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Balanced Scorecards: A Relational Contract Approach

Abstract

Reward systems based on balanced scorecards typically connect pay to an index, i.e. a weighted sum of multiple performance measures. We show that such an index contract may indeed be optimal if performance measures are non-verifiable so that the contracting parties must rely on self-enforcement. Under commonly invoked assumptions (including normally distributed measurements), the optimal self-enforcing (relational) contract between a principal and a multitasking agent is an index contract where the agent gets a bonus if a weighted sum of performance outcomes on the various tasks (the index) exceeds a hurdle. The weights reflect a trade-off between distortion and precision for the measures. The efficiency of the contract improves with higher precision of the index measure, since this strengthens incentives. Correlations between measurements may for this reason be beneficial. For a similar reason, the principal may also want to include verifiable performance measures in the relational index contract in order to improve incentives.

Keywords: incentives, performance measures, relational contracts.

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1 Introduction

Very few jobs can be measured along one single dimension; employees usually multitask. This creates challenges for incentive providers: If the firm only rewards a subset of dimensions or tasks, agents will have incentives to exert efforts only on those tasks that are rewarded, and ignore others. A solution for the firm is to add more metrics to the compensation scheme, but this usually implies some form of measurement problem, leading either to more noise or distortions, or to the use of non-verifiable (subjective) performance measures.

The latter is often implemented by the use of a balanced scorecard (BSC). Kaplan and Norton's (1992, 1996) highly influential concept began with a premise that exclusive reliance on verifiable financial performance measures was not sufficient, as it could distort behavior and promote effort that is not compatible with long-term value creation. Their main ideas were indebted to the canonical multitasking models of Holmström and Milgrom (1991) and Baker (1992). However, their approach was more practical, guiding firms in how to design performance measurement systems that focus not only on short-term financial objectives, but also on long-term strategic goals (Kaplan and Norton, 2001).

While measuring performance is one issue, the question of how to reward performance is a different one. As noted by Budde (2007), there is a general understanding that efficient incentives must be based on multiple performance measures, including non-verifiable ones. Still, the implementation is a matter of controversy. Reward systems based on BSC typically connect pay to an index, i.e. a weighted sum of multiple performance measures. However, there is apparently no formal incentive model that actually derives this kind of index contract as an optimal solution in settings with non-verifiable measures.[1] In fact, Kaplan and Norton (1996) were sceptical to compensation formulas that calculated incentive compensation directly via a sum of weighted metrics. Rather, they proposed to establish different bonuses for a whole set of critical performance measures, more in line with

[1] Banker and Datar (1989) derive conditions under which a contract based on a linear aggregate of verifiable performance measures is optimal in a standard moral hazard problem with a risk averse agent.

the original ideas of Holmström and Milgrom (1991) and Feltham and Xie (1994).

Despite the large literature following the introduction of BSC (see Hoque, 2014, for a review), and the massive use of scorecards in practice, it appears that the index contracts that BSC-firms often prescribe lack a formal contract theoretic justification.[2] We take some steps to fill the gap. Our starting point is that the performance measures are non-verifiable. This means that the incentive contract cannot be enforced by a third party and thus needs to be self-enforcing - or what is commonly termed "relational". Incentive contracts used by firms, including performance measures based on balanced scorecards, often include non-verifiable qualitative assessments of performance (see Ittner et al, 2003, Gibbs et al 2004 and Kaplan and Gibbons, 2015). Moreover, even if some performance measures in principle are verifiable, the costs and uncertainty of taking the contract to court may be so high that the parties in practice need to rely on self-enforcement (see MacLeod, 2007 and references therein).

In the now large literature on self-enforcing relational contracts, relatively few papers have considered relational contracts with multitasking agents (prominent papers include Baker, Gibbons and Murphy, 2002; Budde, 2007, Schottner, 2008; Mukerjee and Vasconcelos, 2011; and Ishihara, 2016). We on the one hand generalize this literature in some dimensions (to an arbitrary number of tasks with stochastic measurements that are possibly correlated and/or distorted), and on the other hand invoke assumptions (notably normally distributed measurements) that make the model quite tractable.[3]

We first show that the optimal relational contract between a principal and a multitasking agent turns out to be an index contract, or what one may call a balanced scorecard. That is, the agent gets a bonus if a weighted sum

[2]According to Hoque (2014), among the more than 100 papers published on BCS theory, only a handful have used principal agent theory to analyze BSC. See also Hesford et al. (2009) for a review.

[3]Our paper is indebted to the seminal literature on relational contracts. The concept of relational contracts was first defined and explored by legal sholars (Macaulay, 1963, Macneil, 1978), while the formal literature started with Klein and Leffler (1981). MacLeod and Malcomson (1989) provides a general treatment of the symmetric information case, while Levin (2003) generalizes the case of asymmetric information. The relevance of the relational contract approach to management accounting and performance measurement is discussed in Glover (2012) and Baldenius et al. (2016).

of performance outcomes on the various tasks (an index) exceeds a hurdle. This is in contrast to the optimal contract in e.g. Holmström and Milgrom (1991), where the agent gets a bonus on each task. The important difference from Holmström and Milgrom is that we consider a relational contracting setting where the size of the bonus is limited by the principal's temptation to renege (rather than risk considerations). In such a setting the marginal incentives to exert effort on each task is higher with index contracts than with bonuses awarded on each task.

The performance measures within a scorecard may well be correlated. We point out that such correlations will affect the efficiency of the contract and we show that the efficiency of the index contract depends on how correlations affect the precision of the overall scorecard measure. In particular, an index contract with non-negative weights on all relevant measures will work even better if the measures are negatively correlated. The reason is that negative correlation reduces the variance of the overall performance measure (the index) in such cases. This is beneficial in our setting not because a more precise measure reduces risk – since the agent is assumed to be risk neutral – but because it strengthens, for any given bonus level, the incentives for the agent to provide effort.[4]

Besides being affected by noise, performance measures are normally also to various degrees distorted, implying that incentives on these measures promote actions that are not perfectly aligned with the firm's true objective. Many firms end up with rewarding performance according to such distorted measures, as long as the performance can be measured precisely. That is, the firm may prefer distorted, but precise performance measures, rather than well-aligned, but vague and imprecise measures. They can find support for this strategy in classic incentive theory where performance measures are verifiable and contracts are court enforceable (e.g. Datar et al. 2001).

A natural solution to this measurement problem may be to rely on subjective performance measures that are better aligned with the true objective, and make the contract self-enforcing. However, as we show in this paper, even in relational contracts, where there are no requirements regarding verifiability, and thus presumably greater scope for subjective and well-aligned

[4]Similar effects appear in Kvaløy and Olsen (2019), which analyzes relational contracts and correlated performances in a model with multiple agents, but single tasks.

performance measures, it may still be optimal to let precision weigh more heavily than alignment in incentive provision.

Our analysis reveals that the optimal weights in the scorecard index reflect a trade-off between distortion and precision, implying that a measure which is well aligned with the firm's true objective may nevertheless get a small weight in the index if that measure is to a large extent affected by noise and therefore highly imprecise. Again, this is not due to risk considerations, but due to incentive effects from the overall precision of the index.

We also consider the case where some measures are verifiable, and some are not. We show that the principal will include verifiable measures in the relational index contract in order to strengthen incentives.[5] This resembles balanced scorecards seen in practice, which often include both verifiable measures such as sales or financial accounting data, and non-verifiable (subjective) measures (see e.g. Kaplan and Norton, 2001 and Ittner et al., 2003). By including a verifiable measure in the relational contract, the variance of the performance index may be reduced, which again strengthens incentives. We also show that the verifiable performance measure is taken into the index as a benchmark, to which the other performances are compared. Moreover, the principal will still offer an explicit bonus contract on the verifiable measure, but this bonus is generally affected by the optimal relational index contract.[6]

A paper closely related to ours is Budde (2007), which investigates incentive effects of a scorecard scheme based on a set of balanced performance measures under both explicit and relational contracts. The paper is important, as it shows that BSC-types of contracts can provide undistorted incentives in settings with no noise and sufficient congruity/alignment between performance measures and the "true" value added. There are, however, important

[5] Our analysis of this issue presumes short-term explicit (court enforced) contracts. Watson, Miller and Olsen (2020) presents a general theory for interactions between relatonal and court enforced contracts when the latter are long term and renegotiable, and show that optimal contracts are then non-stationary. Implications of this for the contracting problems considered in the current paper are left for future research.

[6] Our model thus complements the influential papers by Baker, Gibbons and Murphy (1994) and Schmidt and Schnitzer (1995) on the interaction between relational and explicit contracts. While their results are driven by differences in fallback options created by the explicit contracts, our results stem from correlation between the tasks and (or) misalignment between measurements and true values.

differences which make our paper complementary to Budde's. First, and unlike us, Budde assumes at the outset that the available measurement system is «balanced», «minimal» and without noise. These somewhat strict assumptions imply, among other things, that from an observation of the measurements one can perfectly deduce the agent's action. This means that the action is in essence observable, and simple forcing contracts for the agent are then feasible (and optimal).[7] We do not invoke these restrictive assumptions, but rather allow for both «unbalanced» and noisy measurements. Actions can then not be deduced from observations, which means that there is a real hidden action problem, and the characterization of optimal (relational) incentive contracts becomes a non-trivial task. This characterization is precisely the focus of our paper.

The main focus in Budde's paper is the extent to which a relational contract can supplement an explicit contract to achieve a first-best allocation, in a setting where an explicit contract alone cannot do so due to misalignment between the measures that are verifiable and the true value. The assumptions on the total measurement system imply that the first best can always be achieved if the parties are sufficiently patient.[8] This is generally not the case under our relaxed –and, we believe more realistic – assumptions. We thus complement Budde's analysis by characterizing optimal relational contracts and second-best allocations under more realistic assumptions about the performance measurement system, especially regarding the measurements' precision. Interestingly, an index contract – a scorecard – then emerges as the optimal (relational) contract.

The rest of the paper is organized as follows: In Section 2 we present the basic model and a preliminary result. In Section 3 we introduce distorted performance measures and present our main results, which show that an optimal relational contract takes the form of a BSC (index) contract. The

[7] The paper allows for noisy observations in settings with verifiable measurements, and briefly discusses general noisy observations in a final section. The discussion concludes that "... a subtle tradeoff between the benefits of risk diversification and congruity has to be considered" and ". . a detailed investigation of this tradeoff requires considerable analysis", Budde 2007, p 533. We provide such an investigation here.

[8] The paper characterizes the minimal critical discount factor necessary to achieve the first-best, and importantly shows that this entails restricting informal incentives to that part of the first-best action that cannot be induced by a formal contract. Moreover, all unverifiable measures should be used in the relational contract.

weights on the measures in the index reflect a trade-off between distortion and precision. The results rely on some assumptions, including validity of the "first-order approach"; and we discuss this assumption in two subsections. The discussion reveals that the approach is not valid if measurements are very precise, and a characterization of optimal contracts is thus lacking for such environments. We show that index contracts will nevertheless perform well under such conditions, and in fact become asymptotically optimal when measurement noise vanishes. In Section 4 we extend the model to include both verifiable and non-verfiable performance measures. Section 5 concludes.

2 Model

First, we present the basic model between a principal and a multitasking agent. Consider an ongoing economic relationship between a risk neutral principal and a risk neutral agent. Each period the agent takes an n-dimensional action $a = (a_1, ..., a_n)'$, generating a gross value $v(a)$ for the principal, a private cost $c(a)$ for the agent, and a set of $m \leq n$ stochastic performance measurements $x = (x_1, ..., x_m)'$. These measurements are observable, but not verifiable, with joint density, conditional on action $f(x, a)$. Only the agent observes the action. The gross value $v(a)$ is not observed (as is the case if this is e.g. expected revenue for the principal, conditional on the agent's action). We assume $v(a)$ to be increasing in each a_i and concave, and $c(a)$ to be increasing in a each a_i and strictly convex with $c(0) = 0$ and gradient vector (marginal costs) $\nabla c(0) = 0$. The total surplus (per period) in the relationship is $v(a) - c(a)$.

