff of $0.2143V$ to players 1 and 2 and $0.0091V$ to players 3 and 4.

This completes the equilibrium analysis of the second-round battles. Finally, we analyze the equilibrium efforts in the first-round battles. Once again, our analysis will be divided into two parts, based on the two versions considered. In the former version, each player can anticipate that her continuation payoff would be $0.2454V$ if she wins now and $0.0046V$ if she loses now. One can find that $e_i \approx 0.0602V$ for each player $i \in N$ in the first round. Then, in the latter version, each player can anticipate that her continuation payoff would be $0.2143V$ if she wins now and $0.0091V$ if she loses now. One can find that $e_i \approx 0.0513V$ for each player $i \in N$ in the first round.

Given these results, the expected total effort that will be exerted in equilibrium would be

$$(4\times 0.0602V)+(2\times 0.1898V)+(2\times 0.0046V)+\frac{1}{4}\left(2\times\frac{6V}{27}\right)=0.7407V,$$

if winners of the first round compete against each other in the second round; and

$$\begin{aligned}&(4\times 0.0513V)+(2\times 0.0595V)+(2\times 0.0232V)+\\&\qquad p^2\left(2\times\frac{V}{4}\right)+p(1-p)\left(2\times\frac{6V}{27}\right)+(1-p)^2\left(4\times\frac{V}{8}\right)=0.7585V\end{aligned}$$

where $p = 0.0595/(0.0595+0.0232) = 0.7195$, if a winner of the first round competes with the non-played loser of the first round in the second round. □

To analyze the standard round-robin tournament with an exogenous schedule, we start with the following observations. The equilibrium analysis for the last two rounds would be similar to the corresponding ones above. As for the first-round battles, the equilibrium analysis would be different because of the changes in continuation payoffs. In particular, those payoffs would be written as convex combinations of the continuation payoffs from the two versions of the model with an endogenous schedule. Now, without loss of generality, assume that player 1 wins her battle against player 2 in the first round and that player 1 will compete against player 3 in the second round. Since player 3 has a 50% probability of winning her battle against player 4 in a symmetric equilibrium, the current model evolves into the version in which winners of the first round compete against each other in the second round with a probability of 1/2 and into the other version with a probability of 1/2.

Proposition 2. *There exists a unique symmetric SPNE in a round-robin tournament with an exogenously-given schedule. The expected total effort that will be exerted in this equilibrium is 0.7496V.*

Proof. Utilizing our observations above, we can argue that each player can anticipate that her continuation payoff would be $(0.2454V + 0.2143V)/2 = 0.2298V$ if she wins in the first round and $(0.0046V + 0.0091V)/2 = 0.0068V$ if she loses in the first round. We can then find that $e_i \approx 0.0557V$ for each player $i \in N$ in the first round.

Given these results, the expected total effort that will be exerted in equilibrium would be

$$(4 \times 0.0557V) + \frac{1}{2}\,\mathbf{TE}_W + \frac{1}{2}\,\mathbf{TE}_L = 0.7496V$$

where

$$\mathbf{TE}_W = (2 \times 0.1898V) + (2 \times 0.0046V) + \frac{1}{4}\left(2 \times \frac{6V}{27}\right)$$

is the expected total equilibrium effort (after the first round) in case winners of the first round compete against each other and

$$\mathbf{TE}_L = (2 \times 0.0595V) + (2 \times 0.0232V) + \\ p^2\left(2 \times \frac{V}{4}\right) + p(1-p)\left(2 \times \frac{6V}{27}\right) + (1-p)^2\left(4 \times \frac{V}{8}\right)$$

is the expected total equilibrium effort (after the first round) in case each winner of the first round competes with the non-played loser of the first round, where $p = 0.7195$ as calculated earlier. □

In the following two sections, we formally introduce various versions of our alternative tournament model and identify the optimal one after reporting the expected total equilibrium effort in each version.

