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Adaptive Agents May Be Smarter than You Think: Unbiasedness in Adaptive Expectations

Abstract

Agents forming adaptive expectations generally make systematic mistakes. This characterization has fostered the rejection of adaptive expectations in macroeconomics. Experimental evidence, however, shows that in complex environments human subjects frequently rely on adaptive heuristics – model-consistent expectations being simply too difficult or impossible to implement – but their forecasting performance is not as inadequate as assumed in the characterization above. In this paper we show that adaptive agents may not be as gullible as we used to think. In a model with adaptive expectations augmented with a Belief Correction term (which takes into account the drift of the macroeconomic variable of interest) the average forecasting error is frequently close to zero, hence (belief amended) adaptive expectations are close to unbiasedness.

JEL-Codes: C630, D830, D840, E710.

Keywords: heterogeneous adaptive expectations, belief correction, agent based models.

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1 Introduction

It is a well-known fact that expectations drive actions and the latter affect the former. At the macroeconomic level this loop generates a two-way feedback: agents' expectations drive the dynamics of a macroeconomic variable and the latter, in turn, influences the formation of expectations. In canonical macroeconomic models, agents are rational *à la* Muth (Muth, 1961). In its starkest characterization, the rational agent knows the "true model" of the economy, which, in reduced form can be identified with the *Actual Law of Motion* (ALM) of the macroeconomic variable. The ALM is the function that links the current state of the variable to the aggregate or average expectation of that variable and to an exogenous dynamic stochastic process: a "shock" in macroeconomics – e.g., a rule that describes changes in policy makers' behaviour – or "the fundamental" in finance – e.g., the dynamic process governing dividend payments.

The key assumption in this setting is that the agent recognizes *the stochastic process as the driver* of the macroeconomic variable so that, in the agent's mind, the variable can take on different values with different probabilities. From this assumption follows that the macroeconomic variable is itself stochastic since its objective probability distribution is shaped by the exogenous stochastic process. In the end, therefore, as Hansen and Sargent state at the very beginning of their book, "a model is a stochastic process, that is, a probability distribution over a sequence of random variables, perhaps indexed by a vector of parameters."(Hansen and Sargent, 2014).

Since agents take the probability distribution of the macroeconomic variable into account, rational expectations are *model consistent* [1] and, therefore, characterized by *unbiasedness*: the average expectation[2] coincides with the actual state of the variable. As a consequence agents may make forecasting mistakes but the average forecasting error – i.e., the mean of a sufficiently

[1]We have described in words the "guess and verify" procedure which is the cornerstone of the method of undetermined coefficient to solve a model with rational expectations.

[2]The average expectation can be interpreted as the mean of forecasts generated (i) by a population of individuals in a given period or (ii) by an individual in a given time interval.

large sample of forecasting mistakes – is zero.[3]

Absent a "surprise" agents are always correct. If a surprise occurs, they can indeed be wrong but errors are short-lived. Agents will promptly amend the subjective distribution in the light of the new information coming from the surprise. Hence the shock has no persistence. This prediction has frequently been disproved in reality. For instance, if we assume that the real impact of a monetary disturbance is proportional to the price surprise, the temporary nature of the "rational" error makes also the real effect of a monetary shock short lived. This is not true: it is an established fact that monetary shocks are highly persistent. It is interesting to note that if agents adopted adaptive expectations, the sizable and persistent effects of monetary policy would certainly not be a puzzle: by construction, adaptive expectations magnify the amplitude and persistence of aggregate shocks.

The rational expectations hypothesis in its purest form has also been often disproved in the laboratory. Learning-to-forecast experiments have shown that humans often adopt simple (and generally biased) rules in forming expectations. Brock and Hommes (1997) provide a rationale for this behavior: agents form expectations according to a (limited number of) heuristics and switch from one to the other depending on a measure of "fitness" of the heuristic itself, namely the forecasting error. In this setting adaptive algorithms stand out as very popular expectation formation mechanisms in the lab (see Assenza et al. (2014) for an extensive survey of this class of experiments).

In principle therefore, there are enough empirically corroborated reasons to go back to adaptive expectations as a reliable modelling assumption in macroeconomics. This has not been the case. A huge and still growing literature on information acquisition and expectation formation has explored in detail many different departures from rational expectations – based on bounded rationality, learning or robust control considerations – but has not revived adaptive expectations. To make just one example, Woodford (2003)

[3] Unbiasedness is not sufficient to completely characterize rational expectations. They require also the minimization of the variance of the distribution of errors since the information set the agents use to form expectations incorporates all the relevant information.

argues that agents should take into account *higher order expectations* in expectation formation. In this new and modified rational setting, following a shock, agents may be reasonably confident about the correctedness of their own expectation on the state of the macroeconomy but will continue to be uncertain about whether other agents know that other agents know... This residual uncertainty and the associated slowness of adjustment of higher order expectations may be at the root of the persistence of shocks.

To understand why adaptive expectations have fallen out of the radar screen of the profession, let's consider the effect of an aggregate shock starting from an initial (pre-shock) situation characterized by stationarity of the macroeconomic state variable of interest. Being backward looking, the adaptive agent will *necessarily* fail to immediately notice a sudden change in the state variable, a gap between reality and perception will emerge and – contrary to the case of the rational agent – this gap will not disappear or it will vanish only in the "long run" (if the variable becomes stationary again), i.e., the agent will make systematic errors. Indeed, adaptive expectations can be unbiased but this happens only in the steady state, i.e., only when the dynamics of the macroeconomic variable has come to a halt. Hence, unbiasedness is an *asymptotic property* of adaptive expectations if the steady state of the ALM is globally stable. The rationale for unbiasedness of adaptive expectations is that, when the system is in the stationary state it is "simple to understand" so that expectations on average are correct. If the state variable is increasing (decreasing) over time – i.e., if there is a positive (negative) drift – adaptive expectations based on the lagged values of the state variable systematically underestimate (overestimate) the actual state. If the steady state is absent or unstable adaptive expectations are always biased. In our opinion, it is this characterization of the adaptive agent as a gullible forecaster that has marginalized the idea in the literature.