Given observable (but non-verifiable) measurements, the agent is each period promised a bonus $\beta(x)$ from the principal. Specifically, the stage game proceeds as follows: 1. The principal offers the agent a contract consisting of a fixed payment w and a bonus $\beta(x)$. 2. If the agent accepts, he chooses some action a, generating performance measure x. If the agent declines, nothing happens until the next period. 3. The parties observe performance x, the principal pays w and chooses whether or not to honor the full contract and pay the specified bonus. 4. The agent chooses whether or not to accept the bonus he is offered. 5. The parties decide whether to continue or break off the relationship. Outside options are normalized to zero.

As shown by Levin (2002, 2003), we may assume trigger strategies and stationary contracts. The parties honor the contract only if both parties honored the contract in the previous period, and they break off the relationship and take their respective outside options otherwise. To prevent deviations, the self-enforced discretionary bonus payments must be bounded above and below. As is well known, the range of such self-enforceable payments is defined by the future value of the relationship, hence we have a dynamic enforceability condition given by

$$0 \leq \beta(x) \leq \frac{\delta}{1-\delta}(v(a) - c(a)), \qquad \text{all feasible } x. \tag{1}$$

The optimal relational contract maximizes the surplus $v(a) - c(a)$ subject to this constraint and the agent's incentive compatibility (IC) constraint. The latter is

$$a \in \arg\max_{a'} E(\beta(x)|\, a') - c(a'),$$

with first-order conditions (subscripts denote partials)

$$0 = \frac{\partial}{\partial a_i} E(\beta(x)|\, a) - c_i(a) = \int \beta(x) f_{a_i}(x, a) - c_i(a), \qquad i = 1, ...n.$$

A standard approach to solve this problem is to replace the global incentive constraint for the agent with the local first-order conditions. It is well known that this may or may not be valid, depending on the circumstances (see e.g. Hwang 2016 and Chi-Olsen 2018). In this paper we will mostly assume that it is valid, and subsequently state conditions for which this is true. So we invoke the following:

Assumption A. *The first order approach (FOA) is valid.*

Unless explicitly noted otherwise, we will take this assumption for granted in the following. We then have an optimization problem that is linear in the bonuses $\beta(x)$. The optimal bonuses will then have a bang-bang structure, and hence be either maximal or minimal, depending on the outcome x. Introducing the likelihood ratios

$$l_{a_i}(x, a) = f_{a_i}(x, a)/f(x, a),$$

we obtain the following:

Lemma 1 *There is a vector of multipliers μ such that (at the optimal action $a = a^*$) the optimal bonus is maximal for those outcomes x where $\Sigma_i \mu_i l_{a_i}(x, a) > 0$, and it is zero otherwise, i.e.*

$$\beta(x) = \frac{\delta}{1-\delta}(v(a) - c(a)) \quad \text{if} \quad \Sigma_i \mu_i l_{a_i}(x, a) > 0,$$

and $\beta(x) = 0$ if $\Sigma_i \mu_i l_{a_i}(x, a) < 0$.

The lemma says that there is an index $\tilde{y}(x) = \Sigma_i \mu_i l_{a_i}(x, a)$, with $a = a^*$ being the optimal action, such that the agent should be paid a bonus if and only if this index is positive, and the bonus should then be maximal. This index, which takes the form of a weighted sum of the likelihood ratios for the various action elements, is in this sense an optimal performance measure for the agent.

The index is basically a scorecard for the agent's performance, and since it is optimal, it is (more or less by definition) balanced. In the following we will introduce further assumptions to analyze its properties.

3 Scorecards and distorted measures

Following Baker (1992), Feltham-Xie (1994), and the often used modelling approach in the management accounting literature (e.g. Datar et al. 2001, Huges et al. 2005, Budde, 2007, 2009), in the remainder of the paper we will assume that the measurements x are potentially distorted and given by

$$x = Q'a + \varepsilon, \tag{2}$$

where Q' is an $m \times n$ matrix of rank $m \leq n$, and $\varepsilon \sim N(0, \Sigma)$ is multinormal with covariance matrix $\Sigma = [s_{ij}]$ (i.e. $x \sim N(Q'a, \Sigma)$).[9] Let $q_1, ..., q_m$ be the column vectors of Q, so we have $E(x_i | a) = q_i' a$, $i = 1...m$. As is common in

[9]Budde (2007) assumes in addition "balance", which implies that the first-best action can be implemented by linear bonuses when measurements are verifiable. For the main results (on relational contracts), measurements are also assumed to be noise free.

much of this literature, we assume multinormal noise for tractability The likelihood ratios for this distribution are linear in x, and this implies that the optimal performance index $\Sigma_i \mu_i l_{a_i}(x,a)$ identified in the previous lemma is also linear in x. In particular, the vector of likelihood ratios is given by the gradient $\nabla_a \ln f(x;a) = Q\Sigma^{-1}(x - Q'a)$. Hence, defining vector τ by $\tau' = \mu' Q \Sigma^{-1}$, the index can be written as $\Sigma_i \mu_i l_{a_i}(x,a^*) = \tau'(x - Q'a^*)$; where the expression in accordance with Lemma 1 is evaluated at $a = a^*$. So we have:

Proposition 1 *In the multinormal case, there is a vector τ and a performance index $\tilde{y} = \Sigma_j \tau_j x_j$ such that the agent is optimally paid a bonus if and only if the index exceeds a hurdle ($\tilde{y}_0$). The hurdle is given by the agent's expected performance in this setting ($\tilde{y}_0 = \Sigma_j \tau_j E(x_j | a^*)$), and the bonus, when paid, is maximal: $\beta(x) = \frac{\delta}{1-\delta}(v(a^*) - c(a^*))$.*

This result parallels Levin's (2003) characterization of the single-task case, where the agent optimally gets a bonus if his performance on the single task exceeds a hurdle. Here, in the multitask case, the principal offers an *index* $\tilde{y} = \Sigma_j \tau_j x_j$, i.e. a 'weighted sum' of performance outcomes on the various tasks, such that the agent gets a bonus if and only if this index exceeds a hurdle $\tilde{y}_0$. The optimal hurdle is given as the similar weighted sum of optimal expected performances. Hence, performance x_i is compared to expected performance, given (equilibrium) actions. If the weighted sum of performances exceeds what is expected, then the agent obtains the bonus.[10]

Figure 1 below illustrates the structure of the optimal bonus scheme. The index and its hurdle defines a hyperplane delineating outcomes "above" the plane from those "below", where the former are rewarded with full and maximal bonus while the latter yield no bonus at all. This is clearly different from a structure with separate bonuses and hurdles on each task. Such a structure is illustrated by the blue lines in the figure. In the two-dimensional case this structure defines four regions in the space of outcomes; where either

[10] The characterization given in the proposition relies on our maintained assumption that the first-order approach is valid. This is not innocous in the multinormal case. It is known that in such a setting with a single action ($n = 1$), the approach is not valid if measurements are very precise, i.e. if the variance of the performance measure is sufficiently small. On the other hand, it is valid in that setting if the variance is not too small; and as we will justify below, this is true also in the present multi-action setting.

zero, one or two bonuses are paid, respectively. The analysis shows that the structure defined by the index is better, and in fact optimal.

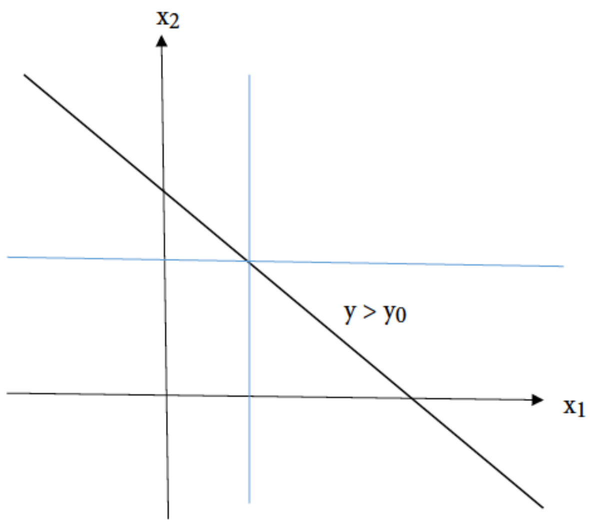

Figure 1. Structure of the optimal index contract.

Proposition 1 characterizes the type of bonus scheme that will be optimal. The next step is to characterize the parameters of the scheme, i.e. the weights τ and the hurdle $\tilde{y}_0$ that will generate optimal actions. To this we now turn.

Given the index $\tilde{y}$ with hurdle $\tilde{y}_0$, and the bonus $\beta = b$ being paid for $\tilde{y} > \tilde{y}_0$, the agent's performance related payoff is

$$b\,\Pr(\tilde{y} > \tilde{y}_0 |\, a) - c(a) = b\,\Pr(\tau' x > \tilde{y}_0 \big|\, a) - c(a)$$

Using the normal distribution we find that the agent's first order conditions for actions at their equilibrium levels ($a = a^*$), then satisfy

$$b\phi_0 \frac{1}{\tilde{\sigma}} Q\tau = \nabla c(a^*) \tag{3}$$

where $\phi_0 = 1/\sqrt{2\pi}$ is a parameter of the distribution, and $\tilde{\sigma}$ is the standard deviation of the performance index:

$$\tilde{\sigma} = SD(\tilde{y}) = (\tau' \Sigma \tau)^{1/2}.$$

Note that incentives, given by the marginal revenues on the left hand side of (3), are inversely proportional to the standard deviation $\tilde{\sigma}$. All else equal, a more precise performance index (lower $\tilde{\sigma}$) will thus enhance the effectiveness of a given bonus in providing incentives to the agent. This indicates that more precise measurements will be beneficial in this setting, and that this will occur not because of reduced risk costs (there are none, by assumption) but because of enhanced incentives. The monetary bonus is constrained by self enforcement, and other factors that enhance its effectiveness will then be beneficial. We return to this below.

The optimal bonus paid for qualifying performance is the maximal one, so

$$b = \frac{\delta}{1-\delta}(v(a^*) - c(a^*))$$

For given action a^* the elements b and τ of the optimal incentive scheme will be given by these relations.

On the other hand, optimal actions must maximize the surplus $v(a) - c(a)$ subject to these conditions. To characterize the associated optimization program for actions, it is convenient to introduce modified weights in the performance index, namely a weight vector θ given by

$$\theta = b\phi_0 \frac{1}{\tilde{\sigma}} \tau$$

Since θ is just a scaling of τ, i.e. $\theta = k\tau, k > 0$, the performance index can be expressed in terms of θ as $y = \theta' x$, and the agent is then given a bonus if this index exceeds its expected value $y_0 = \theta' E(x|\, a^*)$.

Note from the definitions of θ and $\tilde{\sigma}$ that $\theta' \Sigma \theta = (b\phi_0/\tilde{\sigma})^2 \tau' \Sigma \tau = \phi_0^2 b^2$, so we have:

$$\left(\theta' \Sigma \theta\right)^{1/2} / \phi_0 = b = \frac{\delta}{1-\delta}(v(a^*) - c(a^*)) \tag{4}$$

The optimal action a^* must thus satisfy (4) and the agent's first-order condition (3), which now takes the form $Q\theta = \nabla c(a^*)$. As noted, the optimal action must solve the problem of maximizing $v(a) - c(a)$ subject to these constraints. In fact, since the last equality in (4) reflects the dynamic enforcement constraint, we can replace it by weak inequality, and thus state the following result:

Proposition 2 *In the multinormal case, the optimal action a^* solves the following problem:*

$$\max_{a,\theta}(v(a) - c(a))$$

subject to $Q\theta = \nabla c(a)$ and

$$\frac{\delta}{1-\delta}(v(a) - c(a)) \geq \left(\theta'\Sigma\theta\right)^{1/2} / \phi_0 \tag{5}$$

The proposition shows that the general problem of finding a payment function and action can be reduced to the much simpler problem of finding a vector of weight parameters and an action. The optimal solution yields action a^* and associated weight parameters θ^* for the performance index. These weights are (from $Q\theta^* = \nabla c(a^*)$) given by

$$\theta^* = (Q'Q)^{-1}Q'\nabla c(a^*).$$

As noted above, the optimal action can be implemented by rewarding the agent with the largest dynamically enforceable bonus (as given in (4)) if and only if performance measured by the index $y = \theta^{*\prime}x$ exceeds its expected value $y_0 = \theta^{*\prime}E(x|\, a^*)$.