3 An Alternative Tournament Model

Consider four symmetric players in the player set $N = \{1, 2, 3, 4\}$. They compete in a tournament game with multiple rounds. The procedure is as follows. In each round, players are pairwise matched. For each pair of players, there is a component battle in which those players choose how much effort to exert, a Tullock contest success function determines who wins the battle, and the winning player collects one point from this round. Similar to a round-robin tournament, assuming that players $i, j \in N$ are matched in a given round, player i wins the battle with probability

$$\frac{e_i}{e_i + e_j}$$

where e_i and e_j denote the exerted effort levels. Similarly, if both players exert zero effort, then a tie-breaker rule applies: the player with more battle victories wins with probability 1, but if both players have the same number of battle victories, then each player would have a winning probability of 1/2. We also assume that the marginal cost of effort is one for every player and in each round.

Each player's objective is to collect a total of three battle victories before any of the other players succeeds the same. If a player achieves this on her own, then the tournament ends, that player is declared to be the tournament champion, and she collects a winning prize of $V > 0$. Given the tournament design, there might be two players who achieve this at the same time, and for that, we consider two specifications of the model: (i) both three-victory players win the tournament and equally share a total prize of V;[6] and (ii) there is an additional round where those three-victory players compete in a final game to determine the tournament champion who will then collect a prize of V.

This tournament design displays some similarities with a round-robin tournament, a Swiss tournament, and a race. The first three rounds are played as in a round-robin tournament with a Swiss-type endogenous schedule. Notice that if one player wins three component battles in three rounds, the tournament ends exactly in three rounds with that player becoming the tournament champion. However, it is also possible that such a player does not exist, in which case we either have two or three players with two battle victories.[7] In this latter case, our alternative model allows for tie-braking games to be played in additional rounds. This is a direct consequence of defining a threshold number of victories to be achieved by the champion, which is an apparent similarity to a race model.

Before proceeding further, we introduce the following notation. A node is denoted by a quadruple (a, b, c, d) where each entry represents the total number of battle victories achieved by the respective player before arriving that node. For the sake of illustration, victories by players 1 and 4 in that round would move the game to node $(a + 1, b, c, d + 1)$. Moreover, when we refer to a specific node (a, b, c, d), it does not necessarily mean that player 1 has a battle victories, player 2 has b battle victories, and so on. Instead, such a node represents all cases in which one player has a battle victories, another player has b battle victories, and so on. That is to say, we consider *anonymity* among players when reporting our results.

As it was the case earlier, we consider two versions for the second round match-ups: (a) winners of the first round compete against each other, or (b) each winner of the first round competes with the non-played loser of the first round. But now, if the tournament is not finalized in three rounds, there are two possible nodes in the fourth round: $(2, 2, 1, 1)$ or $(2, 2, 2, 0)$. For the former node, there are three versions: (a) the leaders compete against each other, (b) the leader who defeated the other

[6]Compared to a round-robin tournament, this specification tries to break the tie after the first three rounds, but if tie is not broken, there will be two co-champions.

[7]It is worth reminding here that in any version of round-robin tournaments considered in Section 2, those players with two victories would become co-champions.

leader competes with the laggard who defeated the other laggard, or (c) the leader who defeated the other leader competes with the laggard who was defeated by the other laggard. For the latter node, there are three versions: (a) the leader who lost in the first round competes with the laggard, (b) the leader who lost in the second round competes with the laggard, or (c) the leader who lost in the third round competes with the laggard. Furthermore, in case the tournament is not finalized in four rounds, $(2, 2, 2, 2)$ would be the only possible node in the fifth round. Given that this is a symmetric node, we consider random matching at this node. This leads to $2 \times 3 \times 3 = 18$ tournament structures. Considering the two specifications mentioned earlier, we have a total of 36 alternative tournament structures.

4 The Results

In this section, we report our main results on the optimal tournament design in terms of expected total equilibrium effort. In that regard, the complete characterization of equilibrium strategies in each version of our tournament model is necessary. However, considering the length of the equilibrium analyses for all 36 tournament structures, we do not report them in the main body of the paper. The equilibrium efforts and equilibrium winning probabilities on each possible node are reported in a supplementary material available on authors' web pages.