There is more than systematic errors, however, in the behaviour of the adaptive agent. To better understand the latter, let's distinguish between publicly available information and information the agent actually uses to form expectations. The ALM and the properties of the stochastic process may be public information but the adaptive agent does not use it. One obvious reason is that the latter may have limited cognitive capabilities. A different, not necessarily alternative, reason is that the adaptive agent has limited capacity to pay attention to all the information she has free access to ((Sims, 1998, 2003)): due to inattention the adaptive agent does not grasp the ultimate stochastic nature of the variable of interest. It is reasonable to suppose that in the presence of a high degree of "complexity", agents rely on simple heuristics to form expectations because model-consistent expectations are simply too difficult to implement. For instance, in agent based models the properties of the "true model" of the economy emerge from the interaction of a myriad of heterogenenous agents and therefore are unknown to the modeller. It would be paradoxical to assume that they are known to the agents populating the model economy.[4] Summing up: on one hand the characterization of the agent as a boundedly rational individual who cannot manage the complexities of rational forecasting and resort to simple adaptive rules is realistic; on the other hand, the corollary of this characterization, i.e. the tendency of the adaptive agent to be systematically wrong, is clearly unrealistic.

In this paper we propose a simple explanation of this apparent paradox, based on a departure from the standard characterization of the Adaptive agent. On the basis of preliminary experimental evidence,[5] we assume that agents are indeed *inefficient forecasters* – being unable to grasp the true model (if any) of the economy – but they are not backward looking simpletons who are caught by surprise repeatedly in the same direction. They are aware that the source of systematic errors is indeed the presence of a "drift" of the macroeconomic variable[6] that makes the macroeconomic scenario more complicated to understand. Therefore, they augment the simple Friedman-Nerlove adaptive rule with a *belief correction* term which is proportional to

[4]On top of this, (Dosi et al., 2017) show that introducing sophisticated learning schemes – e.g., recursive least squares expectations – in this class of complex models may produce less accurate individual forecasts and also considerably worsen macroeconomic performance.

[5]In laboratory experiments, subjects who adopt an adaptive scheme to form expectations seem to be less prone to systematic errors than the original Friedmanian mechanism suggests (see (Colasante et al., 2017), section 5).

[6]Rational agents who know the true model can trace this drift back to the stochastic dynamic exogenous process. This knowledge is not achievable by the adaptive agent.

the average change of the state variable over a given time window.

By employing the belief correction term agents still use only the past history of the variable to form expectations but they *indirectly* capture the underlying exogenous dynamic stochastic process which drives the macroeconomic variable. We show in a streamlined agent based model that belief correction improves the forecasting performance to a large extent so that adaptive expectations will be not far – under this limited point of view – from the rational solution. In other words, in our setting adaptive agents are smarter than the original Friedman-Nerlove type.

Is expectation unbiasedness too good (and too strange) to be true? In a well-known 1906 experiment Francis Galton pointed out that people as a collective entity are capable of correctly assessing unknown quantities (Galton, 1907; Surowiecki, 2005). He noted that in a competition to evaluate the weight of an ox, while individual opinions differed and frequently missed the mark, the average (median) guess was extremely close to the actual weight. The rationale for this phenomenon may be that, when the system is "simple to understand", expectations' unbiasedness is an emerging property: individual forecasting errors wash out "in the aggregate" (i.e., when individual expectations are aggregated and averaged).[7] The environment is simple to understand when it is stationary. This is the reason why standard adaptive expectations are unbiased in the steady state. When the economy is growing or declining, on the contrary, the environment is difficult to understand and standard adaptive expectations are systematically wrong. Belief correction makes even a dynamic environment understandable to the adaptive agent.

The paper is organized as follows. After a brief review of the literature (section 2), in section 3 we compare and contrast rational and adaptive expectations in a stylized (reduced form) aggregative macroeconomic framework consisting of an Actual Law of Motion (ALM) and an Adaptive Expectation (AE) scheme. In section 4 we present the belief correction term and discuss the way in which it affects the dynamics of the average forecasting error in

[7]Galton's fact is also exploited in the forecasting literature combining different algorithm to produce better forecasting performance (Clements, 2019).The aggregation of expectations may underweight private information but public information is correctly accounted for (Satopää, 2017; Clements, 2019).

the stylized ALM-AE model. Section 5 is devoted to an application of these ideas to an heterogenenous agents model along the lines of Palestrini (2017). In section 6 we explore the results of a policy shock (the imposition of a sales tax to finance Government expenditure in fundamental research which affects TFP growth) with and without belief correction. Section 7 concludes.

2 Related literature

This paper is motivated by two strands of literature. The first one focuses on learning and forecasting in complex environments so that agents use simple heuristics to form expectations. Heterogeneous expectations and *heuristics switching* à la Brock and Hommes are thouroughly surveyed in Hommes (2013). Heuristic switching has been incorporated in "Behavioural New Keynesian" DSGE models (De Grauwe, 2011). This literature is extensively discussed in Branch and McGough (2018).

Heuristic switching has been put to test in Learning to Forecast Experiments (LtFEs hereafter) (see Assenza et al. (2014) for a survey). Three *robust* facts stand out starkly from the experimental evidence: (i) subjects use very few heuristics to form expectations whatever the economic setting (Assenza et al., 2019), (ii) the Friedman-Nerlove adaptive rule features prominently among these heuristics, (iii) learning leads to Rational expectations (after many iterations) only when the dynamic process tends to a stationary state, i.e. when a *negative feedback* is at work.[8] With *positive feedback*, heterogeneous expectations do not converge to the rational outcome.

From our point of view, Colasante et al. (2017) provides particularly interesting insights. The aim of this LtFE, which mimics a financial market with increasing "fundamental" value, in fact, is to test whether subjects using an adaptive rule are able to learn the presence of a drift of the fundamental and to incorporate this acquired knowledge in expectation formation. Section 5 of the above mentioned paper shows that subjects indeed use an adaptive rule and they grasp the presence of a drift so that they are not systematically

[8]Incidentally, as we have shown above, in a steady state adaptive expectations are unbiased.

wrong. This experimental evidence both motivates and corroborates the idea of adaptive expectations with belief correction which is the core of the present paper.

The second strand of literature is *agent based macroeconomics*. A sizable number of agent based macroeconomic frameworks has been developed over the last two decades (see Dawid and Delli Gatti (2018) for an extensive survey). In these models, as in standard DSGE models, expectations drive actions and actions affect expectations. Agents, however, hold heterogeneous expectations and are not rational *à la* Muth: they do not form model-consistent expectations. Since the macrodynamic properties of the "true (agent based) model" of the economy are unknown ex ante, agents cannot take them into account in forming expectations. The default modelling option in agent based macroeconomics therefore is the adaptive expectation scheme. Variants and alternatives to this option within this literature are touched upon in the survey mentioned above and discussed extensively in Salle (2015). For an application of the heuristics switching mechanism to an agent based macroeconomic framework see Dosi et al. (2017).