There are two sources for deviations from the first-best action in this setting, and they are reflected in the two constraints in the optimization problem. The first is due to distorted primary measures x, and will be relevant when the vector of marginal costs at the first-best actions (a^{FB}) cannot be written as $\nabla c(a^{FB}) = Q\theta$, for any θ; i.e. when this vector doesn't belong to the space spanned by (the column vectors of) Q.[11] Implications of distorted measures will be discussed below.

[11] This possibility is precluded in Budde (2007) by the requirement of measurements being balanced.

The second source is self-enforcement, which is reflected in the dynamic enforcement constraint (5). The expression $(\theta'\Sigma\theta)^{1/2}$ on the right-hand side of this constraint represents the standard deviation of the performance index $y = \theta' x$. It can be written as $(\Sigma_i \Sigma_j s_{ij} \theta_i \theta_j)^{1/2}$, where $s_{ij} = cov(x_i, x_j)$. It is clear that any variation in Σ that increases this expression will tighten the constraint, and hence reduce the total surplus. In particular, any increase of a variance in Σ will have this effect and, provided θ has no negative elements, any increase of a covariance in Σ will also have this effect. This substantiates the intuition discussed above about less precise measurements (larger variances) being detrimental in this setting.

It is also noteworthy that, provided θ has no negative elements, then positive correlations among elements in the measurement vector x will be detrimental for the surplus, while negative correlations will be beneficial. This follows because, all else equal, the former increases and the latter reduces the variance of the performance index. In the appendix (Appendix B) we present an example that illustrates these effects.

From the enforcement constraint (5) it may appear that any action a will satisfy this constraint if the standard deviation of the performance index on the right hand side is sufficiently small; and hence that the constraint becomes irrelevant (non-binding) if measurements are sufficiently precise. The result in Proposition 2 builds, however, on the assumption that the first-order approach is valid; and as we will demonstrate below, this is generally not the case for sufficiently precise measurements.

The approach replaces global IC constraints for the agent with a local one, and is only valid if the action (a^*) derived this way is in fact a global optimum for him under the given incentive scheme. Observe that, by choosing action a^* the agent gets a bonus if the index $y = \theta^{*\prime} x$ exceeds its expected value, an event which occurs with probability $\frac{1}{2}$. The agent's expected revenue is then $b/2$, with the bonus b given by (4), and this must strictly exceed the cost $c(a^*)$ in order for the agent to be willing to choose action a^*. This is so because by alternatively choosing action $a = 0$, the agent incurs zero costs but still obtains the bonus with some (small) positive probability. The following condition is thus necessary:

$$\frac{\delta}{1-\delta}(v(a^*) - c(a^*)) > 2c(a^*) \tag{6}$$

If a solution identified by the program in Proposition 2 doesn't satisfy this condition, it is not a valid solution. The reason is that the identified action is not a global optimum for the agent under the associated incentive scheme. A sufficient condition will be given below in Section 3.2.

Remarks on relational vs. classical multitasking contracts. It is of some interest to compare the relational contract in Proposition 2 to the by now classical Holmstrom-Milgrom (1991) and Feltham-Xie (1994) multitask contracts for verifiable measurements. In those models the agent is offered a linear incentive scheme $\beta' x + \alpha$, and for $E(x|\, a) = Q'a$ the IC constraint takes the form $Q\beta = \nabla c(a)$. With a risk averse (CARA) agent the total surplus (in certainty equivalents) is then $v(a) - c(a) - \frac{r}{2}\beta'\Sigma\beta$, where the last term captures risk costs, given by $\frac{r}{2}var(\beta' x)$. Letting $M = (Q'Q)^{-1}Q'$ we have $\beta = M\nabla c(a)$ and surplus

$$v(a) - c(a) - \frac{r}{2}(M\nabla c(a))'\Sigma(M\nabla c(a)),$$

which is to be maximized by choice of a.

In the maximization problem in Proposition 2 we have similarly from the IC constraint $Q\theta = \nabla c(a)$ that $\theta = M\nabla c(a)$, and the Lagrangian for the problem can then be written as $(v(a)-c(a))(1+\lambda)-\lambda\frac{1-\delta}{\delta\phi_0}\left((M\nabla c(a))'\Sigma(M\nabla c(a))\right)^{1/2}$, where λ is the shadow price on the enforcement constraint. Hence the optimal solution maximizes

$$v(a) - c(a) - \zeta\left((M\nabla c(a))'\Sigma(M\nabla c(a))\right)^{1/2},$$

where $\zeta = \frac{\lambda}{1+\lambda}\frac{1-\delta}{\delta\phi_0}$ can be seen as an (endogenous) cost factor.

There is thus a formal similarity between the models for the two contractual settings. But the mechanisms behind the trade-offs are different. When performance measures are verifiable, bonuses can in principle be arbitrarily large, but are optimally constrained due to the risk costs they generate for a risk averse agent. More precise measurements lower the risk costs and consequently make bonuses in a sense more effective instruments to achieve higher surplus. With non-verifiable measures, bonuses are constrained by self-enforcement at the outset, but are more effective in providing incentives if measurements are more precise. More precise measurements are thus beneficial in both settings, but for quite different reasons.

3.1 Distortions, alignment and precision.

We now discuss implications of distorted performance measures in the present setting. Such measures have been studied extensively for the case when these measures are verifiable, see e.g. Feltham-Xie (1994), Baker (1992), Datar et al. (2001), Budde (2007); and particularly in settings where value- and cost-functions are linear and quadratic, respectively:

$$v(a) = p'a + v_0 \qquad \text{and} \qquad c(a) = \frac{1}{2}a'a. \tag{7}$$

Here $\nabla c(a) = a$ and the first-best action, characterized by marginal cost being equal to marginal value, are given by $a^{FB} = p$. If we now neglect the dynamic enforceability constraint (5) in Proposition 2, we are led to maximize the surplus $p'a - a'a/2$ subject to $a = Q\theta$. This maximization yields $\theta = (Q'Q)^{-1}Q'p$ and action, here denoted a_0^* given by $a_0^* = Q(Q'Q)^{-1}Q'p$. The best action, subject only to the agent's IC constraint $a = Q\theta$, is thus generally distorted relative to the first-best action.

It may be noted that the solution a_0^* just derived is also the optimal solution in a setting where the measurements x are verifiable and the agent is rewarded with a linear incentive scheme $\beta'x + \alpha$. This is the setting studied in several papers on distorted measures, and the literature has introduced indicators to measure the degree of distortion. One such indicator is the ratio of second-best to first-best surplus (as in Budde 2007), which for the the second-best solution just derived (and with $v_0 = 0$) amounts to

$$\frac{a_0^{*\prime} a_0^*}{p'p} = \frac{p'Q(Q'Q)^{-1}Q'p}{p'p}$$

In particular, when the measure x is one-dimensional, so Q is a vector, say $Q = q \in R^n$, the ratio is $(p'q/\,|p|\,|q|)^2$ and is thus a measure of the alignment between vectors p and q. Then the first-best can be attained only if the two vectors are perfectly aligned ($q = kp, k \neq 0$).

In the case of non-verfiable measurements x, which is the case analyzed in this paper, the solution must also respect the dynamic enforcement constraint, represented by (5) in the last proposition. When this constraint binds, the action a_0^* is generally no longer feasible. Moreover, since the stochastic properties of the measurements, represented by the covariance matrix Σ, affect the constraint, they will also affect the solution.

This leads to a trade-off between alignment and precision when it comes to incentive provision. To highlight the trade-off, suppose there is a measure which is well aligned with the marginal value vector p, but which is very imprecise in the sense of having a large variance; and another measure which is not as well aligned with p, but is quite precise. In a setting with verifiable measures (and no risk aversion), the optimal solution would then entail strong incentives on the first measure and weak incentives on the second one. In particular, if the first measure, say x_1, is perfectly aligned with p, all incentives would be concentrated on this measure, and the second measure, say x_2, would be neglected (by letting the associated bonus θ_2 be zero). This solution, however, would imply a large variance for the performance index, and hence quite possibly be infeasible under self-enforcement by violating the constraint (5). The constraint may thus imply weaker incentives on measures that are well aligned but imprecise, and stronger incentives on measures that are less well aligned but more precise.

The trade-off emerges very clearly if we consider a limiting case where some measure (say x_2) has a vanishingly small variance. In the limit, with $var(x_2) \to 0$, the first-order conditon for the optimal weight θ_2 in the index is then $(p - a)'q_2 = 0$. (This follows from the surplus being $p'a - \frac{1}{2}a'a$ with[12] $a = Q\theta = \Sigma_{i=1}^{m} q_i\theta_i$ and all terms containing θ_2 becoming zero in the quadratic form on the right-hand side of the constraint (5).) In this limiting case, it is thus optimal that vector $p - a$ is orthogonal to vector q_2, irrespective of how well aligned the other measures are with p.[13]

Such a case is illustrated in the figure below, where there are two available measures, and measure x_1 with associated vector q_1 is much better aligned

[12] Recall that Q has columns $q_1, ...q_m$.

[13] This holds as long as the assumptions behind Proposition 2 are valid. A sufficiently large variance of the total performance index is sufficent, see Proposition 4 below.

with p than measure x_2. If now the variance $var(x_1)$ is large, the enforcement constraint forces the weight θ_1 to be small, and this weight will then, for sufficiently large $var(x_1)$, be smaller than the weight θ_2 on the relatively poorly aligned measure represented by vector q_2 in the figure.

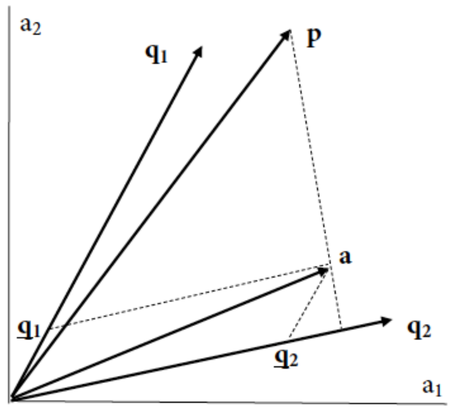

Figure 2. Illustration of distortion vs. precision

In this figure the points $\underline{q}_i = \theta_i q_i,\ i = 1, 2$ represent the components of vector $a = \theta_1 q_1 + \theta_2 q_2$, where θ_1, θ_2 are now determined by the requirement that $p - a$ is to be orthogonal to q_2, plus the binding enforcement constraint (5).[14] A large $var(x_1)$ will force θ_1 to be small, and a picture like that in the figure then emerges: a small weight θ_1 (and thus weak incentives) on the highly aligned but imprecise measure, and a considerably larger weight θ_2 on the very precise but less aligned measure.

It is also worth noting that lower alignment may improve welfare here. If we keep θ_1 fixed in the figure, and then rotate vector q_1 counter-clockwise to become less aligned with p, the dotted line parallel to q_2 will shift up, and vector $p - a$ will become shorter. This increases welfare (due to the identity $p'a - \frac{1}{2}a'a = \frac{1}{2}|p|^2 - \frac{1}{2}|p - a|^2$), and since θ_1 is kept fixed, the enforcement constraint will become slack. Higher welfare is then feasible.

In the appendix (Appendix B) we present a numerical example that illus-

[14]The first requirement implies $p'q_2 - \theta_1 q_1' q_2 - \theta_2 q_2' q_2 = 0$. Substituting for θ_2 in the constraint (5) then determines θ_1.

trates the issues discussed above.