The 4th round		The 2nd round (i.e., Node (1,1,0,0))	
Node (2,2,1,1)	Node (2,2,2,0)	Winner vs. Winner	Winner vs. Loser
a	a	CW1	CL1
a	b	CW2	CL2
a	c	CW3	CL3
b	a	CW4	CL4
b	b	CW5	CL5
b	c	CW6	CL6
c	a	CW7	CL7
c	b	CW8	CL8
c	c	CW9	CL9

Table 1: 18 possible versions under the first specification with co-champions (C)

Here we first report the tournament structure that yields the highest expected total equilibrium effort under each specification of the model: (i) two three-victory players equally share V and (ii) those three-victory players compete in a final game after which the winner collects V. Afterward, we also make a comparison between all tournament structures considered in this paper.

Proposition 3. *In the alternative tournament model, consider the first specification with co-champions. Among the 18 possible versions, the optimal tournament is such that (i) in the second round, a winner of the first round competes with the non-played loser of the first round; (ii) on node* $(2, 2, 1, 1)$ *of the fourth round, a leader competes with either laggard; and (iii) on node* $(2, 2, 2, 0)$ *of the fourth round, the leader who lost in the second round competes with the laggard. The expected total equilibrium effort is 0.7929V.*

Proof. The equilibrium analysis follows as in the proof of Proposition 1. For each version of the tournament, by performing backward induction, starting from the furthest decision node (2,2,2,2) and ending at the first decision node (0,0,0,0), we characterize the unique stationary subgame perfect Nash equilibrium of the model.

Specification: C	W	L
1	0.7732V	0.7539V
2	0.7515V	0.7775V
3	0.7612V	0.7480V
4	0.7732V	0.7558V
5	0.7515V	**0.7929V**
6	0.7612V	0.7520V
7	0.7732V	0.7559V
8	0.7515V	**0.7929V**
9	0.7612V	0.7520V

Table 2: The expected total equilibrium efforts under specification C

The interested reader is referred to the supplementary material for the respective equilibrium efforts and equilibrium winning probabilities on each possible node. Here we summarize the expected total equilibrium efforts in Table 2. □

The intuition is that if in the second round, a winner of the first round competes with the non-played loser of the first round (as specified in *(i)*), then a possible third-round battle is between a player with two battle victories and a player who won the first-round battle but lost the second-round battle. The latter player knows that if she wins in the third round, she will be playing against a totally discouraged player in the fourth round (as specified in *(iii)*). This creates an additional incentive for that player to exert a higher effort in the third round against her opponent who already has a strong incentive to exert high effort due to her two battle victories.

As for *(ii)*, on node (2,2,1,1), if the leaders compete against each other, then they would exert $V/4$ and end up with an expected payoff of $V/4$ each; but if a leader competes with a laggard, then the laggards would be totally discouraged, so that even the leaders would have no reason to exert any effort, but then they would end up with an expected payoff of $V/2$ each. Since the latter expected payoff is much higher, it creates an additional incentive to exert higher efforts in the earlier rounds. Note also that although some positive amount of total effort would be exerted in the former case, as it turns out, its overall effect is not dominant due to the fact that node (2,2,1,1) will not be visited with a high probability.

The 4th round		The 2nd round (i.e., Node (1,1,0,0))	
Node (2,2,1,1)	Node (2,2,2,0)	Winner vs. Winner	Winner vs. Loser
a	a	FW1	FL1
a	b	FW2	FL2
a	c	FW3	FL3
b	a	FW4	FL4
b	b	FW5	FL5
b	c	FW6	FL6
c	a	FW7	FL7
c	b	FW8	FL8
c	c	FW9	FL9

Table 3: 18 possible versions under the second specification with a final game (F)

Proposition 4. *In the alternative tournament model, consider the second specification with a final game. Among the 18 possible versions, the optimal tournament is such that (i) in the second round, winners of the first round compete against each* *other; and (ii) on node $(2,2,2,0)$ of the fourth round, the leader who lost in the first round competes with the laggard. The expected total equilibrium effort is 0.7851V.*

Proof. The equilibrium analysis follows as in the proof of Proposition 1. For each version of the tournament, by performing backward induction, starting from the final game and ending at the first decision node (0,0,0,0), we characterize the unique stationary subgame perfect Nash equilibrium of the model.