3 Expectations and macroeconomic activity

3.1 Rational expectations

Let's assume that the reduced form of a generic macroeconomic model is described by an expectation-driven macro-dynamic function $x_t = f(x_t^e, a_t)$ where x_t is a macroeconomic state variable in t, x_t^e is the expectation of the variable and a_t is an exogenous variable which is governed by a stochastic process. If the economy is populated by agents holding heterogeneous expectations, x_t^e can be interpreted as the aggregate or average expectation, i.e., an aggregator of individual expectations. For simplicity, in this section we assume that the model is linear or linearized so that the macro-dynamic function becomes:

$$x_t = a_t + \alpha x_t^e \tag{1}$$

with $\alpha \in \mathbb{R}$ a parameter. The exogenous variable plays the role of the "fundamental" or the "shock". Equation (1) is based on the assumption that agents make decisions in t without knowing the contemporaneous state variable. For instance, expectations are formed at the beginning of period t, while the current state of the variable will be revealed at the end of the period.[9] Following (Evans and Honkapohja, 2001), we will label (1) the *Actual Law of motion* (ALM) of the economy.

In order to set a benchmark, in this section we study the so called "rational solution" in the simplest possible setting. Let's assume that a_t is a stochastic shock governed by the following law: $a_t = \bar{a} + \tilde{a}_t$ where $\bar{a}$ is finite and $\tilde{a}_t$ is white noise. Hence $E(a_t) = \bar{a}$. The rational solution of (1) – also known as the Minimum State Variable Solution – can be written as follows:

$$x_t = b_0 + b_1 a_t \tag{2}$$

where b_0 and b_1 are (so far) undetermined coefficients. This is the *Perceived Law of Motion* (PLM) of the economy, i.e., the law of motion of the aggregate variable as conceived by the rational agent. The agent knows the "true model" of the economy, i.e., the ALM. Hence she knows that the state variable is ultimately affected by the "fundamental". However the agent does not know the coefficients.[10] In appendix A we derive the rational solution $b_0 = \frac{\alpha}{1-\alpha}\bar{a}$, $b_1 = 1$ so that the state variable is indeed a linear function of the shock:

$$x_t = a_t + \frac{\alpha}{1-\alpha}\bar{a} \tag{3}$$

The forecasting mistake therefore is $\varepsilon_t := x_t - E(x_t) = \tilde{a}_t$. Since $\tilde{a}_t$ is white noise, $E(\varepsilon_t) = 0$ i.e., rational expectations are unbiased. In this setting the *E-stability Principle* is satisfied (i.e., the agent is indeed capable of learning

[9] In section 5 we will consider a setting in which firms plan production at the beginning of a given period while the market for goods opens (and the sale price is revealed) at the end of the period. Hence output should be decided in t on the basis of the expected sale price in the same period.

[10] Notice that in principle agents can hold different priors on these parameters but they use the same procedure and the same data to estimate them and therefore they are bound to "discover" the same numerical values. In other words, expectations will be homogeneous across agents, at least at the end of the learning process.

the true values of the parameters) only if $\alpha < 1$.

In forming expectations, the rational agent focuses exclusively on the exogenous variable in (1), because she knows the structure of the model and therefore correctly conjectures that the state variable is "caused" by the exogenous variable. In the real world, however, people are uncertain (at least) on the structure of the model. As (Hansen and Sargent, 2014) point out: "model uncertainty includes a suspicion that the model is incorrect." Some agents may well have a PLM different from (2). For instance, they may believe that the past history of the variable of interest is playing a role alongside the stochastic process. A *hybrid PLM* which takes this belief into account is the following:

$$x_t = b_0 + b_1 a_t + b_2 x_{t-1} \tag{4}$$

Following the same procedure as above, we infer that under E-stability, these agents will learn that the past history of the variable plays no role (i.e., $b_2 = 0$) and therefore will discard the hybrid PLM (which is *over-parameterized*) and use (2) instead.

An interesting alternative PLM simply abstracts from the shock and focusses exclusively on the past history of the variable:

$$x_t = \beta_0 + \beta_1 x_{t-1} + \tilde{x} \tag{5}$$

with $\tilde{x}$ white noise. In this case, the PLM is *misspecified*. This PLM is at the root of the literature on *misspecification equilibria* under learning (see (Hommes, 2021) for a discussion).[11]

3.2 Adaptive expectations

In this section, we want to explore a specific type of misspecified PLM, that of Friedman-Nerlove adaptive expectations. We assume that agents hold heterogeneous expectations and that the i-th agent forms expectations using

[11] Among the different types of misspecified expectations, it is particularly interesting the class of *natural expectations* proposed by Fuster et al. (2010).

the following adaptive algorithm:

$$x^e_{i,t} = \lambda_i x_{t-1} + (1-\lambda_i)x^e_{i,t-1} = x^e_{i,t-1} + \lambda_i \varepsilon_{i,t-1} \tag{6}$$

where $0 < \lambda_i \leq 1$ is the updating coefficient and $\varepsilon_{i,t-1} := x_{t-1} - x^e_{i,t-1}$ is the individual forecast error. This expression incorporates the notion that agents revise their expectation on the basis of the forecast error they made in the past. Iterating equation (6), it is easy to infer that the expectation of x in t is an autoregressive process $AR(\infty)$ with exponentially declining weights: $x^e_{i,t} = \lambda_i \sum_{s=1}^{\infty}(1-\lambda_i)^{s-1}x_{t-s}$. Only past values of the variable play a role in this expectation formation mechanism. Information which may increase the accuracy of the forecast (for instance the role of the exogenous variable a_t) is ignored by assumption. Since the history of the variable of interest is known with certainty, adaptive expectations are heterogeneous inasmuch as the coefficient λ_i is different from one agent to the other. Subtracting the LHS of equation (6) from x_t, the individual forecasting mistake can be rewritten as

$$\varepsilon_{i,t} = \Delta x_t + (1-\lambda_i)\,\varepsilon_{i,t-1} \tag{7}$$

where $\Delta x_t = x_t - x_{t-1}$. In words: the forecasting mistake is increasing with the first difference of x. The "drift" of the macroeconomic variable drives the forecast error. This consideration will be crucial in motivating belief correction, as we will show in section 4. Taking the mean of (6) we get the average expectation:

$$x^e_t = \lambda x_{t-1} + (1-\lambda)\,x^e_{t-1} \tag{8}$$

where $x^e_t := \frac{1}{F}\sum_{i=1}^{F} x^e_{i,t}$ is the mean of the distribution of expectations and $\lambda = \frac{1}{F}\sum_{i=1}^{F}\lambda_i$ is the mean of the distribution of updating coefficients. Of course $0 < \lambda \leq 1$. Equation (8) is the *Average Expectations (AE) function.*