Consider now an algebraic analysis. Observe first that for the linear-quadratic case, the Lagrangean for the optimization problem in Proposition 2 can be written as $(1+\lambda)(p'a - \frac{1}{2}a'a) - \lambda\frac{1-\delta}{\delta\phi_0}(\theta'\Sigma\theta)^{1/2}$ with $a = Q\theta$. The first-order conditions for the optimal θ^* therefore include

$$Q'(p - Q\theta^*) - \frac{\lambda}{1+\lambda}\frac{1-\delta}{\delta\phi_0}(\theta^{*\prime}\Sigma\theta^*)^{-1/2}\Sigma\theta^* = 0$$

Say that the enforcement constraint is *strictly binding* if the multiplier (shadow cost) λ is non-zero, and thus $\lambda > 0$. If we define $\psi^* = \frac{\lambda}{1+\lambda}\frac{1-\delta}{\delta\phi_0}(\theta^{*\prime}\Sigma\theta^*)^{-1/2}$ we may then state the following result.

Corollary 1 *Let $v(a) = p'a$ and $c(a) = \frac{1}{2}a'a$. An optimal solution in Proposition 2 with the enforcement constraint strictly binding then satisfies*

$$\theta^* = (\psi^*\Sigma + Q'Q)^{-1}Q'p$$

with $\psi^* > 0$.

The trade-off between distortion and precision is captured in this expression for θ^*, and can be nicely illustrated by considering measurements that are uncorrelated and for which the associated vectors in Q are orthogonal, i.e. $q_i'q_j = 0$ for all $i \neq j$. Then the formula yields

$$\theta_i^* = \frac{q_i'p}{\psi^* s_{ii} + q_i'q_i}, \quad i = 1, ..., m$$

All else equal, a measure with better alignment (larger $q_i'p$) will optimally have a larger weight in the index; but also, all else equal, so will a measure with higher precision (smaller variance s_{ii}). A highly precise, but not so well aligned measure may thus get a larger weight than a measure that is better aligned, but quite imprecise.

Remark. A formally similar trade-off between distortion and precision arises in multi-tasking models with verifiable measurements and a risk averse agent, such as Feltham-Xie (1994) or Datar et al. (2001). In fact, the optimal bonuses in the setting of these papers will be given by a formula identical to the formula in Corollary 1, except that ψ^* will be replaced by the agent's coefficient of absolute risk aversion (r). This is not surprising in light of the formal similarity between the two types of models that we pointed out above. But we should keep in mind that the trade-offs arise from two very distinct phenomena: the requirements of self-enforcement and the costs of risk exposure, respectively. Moreover, while comparative statics results are relatively straightforward in the Feltham-Xie setting, they are less straightforward here. For example, we cannot conclude directly from the last displayed formula that θ_i^* is decreasing in the variance s_{ii}, because ψ^* is endogenous and hence also depends on s_{ii}.

It turns out that a two-step procedure is fruitful for deriving comparative statics results. In the first step, consider the problem of finding an index that implements a given surplus V with minimal variance, i.e. the problem

$$\min \theta' \Sigma \theta \quad \text{s.t.} \quad \nabla c(a) = Q\theta \quad \text{and} \quad v(a) - c(a) \geq V$$

Let $\hat{\theta}(\Sigma, V)$ be the optimal solution and $m(\Sigma, V)$ the minimal value. Observe that for $V > v(0)$ the last constraint here must bind, since otherwise $a = 0$ and $\theta = 0$ would solve the minimization problem.

Next observe that if (θ^*, a^*) is a solution to the problem in Proposition 2 with the enforcement constraint strictly binding and with surplus $V^* = v(a^*) - c(a^*)$, then we must have

$$\theta^* = \hat{\theta}(\Sigma, V^*)$$

If this was not true, there would be (a, θ) satisfying the two constraints in the minimization problem and $\theta' \Sigma \theta < \theta^{*\prime} \Sigma \theta^*$. Since the enforcement constraint in Proposition 2 would then be slack, a higher surplus than V^* would be feasible.

From the last formula we now have

$$\frac{\partial \theta_i^*}{\partial s_{ii}} = \frac{\partial \hat{\theta}_i}{\partial s_{ii}} + \frac{\partial \hat{\theta}_i}{\partial V} \frac{\partial V^*}{\partial s_{ii}} \tag{8}$$

This (Slutsky type) formula shows that the effect on the weight θ_i^* in the optimal index can be decomposed in two effects: first an effect induced from a change in s_{ii} with the value V^* held constant; and second an effect generated by the change in V^* induced by the change in s_{ii}.

It turns out that the first effect, i.e. the "own effect" on the weight $\hat{\theta}_i$ of an increase in the variance s_{ii}, has the opposite sign of $\hat{\theta}_i$, and is thus negative if $\hat{\theta}_i$ is positive. This follows from the minimal value $m(\Sigma, V)$ being concave[15] in Σ and the envelope property, which implies

$$0 \geq \frac{\partial^2 m}{\partial s_{ii}^2} = \frac{\partial}{\partial s_{ii}} \hat{\theta}_i^2 = 2\hat{\theta}_i \frac{\partial \hat{\theta}_i}{\partial s_{ii}}$$

Regarding the second effect in the decomposition (8), we know from the discussion following Proposition 2 that the value V^* is decreasing in a variance s_{ii}. We thus have $\frac{\partial V^*}{\partial s_{ii}} \leq 0$, but it appears that the sign of $\frac{\partial \hat{\theta}_i}{\partial V}$ may depend on the parameters, and hence that the total effect in (8) cannot be unambigously signed. For the linear-quadratic case with uncorrelated and orthogonal measurements, however, we can show that $\frac{\partial \hat{\theta}_i}{\partial V}$ has the opposite sign of $\frac{\partial \hat{\theta}_i}{\partial s_{ii}}$, which then implies that the two terms representing the two effects in (8) have equal signs. Thus we have the following result.

Proposition 3 *Let $v(a) = p'a$ and $c(a) = \frac{1}{2}a'a$, and assume that the measurements are uncorrelated and that $q_i' q_j = 0$ for all $i \neq j$. An optimal solution in Proposition 2 with the enforcement constraint strictly binding then satisfies*

$$\theta_i^* \frac{\partial \theta_i^*}{\partial s_{ii}} \leq 0, \quad i = 1, ..., n.$$

The absolute value of the weight θ_i^* on measurement x_i in the optimal index will thus be decreasing in the measurement's variance s_{ii}.

The trade-off between distortion and precision that we have analyzed in this section, implies that scorecards must be constructed to find the best balance between these effects. Scorecards can be based on non-verifiable measures, and among those it may be possible to find one that is well aligned with the principal's true (marginal) values. This does not mean, however, that such a measure should be given a large weight in the scorecard index. If the measure is highly imprecise, a large weight on this measure may make

[15] Concavity of m follows by observing that if $k \in (0, 1)$ and $\Sigma = k\Sigma_1 + (1-k)\Sigma_2$, then $\theta'\Sigma\theta = k\theta'\Sigma_1\theta + (1-k)\theta'\Sigma_2\theta$, and hence $\theta'\Sigma\theta \geq km(\Sigma_1, V) + (1-k)m(\Sigma_2, V)$ holds for any θ that is admissible in the minimization problem.

the relational contract non-sustainable. Then it will be better to shift more weight to measures that are more precise, even if they may be less well aligned with the principal's true value.

3.2 Validity of the first-order approach

We have throughout assumed FOA to be valid. Here we give sufficient conditions for this to be the case.

Let a^*, θ^* be a solution to the optimization problem in Proposition 2. The agent then gets a bonus (b) if the index $y = x'\theta^*$ exceeds the hurdle $y_0 = E(y|\, a^*) = a^{*\prime}Q\theta^*$. By construction, a^* satisfies the first-order conditions for the agent's optimization problem. These conditions are given by $Q\theta^* = \nabla c(a^*)$. We will find conditions guaranteeing that a^* is indeed an optimal choice for the agent. Observe that when the enforcement constraint binds, the necessary condition (6) implies a lower bound for the standard deviation of the performance index: $(\theta^{*\prime}\Sigma\theta^*)^{1/2} > 2c(a^*)\phi_0$.

If the agent chooses an action a, the index y has expectation $e = E(y|\, a) = a'Q\theta^*$ and variance $\sigma^2 = vary = \theta^{*\prime}\Sigma\theta^*$. Given our assumptions, the index y is $N(e, \sigma)$, and thus has a probability distribution that depends on action a only via the (one-dimensional) expectation $e = E(y|\, a)$. The agent's expected revenue ($b \Pr(y > y_0|\, a)$) then also depends on a only via e. In light of this, it is natural to consider the action that induces e with minimal costs for the agent, i.e. action $\hat{a}(e)$ given by

$$\hat{a}(e) = \arg\min_a c(a) \quad \text{s.t.} \quad a'Q\theta^* = e,$$

and let $C(e) = c(\hat{a}(e))$ be the minimal cost. We can then essentially write the agent's payoff as a function $u(e)$ (see the appendix for details), and seek conditions which guarantee that this function has a unique maximum. This yields the following result.

Proposition 4 *Let a^*, θ^* be a solution from Proposition 2 with the enforcement constraint binding. There is $\sigma_0^* > 0$ such that a^* is an optimal choice for the agent, and thus the first-order approach is valid, if and only if $\theta^{*\prime}\Sigma\theta^* \geq \sigma_0^{*2}$. A sufficient condition (for strict inequality, $\theta^{*\prime}\Sigma\theta^* > \sigma_0^{*2}$,)* *is*

$$\frac{\delta}{1-\delta}(v(a^*) - c(a^*)) \geq a^{*\prime}\nabla c(a^*)\sqrt{h(a^*)}/(2\phi_0), \tag{9}$$

where $h(a)$ is a function defined from the cost function $c(a)$, see defnition (18) in the appendix, and which is identically 1 *for a quadratic cost function.*

Observe that for a quadratic cost function the expression on the right-hand side of (9) is $c(a^*)/\phi_0$ with $1/\phi_0 = \sqrt{2\pi} \approx 2.5$. A sufficient condition for the approach employed in Proposition 2 to be valid in this case is thus that the solution entails a cost for the agent that is no larger than 40% of the entire value of the future relationship.

It can be verified that for sufficiently imprecise measurements, a solution from Proposition 2 will indeed, under some regularity conditions, satisfy condition (9). Specifically, assuming $\Sigma = s\Sigma_0$ and $\lim_{a\to 0} a'\nabla c(a)\sqrt{h(a)} = 0$ we can verify that if $s > 0$ is sufficiently large, a solution a^* will satisfy this condition when $v(0) > 0$.[16] This is so because a solution a^* will necessarily become "small" (approach zero) when measurements become very imprecise ($s \to \infty$), and then (9) will be satisfied under the given assumptions.

3.3 Very precise measurements

We have seen that the first-order approach used to derive Proposition 2 may be invalid if measurements are noisy, but very precise. Specifically, the action a_0^* that maximizes surplus subject to the constraint $\nabla c(a) = Q\theta$ will be a solution to the program in Proposition 2 if measurements are sufficiently precise to make the index variance ($\theta'\Sigma\theta$) small enough to satisfy the enforcement constraint. This is true for any $\delta > 0$, but the action a_0^* will not satisfy the necessary condition (6) for a valid solution if δ is sufficiently small. Hence the first order approach is not valid in such a case.

We thus lack a characterization of optimal incentive schemes for settings with noisy but very precise measurements. On the other hand, the optimal scheme for an environment with no noise is known (e.g. Budde 2007). In this subsection we show that *if V^{NF} is the optimal surplus in a setting with*

[16]This will also hold for $v(0) = 0$ if $(v(a) - c(a))/a'\nabla c(a)\sqrt{h(a)}$ is bounded away from zero when $a \to 0$.

no noise, then any surplus value $V < V^{NF}$ can be implemented with an index contract if the measurements are sufficiently precise. Index contracts (scorecards) are in this sense at least approximately optimal for sufficiently precise measurements.