Specification: F	W	L
1	**0.7851V**	0.7447V
2	0.7582V	0.7560V
3	0.7619V	0.7473V
4	**0.7851V**	0.7460V
5	0.7582V	0.7604V
6	0.7619V	0.7479V
7	**0.7851V**	0.7460V
8	0.7582V	0.7604V
9	0.7619V	0.7479V

Table 4: The expected total equilibrium efforts under specification F

The interested reader is referred to the supplementary material for the respective equilibrium efforts and equilibrium winning probabilities on each possible node. Here we summarize the expected total equilibrium efforts in Table 4. □

The intuition is quite similar to the one we provided for Proposition 3 above. The match-ups specified in *(i)* and *(ii)* aim to motivate the player who will be competing against an opponent with two victories in the third round. There are two differences. First, given the current second-round battles, node (2,2,1,1) would not be reached.[8] Accordingly, the match-ups on node (2,2,1,1) will not have any effect on the expected total equilibrium effort. Second, there is now an important final game in which players would exert a total contest effort of $V/2$. Thus, the version where the probability of reaching node (3,3,1,1), which is the only node that leads to a final game, is the highest turns out to be the total-effort-maximizing one.

[8] It is for sure that node (2,1,1,0) is reached from node (1,1,0,0). The player with no victory would be totally discouraged in the third round, so that node (2,2,1,1) cannot be reached.

Given the expected total equilibrium efforts presented in Propositions 1–4 for 39 different tournament structures (i.e., 36 for our alternative tournament model, two for round-robin tournaments with an endogenous schedule, and one for a round-robin tournament with an exogenous schedule), we state our main result in the following corollary. In short, two of the tournament structures we introduce here outperform all the others in the expected total effort they induce in equilibrium.

Corollary 1. *Among all tournament structures considered, CL5 and CL8 maximize the expected total equilibrium efforts.*

At this point, it may be natural to think that the reason why the expected total equilibrium effort is maximized in two versions of our alternative tournament model is that, compared to a standard round-robin tournament, we allow for additional component battles in the fourth and fifth rounds, whenever necessary. However, as we explain in the following remark, this line of thought is not necessarily correct.

Remark 1. *Given a tournament structure, adding more component battles into the model does not necessarily increase the expected total effort in equilibrium. This can be observed in two specifications mentioned above. By definition, the total number of component battles in the final game specification is greater than the total number of component battles in the co-champion specification. However, as shown in Corollary 1, the expected total equilibrium effort is maximized in two versions of the co-champion specification.*

Finally, while conducting our analysis we made some further observations on the equilibrium path, which we think are worthy of presenting here.

- If node $(2,2,1,1)$ is reached and if the leaders do not compete against each other in the current round, then the laggards are totally discouraged in their respective contests. The game moves to node $(3,3,1,1)$ for sure.
- The game never reaches node $(2,2,2,2)$ in equilibrium. So, a maximum of four rounds will be played (neglecting the final game).
- A player who won at most once in the first three rounds is totally discouraged.

- A player who lost the first two rounds is totally discouraged.
- If node $(2, 1, 1, 0)$ is reached and (a) if the leader competes with the laggard, then the game moves to node $(3, 2, 1, 0)$ for sure; but (b) if the leader competes with a one-victory player, then the game moves either to node $(3, 2, 1, 0)$ or to node $(2, 2, 2, 0)$.
- The game never reaches node $(3, 1, 1, 1)$ in an equilibrium.