The macroeconomy is described by the two-dimensional linear map S_0 : $(x_{t-1}; x^e_{t-1}) \to (x_t; x^e_t)$ consisting of the ALM and AE functions (i.e., equations (1) and (8):

$$S_0 : \begin{cases} x_t = a_t + \alpha x_t^e \\ x_t^e = \lambda x_{t-1} + (1-\lambda)\, x_{t-1}^e \end{cases} \tag{9}$$

From the first equation we get $x_{t-s}^e = \frac{x_{t-s} - a_{t-s}}{\alpha}$ with s=0,1. Using these expressions in the AE function and rearranging we get the reduced form:

$$x_t = a_t - (1-\lambda)a_{t-1} + [1 + \lambda(\alpha - 1)]x_{t-1} \tag{10}$$

The reduced form of S_0 is a simple linear *first-order* difference equation. Since the exogenous variable a_t is generally governed by a dynamic stochastic process, the difference equation is generally *non-autonomous* and subject to a stochastic disturbance.

From (7), we derive the following law of motion of the average forecasting error

$$\varepsilon_t = \Delta x_t + (1-\lambda)\, \varepsilon_{t-1} \tag{11}$$

Using (10) we get

$$\Delta x_t = \Delta a_t + \lambda a_{t-1} + \lambda(\alpha - 1)x_{t-1} \tag{12}$$

where $\Delta a_t = a_t - a_{t-1}$ is the drift of the exogenous variable. From the last two equations we conclude that this drift determines the first difference of the state variable, which in turn magnifies the forecasting mistake.

In the following we will assume $a_t = a\ \forall t$, i.e., we will consider for simplicity the scenario in which the exogenous variable is constant. Moreover, we will distinguish between the general case $\lambda \in (0,1)$ and the special case $\lambda = 1$ The special case goes under the name of naive expectation. In symbols: $x_t^e = x_{t-1}$. In this case, by construction, the individual and the average updating coefficient coincide: $\lambda_i = \lambda = 1$. Every agent makes the same forecast – i.e., expectations are homogeneous – and therefore the same forecasting mistake. Naive expectation is a special case of (5).

The system ALM-AE in the general case is

$$S_1 : \begin{cases} x_t = a + \alpha x_t^e \\ x_t^e = \lambda x_{t-1} + (1-\lambda)\, x_{t-1}^e \end{cases} \tag{13}$$

so that the resulting *first order* difference equation is:

$$x_t = a\lambda + [1 + \lambda(\alpha - 1)]x_{t-1} \tag{14}$$

The expression in brackets is the slope of the (linear) phase portrait. The steady state

$$x^* = \frac{a}{1-\alpha} \tag{15}$$

is independent of the average updating coefficient. Notice that the steady state (15) is identical to the rational solution in the absence of shocks (see (58) in appendix A). The steady state exists if $\alpha \neq 1$ and is stable if

$$1 - \frac{2}{\lambda} < \alpha < 1 \tag{16}$$

The law of motion of the average forecasting error is:

$$\varepsilon_t = \lambda a + \lambda(\alpha - 1)x_{t-1} \tag{17}$$

Let's now focus on the special case of naive expectations: $\lambda = 1$. The AE equation simplifies to $x_t^e = x_{t-1}$. In this case, the ALM-AE system becomes

$$S_1' : \begin{cases} x_t = a + \alpha x_t^e \\ x_t^e = x_{t-1} \end{cases} \tag{18}$$

The reduced form of S_1' is the difference equation

$$x_t = a + \alpha x_{t-1} \tag{19}$$

The steady state is the same as before (see (15)) and is stable if $-1 < \alpha < 1$. Convergence is oscillatory if $-1 < \alpha < 0$. Notice that in this case, the law of motion of the average forecasting error specializes to

$$\varepsilon_t = \Delta x_t = a + (\alpha - 1)x_{t-1} \tag{20}$$

In words: in the naive expectations case, the forecasting mistake coincides with the first difference of the variable of interest. On the basis of the discussion of systems S_1 and S_1' we can state the following proposition.

Proposition 1 *In the steady state expectations are unbiased in Galton's sense: individual expectations are wrong but the mean of these biased expectations – i.e., the average expectation – is correct. In fact, setting $x_{t-1} = x^*$ in (17) and using (15) we obtain $\varepsilon^* = 0$. Out of the steady state agents are (on average) wrong – i.e., the average expectation is biased.*

In the general case (system S_1), the steady state is stable if (16) *is satisfied. In particular, if the numerical values of α and λ satisfy $1 - \frac{2}{\lambda} < \alpha < 1 - \frac{1}{\lambda}$ transitional dynamics is characterized by dampened oscillations so that the average forecasting error will alternate in sign period by period but il will decrease in absolute value over time. If $1 - \frac{1}{\lambda} < \alpha < 1$ transitional dynamics will be characterized by monotonic convergence and the average forecasting error will have the same sign in each period but decrease over time in magnitude.*

In the special case (system S_1') convergence is oscillatory if $-1 < \alpha < 0$ and monotonic if $0 < \alpha < 1$.

If the steady state exists and is unstable, out of the steady state the average error will be increasing over time (in absolute value). In the special case $\alpha = 1$, the steady state does not exist, the first difference of x is constant and the average error is also constant.

4 Can individual biased expectations be collectively unbiased?

In section 3.2, we have assumed that agents form adaptive expectations à la Friedman-Nerlove. Their PLM is "misspecified" because they don't realize that the dynamics of the macroeconomic variable is affected by the exogenous variable (parameter a). In this section we explore a modified adaptive setting characterized by the following assumption.