Measurements without noise. As a reference case we first consider measurements with no noise, i.e. of the form

$$x = Q'a.$$

We have then that an action a can be implemented by some bonus scheme $\beta(x)$ if and only if

$$\nabla c(a) = Q\gamma \tag{10}$$

for some $\gamma \in R^m$. The condition is necessary because, if a generating measurement $x = Q'a$ is optimal for the agent, then it must be cost-minimizing among all actions that generate the same x. So it must solve $\min_{\tilde{a}} c(\tilde{a})$ subject to $x = Q'\tilde{a}$, and hence satisfy the first-order condition (10) with Lagrange multiplier γ. Observe that γ is uniquely given by $\gamma = (Q'Q)^{-1}Q'\nabla c(a)$. On the other hand, if a satisfies (10), it is a cost-minimizing action generating measurement $x = Q'a$, and will be chosen by the agent under a bonus scheme with $\beta(x) \geq c(a)$ and $\beta(\tilde{x}) = 0, \tilde{x} \neq x$.

Being discretionary, bonuses must respect a dynamic enforcement constraint. Since the minimal bonus to implement an action a is its cost $c(a)$, the constraint here takes the form

$$c(a) \leq \frac{\delta}{1-\delta}(v(a) - c(a)) \tag{11}$$

The optimal contract in this setting thus maximizes the surplus $v(a) - c(a)$ subject to (10) and (11). Let a^{NF} denote the optimal action and V^{NF} the maximal surplus in this noise-free environment. In the following we will assume that the enforcement constraint binds and implies a surplus V^{NF} strictly less than the optimal surplus obtained without the constraint, thus $V^{NF} < V_0^* = \max\{v(a) - c(a) | \nabla c(a) = Q\theta, \theta \in R^m\}$.

When the enforcement constraint here binds, we have $c(a^{NF}) = \delta v(a^{NF})$. We further have, from (10) that $\nabla c(a^{NF}) = Q\gamma$. In the linear-quadratic case as in (7) with $v_0 = 0$, this yields $a^{NF} = Q\gamma$ and (by optimization of the surplus with respect to γ) $\gamma = k(Q'Q)^{-1}Q'p$ with $k = 2\delta$ when the enforcement constraint binds, and $k = 1$ otherwise. The constraint binds for $\delta < \frac{1}{2}$. The optimal surplus is then $V^{NF} = (k - \frac{1}{2}k^2)p'Q(Q'Q)^{-1}Q'p$. This is a case considered in Budde (2007).

Measurements with noise. Consider again noisy measurements, and recall that the approach behind Proposition 2 is valid only if the solution (action a^*) satisfies condition (6). This condition is stricter than condition (11). This implies that, although noise-free measurements can be seen as a limiting case of noisy measurements when all variances go to zero, a valid solution from Proposition 2 can generally not converge to a^{NF}.

It may be noted that Chi and Olsen (2018) have found that for settings with a univariate action, an index contract derived from the likelihood ratio is still optimal even when the first-order approach is not valid. The only required modification is that the threshold for the index must be adjusted, taking into account not only a local IC constraint for the agent, but also non-local ones, which will be binding. It is an open question whether a similar property holds in settings with multivariate actions.

In the setting of this paper, however, we can show that for noisy but sufficiently precise measurements, any surplus $V < V^{NF}$ can be obtained by means of an index contract. This doesn't mean that such a contract is optimal, but it will at least be approximately optimal for such measurements. Specifically, we will consider actions that satisfy

$$2c(a) \geq \frac{\delta}{1-\delta}(v(a) - c(a)) > c(a), \tag{12}$$

plus $\nabla c(a) = Q\theta$ for some $\theta \in R^n$. Such an action will be feasible for the optimization problem with noise free measurements, but not optimal in that problem, since the enforcement constraint (11) does not bind. Hence it generates a surplus $V < V^{NF}$, but the action a can be chosen such that V is arbitrarily close to V^{NF}.

The first inequality in (12) implies that the necessary condition (6) for FOA to be valid is violated, hence a cannot be implemented by the scheme applied in Proposition 2. Recall that this is a consequence of the scheme being designed such that, for the desired action the agent's expected revenue falls short of his costs. (The hurdle for the index is set to maximize marginal incentives, but this implies that the probability to obtain the bonus is 1/2, and the first inequality in (12) then implies a negative payoff for the agent, relative to choosing action $a = 0$.)

It seems intuitive that this problem can be alleviated by modifying the hurdle so as to make it less demanding for the agent to qualify for the bonus. On the other hand, such a modification will also negatively affect the agent's marginal incentives. It turns out that, if the measurements are sufficiently precise, a modification of the hurdle can achieve both goals: sufficiently strong incentives and a sufficiently large payoff for the agent, so that the desired action can be implemented. This is formally stated as follows.

Proposition 5 *Let action a satisfy $2c(a) \geq \frac{\delta}{1-\delta}(v(a) - c(a)) > c(a)$ and $\nabla c(a) = Q\theta$, for some $\theta \in R^m$. There is $\sigma_0 > 0$ with the following property: If Σ satisfies $\theta'\Sigma\theta \equiv \sigma^2 < \sigma_0^2$, then there is a hurdle $\kappa(\sigma) < E(x'\theta|\, a)$ such that the index $x'\theta$ with hurdle $\kappa(\sigma)$ implements a. Moreover, $\kappa(\sigma) \to E(x'\theta|\, a)$ as $\sigma \to 0$.*

Observe that the second condition in this proposition requires that $\nabla c(a)$ belongs to the span of Q and thus can be written $\nabla c(a) = Q\theta$. By our assumptions regarding Q this implies that θ is unique and given by $\theta = (Q'Q)^{-1}Q'\nabla c(a)$.

Now recall that an action a satisfying the two conditions in the proposition generates a surplus V smaller than the optimal surplus with no noise (V^{NF}), but that a can be chosen such that V is arbitrarily close to V^{NF}. An immediate consequence of the proposition is then the following:

Corollary 2 *Any surplus $V < V^{NF}$ can be obtained by means of an index contract, provided measurements are sufficiently precise.*

The proposition also implies that if an index contract generates a surplus V that is close to V^{NF}, and this contract is optimal in the class of index contracts, then FOA must necessarily be violated, and hence some non-local incentive constraint must bind.[17]

[17] The optimal action yielding surplus V must be close to the action a^{NF} yielding surplus

This implies that characterizing the optimal (linear) index contract can be technically challenging in this setting. Of course this applies also for the overall optimal contract, since it must have non-local incentive constraints binding as well. (Otherwise it would be characterized by Proposition 2, and thus be an index contract with only a local constraint binding.) We leave these issues as topics for future research.

4 Non-verifiable and verifiable measurements

We have so far focused on non-verifiable measurements. But incentive schemes, at least for top management, will typically also include verifiable financial performance measures. Consider then a situation where there are both non-verifiable and verifiable measurements available. To simplify the exposition we will assume that there is one verifiable measure (x_0) in addition to the non-verifiable measures (x) considered above. The latter depends stochastically on effort as in (2) and the former is assumed to have a similar representation:

$$x_0 = q_0' a + \varepsilon_0,$$

where $q_0 \in R^n$ and ε_0 is normally distributed noise generally correlated with the noise variables ε in x. (More precisely, the vector $(\varepsilon_0, \varepsilon)$ is multinormal.)

The agent can now be incentivized by a court enforced (explicit) bonus $b_0 x_0$ on the verifiable measure and a discretionary (relational) bonus $\beta(x_0, x)$ depending on the entire measurement vector (x_0, x). We consider a case where only short term explicit contracts are feasible, which allows us to confine attention to stationary contracts[18].

In each period, the agent will now choose action a to maximize $E(\, b_0 x_0 + \beta(x_0, x) |\, a) -$

V^{NF}, and since the latter action by our assumptions satisfies (11) with equality and thus violates the necessary condition (6) for FOA to be valid, the former action must also violate this condition.

[18] Watson, Miller and Olsen (2020) analyse long term renegotiable court-enforced contracts, and show that it will generally be optimal to renegotiate these contracts each period when in combination with relational contracts.

$c(a)$, yielding first-order conditions[19]

$$\int (b_0 x_0 + \beta(x_0, x)) f_{a_i}(x_0, x, a) - c_i(a) = 0, \quad i = 1, ..., n.$$

Returning to the assumption that FOA is valid, the principal then maximizes the total surplus $v(a) - c(a)$ subject to these constraints and the dynamic enforcement constraint. We assume as before that the parties separate if the relational contract is broken. The enforcement constraint is then the same as (1), just with x now replaced by the entire measurement vector (x_0, x).

From the same principles as before it follows that the agent should be given the discretionary bonus if and only if an index exceeds a hurdle, and from the normal distribution it follows that this index is linear in the measurements; $y = \Sigma_{i=0}^m \tau_i x_i \equiv \tau_0 x_0 + \tau' x$, and moreover that the hurdle is $y_0 = E(\Sigma_{i=0}^m \tau_i x_i | \, a^*)$, where a^* is the equilibrium action. If the magnitude of the bonus is b, this leads to the following first-order conditions for the agent at the equilibrium action:

$$(b_0 + b\frac{\phi_0}{\sigma}\tau_0) q_0 + b\frac{\phi_0}{\sigma} Q\tau = \nabla c(a^*)$$

where now $\sigma^2 = var \Sigma_{i=0}^m \tau_i x_i = var(\tau_0 x_0 + \tau' x)$ is the variance of the performance index in this setting.

As before, it is convenient to introduce modified weights in the index:

$$\theta_0 = b\frac{\phi_0}{\sigma}\tau_0, \qquad \theta = b\frac{\phi_0}{\sigma}\tau.$$

This yields $var(\Sigma_{i=0}^m \theta_i x_i)/\phi_0^2 = (b\frac{1}{\sigma})^2 var(\Sigma_{i=0}^m \tau_i x_i) = b^2$, and implies that the IC condition and the dynamic enforcement condition can be written as, respectively; the following relations:

$$(b_0 + \theta_0) q_0 + Q\theta = \nabla c(a)$$

$$\frac{\delta}{1-\delta}(v(a) - c(a)) \geq \frac{1}{\phi_0}(var(\theta_0 x_0 + \theta' x))^{1/2}$$

The principal maximizes the total surplus $v(a) - c(a)$ subject to these con-

[19] Here we use $f(x_0, x, a)$ to denote the joint density of all measurements, conditional on action.

straints.

Since the court-enforced bonus b_0 can be chosen freely, while the elements θ_0, θ of the discretionary bonus scheme are constrained by self-enforcement, we see that θ_0 should be chosen so as to minimize the variance appearing in the enforcement constraint. (If not, then for given θ we could modify b_0 and θ_0 so that the IC constraint holds and the enforcement constraint becomes slack.)

The variance is minimized for $\theta_0 = -cov(x_0, \theta' x)/s_0^2$, where $s_0^2 = var(x_0)$, and this implies in turn that the performance index takes the form

$$\theta_0 x_0 + \theta' x = \Sigma_{i=1}^m \theta_i (x_i - \frac{cov(x_0, x_i)}{s_0^2} x_0).$$

This shows that for correlated measurements ($cov(x_0, x_i) \neq 0$), performance on the verifiable measure is taken into the index as a benchmark, to which the other performances are compared.

The hurdle for the index is the expected value $\Sigma_{i=1}^m \theta_i (e_i^* - \frac{cov(x_0, x_i)}{s_0^2} e_0^*)$, where $e_i^* = E(x_i | \, a^*), i = 0, ..., m$. Since $e_i^* + \frac{cov(x_0, x_i)}{s_0^2}(x_0 - e_0^*)$ is the conditional expectation of x_i, given x_0 (and a^*), it follows that we can write the condition for the index to pass the hurdle as

$$\Sigma_{i=1}^m \theta_i (x_i - E(\, x_i | \, x_0, a^*)) > 0.$$

Performance x_i is thus compared to expected performance, given (equilibrium) actions and the outcome on the verifiable measure. If the performance exceeds what is expected, given this information, then it contributes positively to making the index exceed the hurdle, and thus for the agent to obtain the bonus.