5 Conclusion

In this paper we propose a novel tournament design that incorporates the main properties of a round-robin tournament, a Swiss tournament, and a race. We conduct equilibrium analyses for various versions of our alternative model as well as three versions of a round-robin tournament. Utilizing expected total equilibrium effort as a comparison criterion, we show that two of the tournament structures we introduce outperform all the other tournament structures considered in this paper. We provide intuition for differences in the expected total equilibrium efforts and present some further observations on the equilibrium behavior in our tournament design. Future work may study fairness properties of our tournament, conduct comparisons using different objective criteria, and develop computational methods to extend it to more than four players and three victory thresholds.

Acknowledgments

Emin Karagözoğlu and Çağrı Sağlam would like to thank the Scientific and Technological Research Council of Turkey (TÜBİTAK) for financial support under grant number 118K239. The usual disclaimers apply.

References

[1] Borland J, MacDonald R (2003). Demand for sport. *Oxford Review of Economic Policy*, **19**, 478–502.

[2] Dasgupta A, Nti KO (1998) Designing an optimal contest. *European Journal of Political Economy*, **14**, 587–603.

[3] Doğan S, Karagözoğlu E, Keskin K, Sağlam Ç (2018). Multi-player race. *Journal of Economic Behavior and Organization*, **149**, 123–136.

[4] Fleurent C, Ferland JA (1993). Allocating games for the NHL using integer programming. *Operations Research*, **41**, 649–654.

[5] Harbaugh R, Klumpp T (2005). Early round upsets and championship blowouts. *Economic Inquiry*, **43**, 316–329.

[6] Henz M, Muller T, Thiel S (2004). Global constraints for round robin tournaments scheduling. *European Journal of Operations Research*, **153**, 92–101.

[7] Klumpp T, Polborn MK (2006). Primaries and the New Hampshire effect. *Journal of Public Economics*, **90**, 1073–1114.

[8] Konrad K, Kovenock D (2009). Multi-battle contests, *Games and Economic Behavior*, **66**, 256–274.

[9] Krumer A, Megidish R, Sela A (2017a). First-mover advantage in round-robin tournaments. *Social Choice and Welfare*, **48**, 633–658.

[10] Krumer A, Megidish R, Sela A (2017b). Round-robin tournaments with a dominant player. *Scandinavian Journal of Economics*, **119**, 1167–1200.

[11] Krumer A, Megidish R, Sela A (2020). The optimal design of round-robin tournaments with three players. *Journal of Scheduling*, **23**, 379–396.

[12] Laica C, Lauber A, Sahm M (2017). Sequential round-robin tournaments with multiple prizes. *CESifo Working Paper*, No. 6685.

[13] Lazear EP, Rosen S (1981). Rank-order tournaments as optimum labor contracts. *Journal of Political Economy*, **89**, 841–864.

[14] Moldovanu B, Sela A (2001) The optimal allocation of prizes in contests. *American Economic Review*, **91**, 542–558.

[15] Nemhauser GL, Trick MA (1998). Scheduling a major college basketball conference. *Operations Research*, **46**, 1–8.

[16] Nti KO (2004) Maximum efforts in contests with asymmetric valuations. *European Journal of Political Economy*, **20**, 1059–1066.

[17] Prendergast C (1999). The provision of incentives in firms. *Journal of Economic Literature*, **37**, 7–63.

[18] Rasmussen RV, Trick MA (2008). Round robin scheduling – A survey. *European Journal of Operations Research*, **188**, 617–636.

[19] Rosen S (1986). Prizes and incentives in elimination tournaments. *American Economic Review*, **76**, 701–715.

[20] Russel RA, Leung JMY (1994). Devising a cost effective schedule for a baseball league. *Operations Research*, **42**, 614–625.

[21] Sahm M (2019). Are sequential round-robin tournaments discriminatory? *Journal of Public Economic Thoery*, **21**, 44–61.

[22] Sela A, Krumer A, Megidish R (2020). Strategic manipulations in round-robin tournaments. *CEPR Discussion Paper*, No. DP14412.

[23] Szymanski S (2003). The economic design of sporting contests. *Journal of Economic Literature*, **41**, 1137–1187.