Assumption 1 *Agents realize that they do not make mistakes (at least on average) in a stationary state while they are off the mark when the state* *variable grows or declines. Therefore they conjecture that the origin of the mistakes they make is the change over time (first difference) of the state variable. The i-th agent follows an* error mitigation strategy *which consists in augmenting her expectation by a* Belief Correction (BC) *term proportional to the* estimated first difference Δ_t^e*:*

$$BCT_{i,t} = \gamma_i \Delta_t^e \tag{21}$$

where $0 < \gamma_i \leq 1$ *is the BC parameter.*[12]

The expectation formation equation (6) becomes: $x_{i,t}^e = \lambda_i x_{t-1} + (1-\lambda_i) x_{i,t-1}^e + \gamma_i \Delta_t^e$ and the average expectation will be:

$$x_t^e = \lambda x_{t-1} + (1-\lambda) x_{t-1}^e + \gamma \Delta_t^e \tag{22}$$

where $\gamma := \frac{1}{F}\sum_{i=1}^{F} \gamma_i$ is the mean of the individual BC parameters and $BCT_t = \gamma \Delta_t^e$ is the average BC term. By construction $0 < \gamma \leq 1$.

We assume that agents estimate the current change in x applying a filter to observed past differences up to order N: $\Delta_t^e = F(\Delta x_{t-1}, \Delta x_{t-2}, ..., \Delta x_{t-N})$. As a first approximation, suppose $\Delta_t^e = \Delta x_{t-1} := x_{t-1} - x_{t-2}$. In this *simple belief correction* setting, $BCT_{i,t} = \gamma_i \Delta x_{t-1}$ and the the average expectations (AE) equation becomes:

$$x_t^e = \lambda x_{t-1} + (1-\lambda) x_{t-1}^e + \gamma \Delta x_{t-1} \tag{23}$$

In this setting the ALM-AE system consists of equations (1) and (23). Substituting ALM into AE we get a linear *second-order* difference equation:

$$x_t = a\lambda + [\alpha(\lambda+\gamma) + (1-\lambda)] x_{t-1} - \alpha\gamma x_{t-2} \tag{24}$$

The dynamic properties of (24) depend on the numerical values of three parameters: α, λ, γ. To simplify the analysis, let's consider the special case of naive expectations $\lambda_i = 1\ \forall i$. This allows to reduce the number of key parameters. In the presence of naive expectations and simple belief correction, the average expectation (AE) equation becomes:

$$x_t^e = x_{t-1} + \gamma \Delta x_{t-1} = (1+\gamma) x_{t-1} - \gamma x_{t-2} \tag{25}$$

[12] At the moment they form expectations for period t (beginning of period t), agents do not know the actual first difference $\Delta x_t = x_t - x_{t-1}$ because they do not observe x_t. Therefore they have to estimate the first difference.

The ALM-AE system therefore is:

$$S_2 : \begin{cases} x_t = a + \alpha x_t^e \\ x_t^e = (1+\gamma)x_{t-1} - \gamma x_{t-2} \end{cases} \tag{26}$$

The reduced form of this system is the linear second-order difference equation:

$$x_t = a + \alpha(1+\gamma)x_{t-1} - \alpha\gamma x_{t-2} \tag{27}$$

whose properties depend on the numerical values of α and γ. In the special case $\gamma_i = \gamma = 1 \ \forall i$, (full belief correction) system S_2 boils down to

$$S_2' : \begin{cases} x_t = a + \alpha x_t^e \\ x_t^e = 2x_{t-1} - x_{t-2} \end{cases} \tag{28}$$

The reduced form of this system is:

$$x_t = a + 2\alpha x_{t-1} - \alpha x_{t-2} \tag{29}$$

The steady state of S_2 and S_2' is (15), i.e., it coincides with the steady state of S_1' and of S_1. In appendix B we prove the following proposition.

Proposition 2 *We define the* parameter space *as the region of the* (γ, α) *plane such that* $\alpha \in \mathbb{R}$ *and* $\gamma \in [0, 1]$. *The* stability region *is the portion of the parameter space such that the steady state is stable, i.e., the state variable converges to the steady state and expectations tend to unbiasedness.*

In the absence of BC – i.e., with $\gamma = 0$ *– the dynamic system is* S_1'*, the law of motion of* x *is (19) and the stability region is the interval* $A = \{\alpha | -1 < \alpha < 1\}$. *If* α *falls in the sub-interval* $B = \{\alpha | -1 < \alpha < 0\}$*, then convergence is oscillatory and the average forecasting error will alternate in sign period by period but il will decrease in absolute value over time. If* α *falls in the sub-interval* $A - B = \{\alpha | 0 < \alpha < 1\}$*, then convergence is monotonic and the average forecasting error will have the same sign in each period but decrease over time. If* $\alpha \notin A$ *the steady state is unstable and* x *follows a diverging path so that expectations do not tend to unbiasedness.*

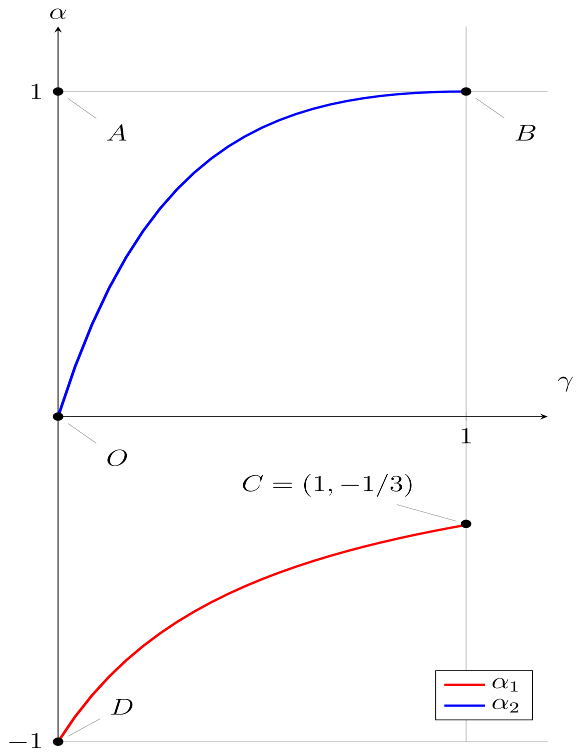
With incomplete BC – i.e., with $\gamma \in (0, 1)$ *– the economy is represented by* S_2 *and the law of motion of* x *is (27). The stability region is* $A = \{(\alpha, \gamma) | \alpha_1 < \alpha < 1, 0 < \gamma < 1\}$ *where* $\alpha_1 = -\frac{1}{1+2\gamma}$. *In particular, if* α *and* γ *fall in the sub-region* $B = \{(\alpha, \gamma) | \alpha_1 < \alpha < \alpha_2), 0 < \gamma < 1\}$ *where* $\alpha_2 = \frac{4\gamma}{(1+\gamma)^2}$, *then convergence is oscillatory. If* $\alpha \notin A$ *the steady state is unstable and expectations do not tend to unbiasedness.*

Finally, with full BC ($\gamma = 1$) the dynamic system is S_2', the law of motion of x is (29) and the stability region is the interval $A = \{\alpha| -\frac{1}{3} < \alpha < 1\}$. Convergence is always oscillatory in this interval. If $\alpha \notin A$ the steady state is unstable and expectations do not tend to unbiasedness.[13]

In figure 1, we compare the properties of the transitional dynamics in system S_1' with that of S_2 and S_2'. The stability region of S_1' (without BC) is the segment AD (extremes excluded). The stability region of S_2' (full BC) is the segment BC.