By benchmarking the agent's performance on the non-verifiable measures to her performance on the verifiable one, the precision of the performance index can be increased, and thereby the dynamic enforcement constraint can be relaxed and the surplus increased. The minimized index variance is

$$\min_{\theta_0} var(\theta_0 x_0 + \theta' x) = var(\Sigma_{i=1}^m \theta_i \tilde{x}_i) = \theta' \tilde{\Sigma} \theta,$$

where $\tilde{x}_i = x_i - \frac{cov(x_0 x_i)}{s_0^2} x_0$, $i = 1, ..., m$, and $\tilde{\Sigma}$ is the covariance matrix for $\tilde{x}$. We have $cov(\tilde{x}_i, \tilde{x}_j) = s_{ij} - \rho_{0i}\rho_{0j}(s_{ii}s_{jj})^{1/2}$, where $\rho_{0i} = corr(x_0, x_i)$, $i = 1, ..., m$ are the correlation coefficients between the verifiable and the non-verifiable measures. We see that if all of these have the same sign, then all elements in the new covariance matrix $\tilde{\Sigma}$ are reduced relative to the elements of matrix Σ. Moreover, the stronger these correlations are in such a case, the smaller are the elements of $\tilde{\Sigma}$, and the smaller is then the variance $\theta'\tilde{\Sigma}\theta$ if all elements of θ are non-negative. This will then relax the enforcement constraint and increase the surplus. Stronger correlations, either all positive or all negative, between the verifiable and each non-verifiable measure, will thus increase the surplus in such a case.

We finally outline an approach to solve for the optimal contract in the setting considered here, and apply this to the linear-quadratic case. First define $\tilde{b}_0 = b_0 + \theta_0$, so that the IC constraint takes the form $\tilde{b}_0 q_0 + Q\theta = \nabla c(a)$, and next define

$$S(\theta) = \max_{\tilde{b}_0, a} \{ v(a) - c(a) | \, \tilde{b}_0 q_0 + Q\theta = \nabla c(a) \}.$$

Then $S(0)$ would be the optimal surplus the parties could achieve if only the verifiable measure x_0 were available. The relational contract allows the parties to achieve

$$\max_{\theta} S(\theta) \quad \text{s.t.} \quad \frac{\delta}{1-\delta} S(\theta) \geq (\theta'\tilde{\Sigma}\theta)^{1/2}/\phi_0$$

In the linear-quadratic case ($v(a) = p'a$ and $c(a) = \frac{1}{2}a'a$), the IC constraint is $\tilde{b}_0 q_0 + Q\theta = a$, and using this to substitute for a, we find that the surplus to be maximized in the first step (with respect to $\tilde{b}_0$) is

$$\tilde{b}_0 p' q_0 - \frac{1}{2}\tilde{b}_0^2 q_0' q_0 - \tilde{b}_0 \theta' Q' q_0 + p' Q\theta - \frac{1}{2}\theta' Q' Q\theta$$

We see that, except if q_0 is orthogonal to all the columns of Q, i.e. $Q' q_0 = 0$, then the optimal bonus $\tilde{b}_0$ will depend on θ and hence be different from the optimal bonus for the verifiable measure alone.

The optimal value in this step is

$$S(\theta) = \frac{1}{2q_0'q_0}(p'q_0 - \theta'Q'q_0)^2 + p'Q\theta - \frac{1}{2}\theta'Q'Q\theta$$

The formula illustrates that, relative to a situation with only non-verifiable measures, the verifiable one helps by (i) providing incentives that generate value (the first term in $S(\theta)$), and (ii) by relaxing the enforcement constraint; partly via the higher value, and partly by allowing for valuable benchmarking in the performance index. Conversely, relative to a setting with only the verifiable measure available, the non-verifiable ones generally allow for a higher surplus to be achieved.

This is illustrated in the figure below for the case of one verifiable (x_0) and one non-verifiable measure (x_1), with associated vectors q_0 and q_1, respectively. If only x_0 is available, only action vectors on the line L_0 can be implemented, and the optimal action is then the projection of p on this line, defined by action $a_0 = b_0q_0$ such that $(p - a_0)'q_0 = 0$. When also x_1 is available, action vectors on a parallel line such as L_1 (given by $a = \theta_1q_1 + \tilde{b}_0q_0$) can be implemented. This allows for implementation of action vectors with a smaller distance to p, and hence a larger surplus. The optimality condition for the bonus $\tilde{b}_0$ on the verifiable measure still implies $(p - a)'q_0 = 0$, and hence that $p - a$ should be orthogonal to q_0 (and line L_1). The figure makes clear that, unless q_0 and q_1 are orthogonal, the bonus $\tilde{b}_0$ defining the component $\underline{q}_0 = \tilde{b}_0q_0$ of action a will be different from the optimal bonus when only measure x_0 is available. Adding a scorecard with non-verifiable measures will thus generally require an adjustment of formal incentives in the agent's total compensation package.

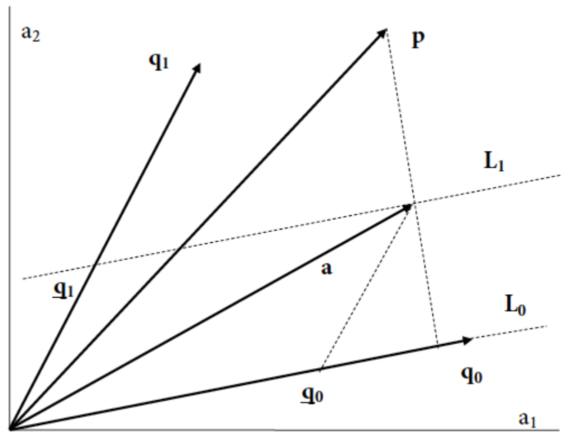

Figure 3. Illustration of incentives on verifiable and non-verifiable measurements.

5 Conclusion

Employees are often evaluated along many dimensions, and at least some of the performance measures will typically be non-verifiable to a third party. They may also be misaligned with (distorted from) the true values for the principal, and be stochastically dependent. The aim of this paper is to study this environment: Optimal incentives for multitasking agents whose performance measures are non-verifiable and potentially distorted and correlated. We extend and generalize the received literature in some important dimensions (to an arbitrary number of tasks with stochastic measurements that are possibly correlated and/or distorted), and we invoke assumptions (normally distributed measurements) that make the model quite tractable.

We show that under standard assumptions, the optimal relational contract is an index contract. That is, the agent gets a bonus if a weighted sum of performance outcomes on the various tasks (an index) exceeds a hurdle. The weights reflect a trade-off between precision and distortion for the various measures. The efficiency of this contract improves with higher precision of the index measure, since this strengthens incentives. Correlations between measurements may for this reason be beneficial. For a similar reason, the principal may also want to include verifiable performance measures in the relational index contract in order to improve incentives. These are then included as benchmarks, to which the other performances are compared.

We point out that for very precise, but still noisy measurements, the standard first-order approach breaks down, and we show that, although index contracts may no longer be optimal in such settings, they can be adjusted to become asymptotically optimal.

The index contracts in our model bear resemblance to key features of the performance measurement system known as balanced scorecards. Reward systems based on BSC typically include non-verifiable measures and connect pay to an index. In that sense, our paper provides a contract theoretic rationale for the way BSC schemes are implemented. However, while the scheme we present is a bonus contract with just one threshold (or 'hurdle'), scorecards in practice often have several thresholds and bonus levels, where the size of the bonus depends on the score. Future research can extend the model we present to incorporate e.g. risk aversion or limited liability, in order to study under which broader conditions the index contract is optimal, and what kind of index contracts that are optimal under various model specifications.

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APPENDIX A: PROOFS.

Proof of Lemma 1. The lemma follows directly from the Lagrangian for the problem, which takes the form

$$L = v(a)-c(a)+\Sigma_i\mu_i(\int \beta(x)f_{a_i}(x,a)-c_i(a))+\int \lambda(x)(\tfrac{\delta}{1-\delta}(v(a)-c(a))-\beta(x))$$

At the optimal action $a = a^*$, the optimal bonus satisfies

$$\tfrac{\partial L}{\partial \beta(x)} = \Sigma_i\mu_i f_{a_i}(x,a) - \lambda(x) = 0 \text{ if } \beta(x) > 0, \qquad \leq 0 \text{ if } \beta(x) = 0$$

Hence we have

If $\Sigma_i \mu_i f_{a_i}(x,a) > 0$ then $\lambda(x) > 0$ and hence $\beta(x) = \frac{\delta}{1-\delta}(v(a) - c(a))$.

If $\Sigma_i \mu_i f_{a_i}(x,a) < 0$ then $\frac{\partial L}{\partial \beta(x)} < 0$ and hence $\beta(x) = 0$ (implying $\lambda(x) = 0$).

Verification of (3). Given that $\tilde{y} = \tau' x$ is nomal with expectation $E(\tilde{y}|\, a)$ and variance $\tilde{\sigma}^2 = \tau' \Sigma \tau$, we have

$$\Pr(\tilde{y} > \tilde{y}_0|\, a) = \Pr(\frac{\tilde{y} - E(\tilde{y}|\, a)}{\tilde{\sigma}} > \frac{\tilde{y}_0 - E(\tilde{y}|\, a)}{\tilde{\sigma}}\Big|\, a) = 1 - \Phi(\frac{\tilde{y}_0 - E(\tilde{y}|\, a)}{\tilde{\sigma}}) \tag{13}$$

where $\Phi(\cdot)$ is the standard normal CDF. Since $E(\tilde{y}|\, a) = \tau' Q' a$ has gradient $\nabla_a E(\tilde{y}|\, a) = Q\tau$, we then obtain

$$\nabla_a \Pr(\tilde{y} > \tilde{y}_0|\, a) = \phi(\frac{\tilde{y}_0 - E(\tilde{y}|\, a)}{\tilde{\sigma}}) \frac{1}{\tilde{\sigma}} Q\tau$$

where $\phi = \Phi'$ is the standard normal density. This verifies (3), since $\tilde{y}_0 = E(\tilde{y}|\, a^*)$.

Proof of Proposition 3. We will show that $\hat{\theta}_i \frac{\partial \hat{\theta}_i}{\partial V} \geq 0$. Since $\hat{\theta}_i \frac{\partial \hat{\theta}_i}{\partial s_{ii}} \leq 0$ and $\frac{\partial V^*}{\partial s_{ii}} \leq 0$, this implies from (8) that $\hat{\theta}_i \frac{\partial \theta_i^*}{\partial s_{ii}} \leq 0$, which verifies the formula in Proposition 3 since $\hat{\theta}_i = \theta_i^*$ when $V = V^*$.

Consider the Lagrangean for the optimization problem that defines $\hat{\theta}$:

$$L = -\theta' \Sigma \theta + \lambda(p'a - a'a/2 - V), \qquad a = Q\theta.$$

The first-order conditions are

$$(-2\Sigma - \lambda Q'Q)\hat{\theta} + \lambda Q'p = 0$$

$$p'Q\hat{\theta} - \tfrac{1}{2}\hat{\theta}' Q'Q\hat{\theta} - V = 0$$

Differentiating this system with respect to V yields

$$\begin{bmatrix} -2\Sigma - \lambda Q'Q & Q'(p - Q\hat{\theta}) \\ (p - Q\hat{\theta})'Q & 0 \end{bmatrix} \begin{bmatrix} \nabla_V \hat{\theta} \\ \frac{\partial \lambda}{\partial V} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{14}$$

By Cramer's rule we have

$$\frac{\partial \hat{\theta}_1}{\partial V} = \frac{1}{D} D_1$$

where D is the determinant of the bordered Hessian in (14), and D_1 is the determinant of the same matrix with the first column replaced by the column on the right-hand side of (14). The determinant D has the same sign as $(-1)^m$.