The curve labelled α_2 has equation $\alpha = \frac{4\gamma}{(1+\gamma)^2}$. The curve labelled α_1 has equation $\alpha = -\frac{1}{1+2\gamma}$. The stability region of S_2 is the approximately trapezoidal area ABCD. If the parameters fall in the area OBCD there is oscillatory convergence. If the parameters fall in the approximately triangular area OAB, there is monotonic convergence. All the other points in the parameter space are associated with divergence.

Figure 1: **Parameter space and transitional dynamics**



[13]Goodwin (1947) explores a similar setting.

From figure 1 we infer the following proposition.

Proposition 3 *Comparing S_2' with S_1' we observe that the range of values of α that implies convergence (and tendency of expectations to unbiasedness) is* smaller *with full BC ($-\frac{1}{3} < \alpha < 1$) than without BC ($-1 < \alpha < 1$): a portion of the parameter space which was characterized by stability in the absence of belief correction turns to instability with full BC. Moreover, convergence occurs always by means of dampened oscillations, while it may be monotonic in the absence of BC. With incomplete BC (system S_2) the stability region becomes a portion of the (α, γ) plane, namely the geometric figure ABCD. The area of the subregion of oscillatory convergence (OBCD) is bigger than the area of the subregion of monotonic convergence (OAB). With full BC, therefore, convergence occurs* always *with oscillations, while with incomplete BC convergence occurs* predominantly *in an oscillatory manner.*

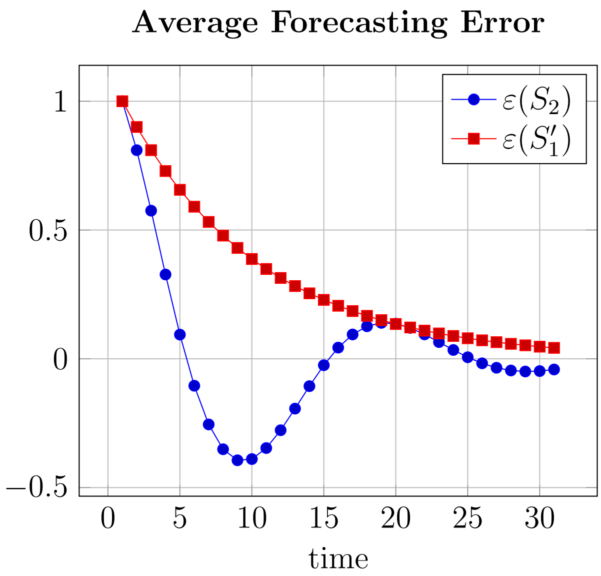
Let's now pause briefly on the Average Forecasting Error (AFE) ε. The law of motion of the AFE with incomplete BC is

$$\varepsilon_t = \Delta x_t - \gamma \Delta x_{t-1} \tag{30}$$

Therefore, expectations are unbiased when $\Delta x_t = \gamma \Delta x_{t-1}$. This condition is satisfied not only in the steady state (where the current and the lagged first differences are zero) but also, in specific cases, out of the steady state.

To illustrate the effects of belief correction, in figure 2 we plot the AFE obtained in S_1' (absence of BC) with $\alpha = 0.9$ (monotonic convergence) and the AFE resulting from S_2 with $\alpha = 0.9$ and $\gamma = 0.9$ (incomplete but almost full BC). We assume that $a = 0$ and the state variable is equal to zero in periods $t_0 - 1$ and $t_0 - 2$. A permanent shock (with a jumping to $a' = 1$) occurs in period $t_0 = 1$. Expectations being naive, the shock is unexpected. Given this configuration of parameters, in the absence of BC the error is always positive (people make systematic mistakes) but decreasing over time and tending asymptotically to zero. With BC, the AFE converges (asymptotically) to zero by means of dampening oscillations. The AFE therefore takes on opposite signs in different intervals: on average people make positive errors in a finite number of periods in a given time window and negative mistakes in the subsequent window. BC therefore implies that people do make mistakes

Figure 2: Average Forecasting Error without BC (system S_1';$\alpha = 0.9$) and with incomplete BC (system S_2; $\alpha = \gamma = 0.9$)



(also on average) but they do not make *systematic* average mistakes over the entire time horizon. In each period, *given this parameterization*, the AFE with BC is smaller (in absolute value) than the AFE without BC.

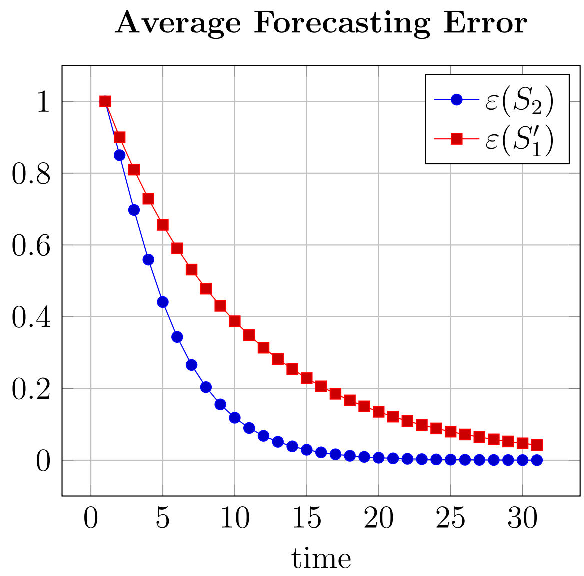
In figure 3 we represent the AFE setting $\alpha = 0.9$ and $\gamma = 0.5$. Given this configuration of parameters, the AFE is always positive both with and without BC: belief correction does not prevent people from making systematic mistakes. In each period, however, the AFE with BC is smaller (in absolute value) than the AFE without BC. A low degree of belief correction (γ relatively small), therefore, allows to reduce the magnitude of the forecasting error but does not prevent people from making systematic mistakes.