For uncorrelated and orthogonal measurements we have

$$-2\Sigma - \lambda Q'Q = diag\,\{d_1, ..., d_m\}\,, \qquad d_i = -2s_{ii} - \lambda q_i' q_i < 0, \quad i = 1...m.$$

Using this special structure to compute the determinant D_1 (by expansion along the first column and then the first row), we obtain

$$D_1 = (-1)q_1'(p - Q\hat{\theta})d_2 \cdot ... \cdot d_m,$$

and thus

$$\hat{\theta}_1 \frac{\partial \hat{\theta}_1}{\partial V} = \frac{-1}{D} \hat{\theta}_1 q_1'(p - Q\hat{\theta})d_2 \cdot ... \cdot d_m$$

Since each d_i is negative, their product has the same sign as $(-1)^{m-1}$, and hence we see that $-d_2 \cdot ... \cdot d_m/D$ is positive. Finally, for uncorrelated and orthogonal measurements we see from the first-order conditions that we have

$$\lambda q_1'(p - Q\hat{\theta}) = 2s_{11}\hat{\theta}_1$$

For $V > 0$ we must have $\lambda > 0$ since $\lambda = 0$ will imply $\hat{\theta} = 0$ and thus $a = 0$ and $V = 0$. Hence we see that $\hat{\theta}_1 q_1'(p - Q\hat{\theta}) = 2s_{11}\hat{\theta}_1^2/\lambda > 0$, which now implies $\hat{\theta}_1 \frac{\partial \hat{\theta}_1}{\partial V} \geq 0$. The same argument obviously holds for any $i > 1$, and the proof is then complete.

Proof of Proposition 4. For an action a the index $y = x'\theta^*$ has variance $\sigma^2 = \theta^{*\prime}\Sigma\theta^*$ and expected value $e = E(y|\,a) = a'Q\theta^*$. For given e, let $C(e)$ be the minimal cost for the agent to achieve this expected value, i.e.

$$C(e) = \min_a c(a) \;\text{ s.t. } \quad a'Q\theta^* = e. \tag{15}$$

From a formula corresponding to (13) we see that the agent's expected revenue depends on a only via $e = E(y|\,a)$, hence consider the payoff

$$u(e) = b(1 - \Phi(\frac{y_0 - e}{\sigma})) - C(e) = \frac{\sigma}{\phi_0}(1 - \Phi(\frac{e^* - e}{\sigma})) - C(e),$$

where we have used $b = \sigma/\phi_0$ and defined $e^* = a^{*\prime}Q\theta^* = y_0$. Note that for $e = e^*$ we have $C(e^*) = c(a^*)$, since a^* satisfies the first-order condition in the convex cost-minimization problem. Hence the agent's payoff from a^* is $u(e^*)$, which equals $b\frac{1}{2} - c(a^*)$.

It is clear that if $u(e) \leq u(e^*)$ for all feasible e, then action a^* is an optimal choice for the agent. (If not, there exists an action $\tilde{a}$ yielding a higher payoff. This payoff is $u(\tilde{e})$, where $\tilde{e} = \tilde{a}'Q\theta^*$, and thus $u(\tilde{e}) > u(e^*)$, a contradiction.) Observe that

$$u'(e) = \phi(\frac{e^* - e}{\sigma})/\phi_0 - C'(e),$$

where $\phi = \Phi'$ is the standard normal density.

Since $Q\theta^* = \nabla c(a^*)$, the first-order conditions for the cost minimization problem defining $C(e)$ are

$$\nabla c(\hat{a}) = \gamma \nabla c(a^*) \qquad \text{and} \qquad e = \hat{a}'\nabla c(a^*), \tag{16}$$

where $\hat{a} = \hat{a}(e)$ is the optimal action and γ is a Lagrange multiplier. Differentiation wrt e yields

$$H(\hat{a})d\hat{a} = d\gamma \nabla c(a^*) \qquad \text{and} \qquad \nabla c(a^*)'d\hat{a} = de,$$

where $H(a) = [c_{ij}(a)]$ is the Hessian of the cost function $c(a)$. Hence $d\hat{a} = H(\hat{a})^{-1}\nabla c(a^*)d\gamma$ and so

$$\frac{d\gamma}{de} = (\nabla c(a^*)'H(\hat{a})^{-1}\nabla c(a^*))^{-1} > 0,$$

where the inequality follows from H being positive definite. From the envelope property we have $C'(e) = \gamma$ and so $C''(e) = \frac{d\gamma}{de} > 0$.

Observe for later use that from conditions (16) we have $e = \hat{a}'\nabla c(a^*)$ and $\gamma = a^{*\prime}\nabla c(\hat{a})/(a^{*\prime}\nabla c(a^*))$, and hence

$$\eta(e) \equiv e\frac{C''(e)}{C'(e)} = \hat{a}'\nabla c(a^*)\frac{a^{*\prime}\nabla c(a^*)}{a^{*\prime}\nabla c(\hat{a})}\frac{1}{\nabla c(a^*)'H(\hat{a})^{-1}\nabla c(a^*)}. \tag{17}$$

Now consider $u(e)$ for $e > e^*$. Here we have $u'(e) < u'(e^*) = 0$ since $\phi(\frac{e^*-e}{\sigma})$

is decreasing and $C'(e)$ is increasing in e, where the latter property follows from $C''(e) = \frac{d\gamma}{de} \geq 0$. This verifies $u(e) < u(e^*)$ for $e > e^*$.

Next consider $u(e)$ for $e < e^*$. We will show that $u(e) \leq u(e^*)$ for all $e \leq e^*$ iff $\sigma \geq \sigma_0^*$. To this end we first state and prove the following claim.

Claim. Let $h^* = \sup_e \{1/\eta(e) | \, 0 < e \leq e^*\}$. If $\sigma \geq e^* \sqrt{h^*}/2 \equiv \sigma_m$, then $u'(e) \geq 0$ for all $e < e^*$.

The last statement obviously implies $u(e) \leq u(e^*)$ for all $e < e^*$ if $\sigma \geq \sigma_m$. To prove the claim, observe first that $u'(0) > 0 = u'(e^*)$ (since $C'(0) = 0$ due to $\hat{a}(0) = 0$ and therefore $\gamma = 0$ for $e = 0$). If $u'(e)$ has no local minimum in $(0, e^*)$, then $u'(e)$ is non-negative on this interval. So consider a local minimum, where then $u''(e) = 0$. Using $\phi'(z) = -z\phi(z)$ we have

$$0 = u''(e) = -\phi'(\frac{e^* - e}{\sigma})\frac{1}{\phi_0 \sigma} - C''(e) = \frac{e^* - e}{\sigma}\phi(\frac{e^* - e}{\sigma})\frac{1}{\phi_0 \sigma} - C''(e).$$

This yields $\phi(\frac{e^*-e}{\sigma})/\phi_0 = \frac{\sigma^2}{e^*-e}C''(e)$ and thus, from the definition of the elasticity $\eta(e)$ above:

$$u'(e) = \phi(\frac{e^* - e}{\sigma})/\phi_0 - C'(e) = C''(e)(\frac{\sigma^2}{e^* - e} - \frac{e}{\eta(e)})$$

By the definition of h^* we have $h^* \geq 1/\eta(e)$ and hence

$$\frac{\sigma^2}{e^* - e} - \frac{e}{\eta(e)} \geq \frac{\sigma^2}{e^* - e} - eh^*.$$

The last expression is non-negative if $\sigma^2/h^* \geq \max_e e(e^* - e) = (e^*/2)^2$, i.e. if $\sigma \geq e^* \sqrt{h^*}/2 \equiv \sigma_m$. This verifies that $u'(e) \geq 0$ for all $e \leq e^*$ if $\sigma \geq \sigma_m$, and thus proves the claim.

So we have $u'(e) \geq 0$ for all $e < e^*$ when $\sigma \geq \sigma_m$. Let σ_l be the smallest σ for which $u'(e) \geq 0$ for all $e < e^*$. (We must have $\sigma_l > 0$ since otherwise the necessary condition (6) would be violated.) So for $\sigma < \sigma_l$ there is $e < e^*$ such that $u'(e) < 0$. Then, since $u'(0) > 0$ as noted above, $u(e)$ must have a local maximum at some $e^0 \in (0, e^*)$. Since both e^0 and e^* are local maxima, we have then, for $\sigma < \sigma_l$

$$\frac{d}{d\sigma}(u(e^*) - u(e^0))\phi_0 = \Phi(\frac{e^* - e^0}{\sigma}) - \Phi(0) - \sigma\Phi'(\frac{e^* - e^0}{\sigma})\frac{e^* - e^0}{\sigma^2} > 0,$$

where the inequality follows from $\Phi(z)$ being strictly concave for $z > 0$, and thus $\Phi(z) - \Phi(0) - \Phi'(z)z > 0$.

Hence, the smaller is σ, the smaller is the payoff difference $u(e^*) - u(e^0)$. Let σ_0^* be the smallest σ for which $u(e^*) - u(e^0) \geq 0$. By the monotonicity of $u(e^*) - u(e^0)$, we have $u(e^*) \geq u(e^0)$ iff $\sigma \geq \sigma_0^*$. This proves the first statement in the proposition.

To verify the last statement in the proposition, recall the definition of $\hat{a}(e)$ as the cost minimizing action in problem (15) and define

$$h(a^*) = \sup_e \left\{ \frac{a^{*\prime}\nabla c(a)}{a'\nabla c(a^*)} \frac{\nabla c(a^*)' H(a)^{-1} \nabla c(a^*)}{a^{*\prime}\nabla c(a^*)} \middle| a = \hat{a}(e), \;\; 0 < e \leq a^{*\prime}\nabla c(a^*) \right\} \tag{18}$$

From the definition of h^* in the Claim, the expression for $\eta(e)$ in (17), and the fact that $e^* = a^{*\prime}Q\theta^* = a^{*\prime}\nabla c(a^*)$, we then have $h^* = h(a^*)$.

The sufficient condition stated in the Claim can thus be written $\sigma \geq e^*\sqrt{h(a^*)}/2$. Since $e^* = a^{*\prime}\nabla c(a^*)$ and $\sigma = (\theta^{*\prime}\Sigma\theta^*)^{1/2}$ it follows from the binding enforcement constraint (5) that the sufficient condition can equivalently be wrtten as the condition (9) stated in the proposition.

We finally note that for a quadratic cost function $c(a) = a'Ka/2$ the expression in (18) yields [20] $h(a^*) = 1$. This completes the proof.

Remark. The proof uses only two properties of a^* and θ^*; namely that they satisfy $\nabla c(a^*) = Q\theta^*$ and the binding enforcement constraint (5). Its conclusions regarding a^* being implementable (an optimal choice for the agent) with index $x'\theta^*$ are therefore valid for any a^* and θ^* that satisfy these conditions

Proof of Proposition 5. To take advantage of the notation developed in the previous proofs, in this proof we will denote the given a and θ by

[20] For $c(a) = (a'Ka)^r/2r$, $r \geq 1$, we find $h(a^*) = 2r - 1$.

a^* and θ^*, respectively. We thus consider a^* and θ^* that satisfy $2c(a^*) \geq \frac{\delta}{1-\delta}(v(a^*) - c(a^*)) > c(a^*)$ and $\nabla c(a^*) = Q\theta^*$.

We will consider the index $y = x'\theta^*$ with a hurdle $\kappa < E(y|\, a^*)$, and with bonus b paid for qualifying performance ($y > \kappa$). The bonus is

$$b = \frac{\delta}{1-\delta}(v(a^*) - c(a^*)).$$

The proof will show that the hurdle κ can be chosen such that this index scheme implements a^*, provided the index has a sufficiently low variance.

By assumption we have $c(a^*) < b$. Choose $\xi_0 > 0$ and σ_0 such that

$$c(a^*) = (\Phi(\xi_0) - \Phi(-\xi_0))b \quad \text{ and } \quad \sigma_0 = b\phi(-\xi_0)$$

The index $y = x'\theta^*$ has variance $\sigma^2 = \theta^{*\prime}\Sigma\theta^*$, and assume now $\sigma < \sigma_0$. Define $\xi > \xi_0$ by

$$\sigma = b\phi(-\xi),$$

and let the hurdle for the index be $\kappa = E(y|\, a^*) - \xi\sigma = \theta^{*\prime}Q'a^* - \xi\sigma$.