Let's consider now a more sophisticated filter, which we will label the *mean difference* BC term:

$$\Delta_t^e = \frac{1}{N}\sum_{s=1}^{N} \Delta x_{t-s} \tag{31}$$

In this case agents estimate the first difference of variable x by computing the mean of the past (first) differences of the variable over the time window [t-1:t-

Figure 3: Average Forecasting Error without BC (system S_1';$\alpha = 0.9$) and with incomplete BC (system S_2; $\alpha = 0.9; \gamma = 0.5$)



N]. Averaging across agents we get the following aggregate mean difference BC term:

$$BCT_t = \gamma \frac{1}{N} \sum_{s=1}^{N} \Delta x_{t-s}$$

Suppose agents want to fully correct their belief ($\gamma = 1$). In the presence of naive expectations and full correction, the average expectation (AE) equation becomes:

$$x_t^e = x_{t-1} + \frac{1}{N} \sum_{s=1}^{N} \Delta x_{t-s} \tag{32}$$

Using this filter, the ALM-AE system will be

$$S_3 : \begin{cases} x_t = a + \alpha x_t^e \\ x_t^e = x_{t-1} + \frac{1}{N} \sum_{s=1}^{N} \Delta x_{t-s} \end{cases} \tag{33}$$

Substituting ALM in AE we get the reduced form of S_3 which is a linear difference equation of order N+1:

$$x_t = a + \alpha x_{t-1} + \alpha \frac{1}{N} \sum_{s=1}^{N} \Delta x_{t-s} \tag{34}$$

The steady state is (15), i.e., it coincides with the steady state of S_2 (and of all the other systems considered so far). The characteristic equation will be of order N+1. Of course the dynamics generated by (34) depend on the numerical value of the roots $\theta_n; n = 1, 2, ..., N+1$ – i.e., the eigenvalues of the companion matrix of (34) – which will be functions of α only (thanks to the assumptions of naive expectations and full correction).

In order to explore the nature of the roots, we will exploit the fact that the region of stability of the reference system S_1' is $-1 < \alpha < 1$, a bounded region. Hence we can use a grid search to solve numerically for the eigenvalues. The search step used for the parameter α is $\delta = 0.01$; i.e., $\alpha \in (-1+\delta, \ldots, 1-\delta)$. We will consider 4 cases: $N = 1, 2, 3, 4$. The case $N = 1$ coincides with the analysis carried out above for S_2'. We summarize the results of this numerical exploration in the following result.

Result 1 *From numerical exploration in the stability region of* S_1' *(*$-1 < \alpha < 1$*) we found that the ALM-AE system with full BC* predominantly *leads to oscillatory behaviour. In particular, for N=2 and N=4, convergence occurs* always *– i.e.,* $\forall \alpha \in (-1, 1)$ *– with oscillations. For N=1, convergence occurs with oscillations in the interval* $-\frac{1}{3} < \alpha < 1$*. For N=3, convergence occurs with oscillations in the interval* $-\frac{3}{5} < \alpha < 1$*.*

The case $N = 4$ will be used in the following section. In appendix C we explore numerically the properties of S_3 with $N = 4$, $\alpha \in (-1, 1)$ (except the trivial case $\alpha = 0$) and (γ, λ) in the unit square.

5 A streamlined Heterogeneous Agents model

In this section we discuss the effects of belief correction in a streamlined agent based setting characterized by heterogeneous adaptive expectations. We consider a closed economy populated by households and firms. Households (not explicitly modelled) supply labour, earn wages (if employed) and spend on consumption goods their wages entirely (they are "hand to mouth" consumers).

There are F firms which produce a homogeneous consumption good. The production function of the i-th firm ($i = 1, 2, ..., F$) is:

$$Q_{i,t} = A_t N_{i,t}^{\frac{1}{\delta}} \tag{35}$$

where Q_i and N_i are output and employment, $\delta > 1$ and A is Total Factor Productivity. TFP grows at the exogenous rate $g_A > 0$ with probability p_A, which in turn is increasing with the sales tax rate τ. We assume, in fact, that sales taxes finance public investment in fundamental research[14] and fundamental research, in turn, affects the rate of growth of TFP. For simplicity, we assume $p_A = \gamma_A \tau$ with $\gamma_A > 0$. When $\tau = 0$, there is no Government expenditure in research and TFP is stationary. We assume $A = 1$ in this case. The stationarity of TFP characterizes the baseline scenario for simulations, which we will present in section 5.1. We will discuss the consequences of TFP growth in section 6.

Due to labour market "frictions", the real wage w_t does not adjust to imbalances between demand and supply of labour but follows an exogenous AR(1) process with drift:

$$w_t = \rho_w w_{t-1} + d + \sigma_\varepsilon \varepsilon_W \tag{36}$$

where $d > 0$, $\rho_w \in [0, 1)$ and $\varepsilon_W \sim \mathcal{N}(0, \sigma_\varepsilon)$ is a wage shock. In the absence of the wage shock, the real wage tends to $\bar{w} = \frac{d}{1-\rho_w}$. Labour supply (not modelled) is always abundant: labour shortages are ruled out by assumption.

At the beginning of period t, the firm decides the quantity to be produced (and the workers to be employed). Production takes time and output will be available for sale only at the end of the period, when the market for consumption goods opens (and transactions are carried out). At the beginning of the period, the firm is uncertain on the sale price $P_{s,t}$ which will be

[14] We assume that the Government budget is always balanced: $\tau \sum_i Q_{it} = R_t$ where R is public expenditure in fundamental research. Hence the latter is endogenous and proportional to total output.

revealed only at the end of the period. In order to make production and employment decisions, therefore, the firm has to form expectations on the sale price.

We will denote the expectation (formed by agent i at the beginning of period t) of the sale price with $P^e_{i,t}$. Under risk neutrality and perfect competition, the firm chooses at the beginning of t the optimal quantity by maximizing expected profits (net of taxes):

$$\max_{Q_{i,t}} \Pi^e_{i,t} = x^e_{i,t}(1-\tau)Q_{i,t} - w_t A_t{}^{-\delta} Q^{\delta}_{i,t}$$

where

$$x^e_{i,t} := \frac{P^e_{i,t}}{P_t} \tag{37}$$

is the within-period (gross) rate of change of the price (inflation for short) expected by the i-th firm. P_t is the price at the beginning of period t, which has been "inherited" from transactions carried out at the end of period t-1. Optimal output is:

$$Q_{i,t} = \eta\zeta_t \left(x^e_{i,t}\right)^{\frac{1}{\delta-1}} \tag{38}$$

where

$$\zeta_t := A_t^{\frac{\delta}{\delta-1}} w_t^{-\frac{1}{\delta-1}}$$

$$\eta := \left(\frac{1-\tau}{\delta}\right)^{\frac{1}{\delta-1}}$$

Plugging (38) into (35) and rearranging we get the demand for labour

$$N_{i,t} = \frac{1}{w_t}\eta^{\delta}\zeta_t \left(x^e_{i,t}\right)^{\frac{\delta}{\delta-1}} \tag{39}$$

Output and employment are increasing with inflation expectations and are heterogeneous because of the heterogeneity of the latter.