The agent's payoff from an action a is then $b(1 - \Phi(\frac{\kappa - E(y|a)}{\sigma})) - c(a)$ with gradient $b\frac{1}{\sigma}\phi(\frac{\kappa-\theta^{*\prime}Q'a}{\sigma})Q\theta^* - \nabla c(a)$. It follows that action a^* satisfies the first-order condition for an optimum, since we have $\kappa - \theta^{*\prime}Q'a^* = -\xi\sigma$, $b\frac{1}{\sigma}\phi(-\xi) = 1$ and $Q\theta^* = \nabla c(a^*)$ Since $\xi > 0$, we can also verify that the Hessian at a^* is positive definite, hence action a^* is a local optimum for the agent under the given incentive scheme.

It remains to show that a^* is a global optimum. As in the proof of Proposition 4, it suffices to consider the payoff

$$u(e) = b(1 - \Phi(\frac{\kappa - e}{\sigma})) - C(e).$$

where e is the expected index value ($e = E(y|\, a)$), $C(e)$ is the minimal cost to obtain a given expected value e, see (15); and κ is here the hurdle for the index. For action a^* this payoff is $u(e^*)$, where $e^* = E(y|\, a^*) = \theta^{*\prime}Q'a^*$. The proof is complete if we show $u(e) \leq u(e^*)$ for all feasible e.

First note that by the definition of κ we have $\frac{\kappa - e^*}{\sigma} = -\xi$ and so

$$u(e^*) = b(1 - \Phi(-\xi)) - c(a^*),$$

where we have used the fact that $C(e^*) = c(a^*)$, by virtue of a^* being the cost-minimizing action to generate expectation $e^* = \theta^{*\prime} Q' a^*$.

Next consider $e < e^*$. Since $u'(0) > 0$ (by virtue of $C'(0) = 0$, see the previous proof), we have $u(e) \leq u(e^*)$ for all $e \in [0, e^*]$ if $u(\cdot)$ has no local maximum in the interior of the interval. So suppose $u(\cdot)$ has a local maximum at some $e^0 \in (0, e^*)$. Then $u'(e^0) = 0$ and so $b \frac{1}{\sigma} \phi(\frac{\kappa - e^0}{\sigma}) = C'(e^0)$. Since $C'(e^0) < C'(e^*)$, and e^* is also a local maximum, we then have $\phi(\frac{\kappa - e^0}{\sigma}) < \phi(\frac{\kappa - e^*}{\sigma})$. Since $\phi(\cdot)$ is symmetric around zero, this implies $\kappa - e^0 > e^* - \kappa$ and hence, by definition of $\kappa = e^* - \xi\sigma$, that $\kappa - e^0 > \xi\sigma$. This yields

$$u(e^0) = b(1 - \Phi(\frac{\kappa - e^0}{\sigma})) - C(e^0) \leq b(1 - \Phi(\xi)),$$

and hence

$$u(e^*) - u(e^0) \geq b(1 - \Phi(-\xi)) - c(a^*) - b(1 - \Phi(\xi)).$$

The last expression is increasing in ξ and is (by definition of ξ_0) zero for $\xi = \xi_0$. Hence $u(e^*) - u(e^0) \geq 0$, since $\xi > \xi_0$. This verifies $u(e) \leq u(e^*)$ for all feasible $e < e^*$.

Now consider $e > e^*$. As in the proof of Proposition 4, we have $u'(e) < u'(e^*) = 0$ when $e > e^*$ This follows because $C'(e)$ is increasing (as shown in the proof of Proposition 4), and because $\phi(\frac{\kappa - e}{\sigma})$ is decreasing in e when $e > e^*$, since $e^* > \kappa$ and thus $\kappa - e < 0$. This verifies $u(e) < u(e^*)$ for $e > e^*$.

We finally verify that $\kappa \to E(y|\, a^*)$ when $\sigma \to 0$. From the definition of κ and ξ we have $E(y|\, a^*) - \kappa = \xi\sigma = \xi\phi(-\xi)b$, where $\xi \to \infty$ when $\sigma \to 0$. The density $\phi(\cdot)$ has the property that $\xi\phi(-\xi) \to 0$ when $\xi \to \infty$, and this completes the proof.

APPENDIX B: EXAMPLES

Example 1. This example illustrates an application of Proposition 2. Suppose $n = 3$ and that we have $m = 2$ measurements, given by

$$x_1 = a_1 + \varepsilon_1, \qquad x_2 = k \cdot (a_2 + a_3) + \varepsilon_2, \quad k > 0,$$

Then Q' has rows $(1, 0, 0)$ and $(0, k, k)$, and we have $Q'Q = I$ (the identity matrix) if $k = 1/\sqrt{2}$. To simplify the algebra we will invoke this assumption regarding k. Assume also linear-quadratic value- and cost-functions: $v(a) = p'a$ and $c(a) = a'a/2$.

Substituting from the agent's IC condition $a = Q\theta$ into the objective and the enforcement constraint in Proposition 2, we are led to choose θ to maximize $p'Q\theta - \frac{1}{2}\theta'\theta$ subject to

$$\frac{\delta}{1-\delta}(p'Q\theta - \frac{1}{2}\theta'\theta) \geq (\theta'\Sigma\theta)^{1/2}/\phi_0$$

Given our assumptions about the measurements, we have $p'Q = (p_1, (p_2 + p_3)k)$. To simplify further, assume $p_1 = (p_2 + p_3)k$ and $var(\varepsilon_1) = var(\varepsilon_2) = s^2$, which implies that the objective and the constraint are entirely symmetric in θ_1 and θ_2. The optimal solution is then also symmetric, i.e. $\theta_1 = \theta_2$, and the (binding) enforcement constraint for the common value θ_1 takes the form

$$\frac{\delta}{1-\delta}(2p_1\theta_1 - \theta_1^2) = s\theta_1(2 + 2\rho)^{1/2}/\phi_0$$

where $\rho = corr(\varepsilon_1, \varepsilon_2)$. The optimal action is then $a^* = Q\theta = (1, k, k)'\theta_1$, and the associated surplus per period is $2p_1\theta_1 - \theta_1^2$. We see that a higher variance (s^2) or a higher correlation (ρ) for the observations will reduce θ_1 and reduce the surplus.

Given our assumptions about measurements in this example, we can promote action a_1 via incentives on x_1, and we can promote the sum $a_2 + a_3$ via incentives on x_2. As we have seen, the optimal incentive scheme rewards the agent with a fixed bonus (b) if performance measured by an index – a scorecard – $\theta_1 x_1 + \theta_2 x_2$ exceeds a hurdle. The agent will then clearly choose $a_2 = a_3$, since the marginal revenues on these two action elements are equal. This will entail a distortion from the first-best if the marginal values of these two elements for the principal are not equal ($p_2 \neq p_3$). The first best action is here $a^{FB} = (p_1, p_2, p_3)'$.

If this were the only distortion, the weight vector θ would be chosen to maximize the surplus, subject to the IC constraints, which would constrain actions such that $a_2 = a_3$. In our setting the enforcement constraint puts further bounds on these weights. In this example we have invoked an additional assumption ($p_1 = (p_2 + p_3)k$) that ensures equal weights $\theta_1 = \theta_2$ in the optimal index. The magnitude of this common weight, and therefore the strength of the agent's incentives, is bounded by the dynamic enforcement constraint. And as we have seen, the noise parameters s and ρ have negative influences in this respect.

Example 2. This example illustrates the trade-off between distortion and precision discussed in Section 3.1. Assume $m = n = 2$ and

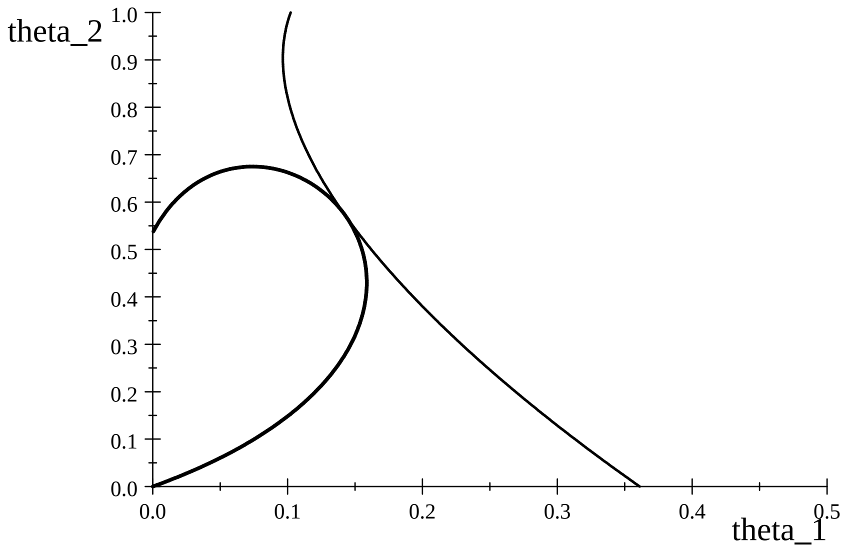
$$q_1 = p = (1,1)' \qquad \textit{and} \qquad q_2 = (1,0)'.$$

Here q_1 is perfectly aligned with the true marginal value p. Assume also that the measurements are uncorrelated with variances

$$s_{11} = 16 \qquad \textit{and} \qquad s_{22} = 1.$$

Measure x_1 is thus considerably less precise than measure x_2. For a value of δ to be specified below, we will see that the optimal weight on the perfectly aligned but imprecise measure x_1 is considerably smaller than the optimal weight on the more precise but also more distorted measure x_2. Specifically, these optimal weights turn out to be $\theta_1^* = \frac{1}{7}$ and $\theta_2^* = \frac{4}{7}$.

The figure below illustrates this case. The left-most curve depicts the binding enforcement constraint, and the other curve is an isoquant for the total surplus. The optimal weight combination is at the tangency point of the two curves.



The specifics of this example are as follows. For given θ the agent will choose action $a = \theta_1 q_1 + \theta_2 q_2$. For the given parameters the surplus $p'a - a'a/2$ then amounts to

$$2\theta_1 - \theta_1^2 - \theta_1\theta_2 + \theta_2 - \tfrac{1}{2}\theta_2^2 \equiv S(\theta)$$

The index $\theta' x$ has variance $16\theta_1^2 + \theta_2^2$, and the enforcement constraint can be written as

$$S(\theta) \geq \tfrac{1-\delta}{\delta\phi_0}(16\theta_1^2 + \theta_2^2)^{1/2}.$$

We find that the tangency condition illustrated in the figure is fullfilled at $\theta^* = (\frac{1}{7}, \frac{4}{7})$ when δ satisfies

$$\tfrac{1-\delta}{\delta\phi_0} = \tfrac{6-13/7}{4\sqrt{2}}, \qquad i.e. \qquad \delta = (\tfrac{6-13/7}{4\sqrt{2}} \tfrac{1}{\sqrt{2\pi}} + 1)^{-1} = 0.773\,89.$$

The value of the optimal surplus can then be computed to be $S(\theta^*) = \frac{29}{49}$.

We will next illustrate that a more distorted measure can be advantageous. Suppose everything is as above, except that measure x_1 has vector

$$\tilde{q}_1 = (\tfrac{1}{2}, \tfrac{1}{2}\sqrt{7})$$

This is a vector of the same length as q_1 (i.e. $\tilde{q}_1'\tilde{q}_1 = q_1' q_1 = 2$), but rotated away from p towards the vertical axis. It is thus more distorted than q_1 relative to vector p. Action $a = \theta_1 \tilde{q}_1 + \theta_2 q_2$ then yields surplus

$$\tilde{S}(\theta) = \frac{1}{2}(1+\sqrt{7})\theta_1 + \theta_2 - \theta_1^2 - \frac{1}{2}\theta_2^2 - \frac{1}{2}\theta_1\theta_2.$$

We now find that for $\theta = \theta^*$ given above we have $\tilde{S}(\theta^*) = \frac{1}{14}\sqrt{7}+\frac{41}{98} > S(\theta^*)$. An index with the same weights as before thus yields a higher surplus. It will then be feasible index in the new situation, since it satisfies the enforcement constraint (with slack). The optimal surplus is therefore stricly higher with vector $\tilde{q}_1$ than with vector q_1.