Since firms produce a homogeneous consumption good, we assume that at the end of the period a fraction $1/F$ of the wage bill is spent at each firm by each household. Aggregate demand in real terms, in fact, is equal to the total wage bill $w_t L_t$ where $L_t := \sum_{i=1}^{F} N_{i,t}$ is total employment. When the market opens actual sales for the i-th firm ($S_{i,t}$) may be different from output planned at the beginning of period t: $S_{i,t} = \min(Q_{i,t}, \frac{w_t L_t}{F})$. Firms with relatively "high" inflation expectations will produce more than firms holding "low" inflation expectations and viceversa.

Overall excess demand will be $ED_t = w_t L_t - Y_t$ where $Y_t := \sum_{i=1}^{F} Q_{i,t}$ is aggregate supply. If excess demand is positive, all the output will be sold and there may be a fringe of unsatisfied consumers at some firms. In this case, it is reasonable to assume that the sale price at the end of the period will be higher than the price at the beginning of the period. If, on the contrary, excess demand is negative the end-of-period price will be lower than P_t. However, *there is no guarantee of market clearing* as we do not assume the presence of a top-down coordinating mechanism such as the auctioneer which brings necessarily demand into equality with supply. On the basis of these consideration we assume the following market protocol.

Assumption 2 *Let's denote with $x_t := \frac{P_{s,t}}{P_t}$ the ratio of the end of period (sale) price to the beginning of period price (i.e., the intra-period gross inflation rate). We assume that the market price evolves according to the stochastic adjustment process:*

$$x_t = \exp\left(\gamma_p ED_t\right) \exp(\varepsilon_P) \tag{40}$$

where $\gamma_p > 0$ is the semi-elasticity of (gross) inflation x to aggregate excess demand

$$ED_t = w_t \sum_{i=1}^{F} N_{i,t} - \sum_{i=1}^{F} Q_{i,t} \tag{41}$$

and $\varepsilon_P \sim \mathcal{N}(0, \sigma_\varepsilon)$ is a price shock, $E\left(\exp(\varepsilon_P)\right) \approx 1$. Excess demand is known in t because all the variables involved (total output and the total wage bill) are determined in t on the basis of individual inflation expectations. Hence the uncertainty concerning x_t is rooted in the shock to price dynamics ε_P. In the absence of the price shock, the price is stationary ($x_t = 1$) when there is market clearing ($ED_t = 0$), i.e., when total output is paid out as wages and spent entirely in consumption goods.

Substituting optimal output (38) and employment into excess demand (41) and the latter into (40) we obtain the ALM of the Heterogeneous Agents model:

$$x_t = \exp\left\{\gamma_p \zeta_t \left[\eta^\delta \sum_i (x^e_{i,t})^{\frac{\delta}{\delta-1}} - \eta \sum_i (x^e_{i,t})^{\frac{1}{\delta-1}}\right]\right\} \exp(\varepsilon_P) \quad (42)$$

Actual inflation therefore is a non linear function of individual inflation expectations for any combination of the real wage and TFP. Non linearity is due exclusively to decreasing returns, an assumption which allows to obtain close form solutions for optimal output and employment.

We assume that agents know the beginning of period price P_t and form expectations on the sale price at the end of the period $P^e_{i,t}$ according to the following adaptive mechanism:

$$P^e_{i,t} = \lambda_i P_t + (1-\lambda_i) P^e_{i,t-1} + \mathbf{1}_P BCT_t \quad (43)$$

where λ_i is drawn from a uniform distribution with support $(\lambda_0, 1]$; $\lambda_0 > 0$. For simplicity, we assume that (i) the corporate sector *in its entirety* can either apply or not apply the BC term;[15] (ii) when applying belief correction, firms opt for full correction; (iii) the BC term is the same for all firms and is the mean of the price changes over the four previous periods: $BCT_t = \frac{1}{4}\sum_{s=1}^{4} \Delta P_{t-s}$. $\mathbf{1}_P$ is an indicator function equal to 1 when all the agents apply a BC term, 0 otherwise. Iterating (43) we get:

$$P^e_{i,t} = \bar{P}_{i,t} + \mathbf{1}_P \overline{BCT}_{i,t} \quad (44)$$

where $\bar{P}_{i,t} := \lambda_i \sum_{s=0}^{\infty} (1-\lambda_i)^s P_{t-s}$ and $\overline{BCT}_{i,t} := \sum_{s=0}^{\infty} (1-\lambda_i)^s BCT_{t-s}$ are the weighted averages of past prices and past BC terms with exponentially declining weights. Heterogeneity is due exclusively to the updating parameter.

Dividing both sides of (44) by P_t we can write the expectation of inflation held by the i-th agent as follows:

$$x^e_{i,t} = \bar{x}_{i,t} + \mathbf{1}_P \Delta^P_{i,t} \quad (45)$$

[15] For simplicity we will rule out the scenario in which belief correction is applied only by a fraction of the population of firms.

where

$$\bar{x}_{i,t} := \frac{\bar{P}_{i,t}}{P_t} = \lambda_i \sum_{s=0}^{\infty} (1-\lambda_i)^s \frac{P_{t-s}}{P_t} \tag{46}$$

and

$$\Delta_{i,t}^P = \frac{\overline{BCT}_{i,t}}{P_t} = \sum_{s=0}^{\infty} (1-\lambda_i)^s \frac{BCT_{t-s}}{P_t} \tag{47}$$

Hence the average inflation expectation is

$$x_t^e = \frac{1}{F} \left(\sum_{i=1}^{F} \bar{x}_{i,t} + \mathbf{1}_P \sum_{i=1}^{F} \Delta_{i,t}^P \right) \tag{48}$$

5.1 The baseline

In this section we simulate the baseline scenario characterized by $\tau = 0$ so that there is no investment in fundamental research.[16] Hence TFP is constant. As we already said above, for simplicity we normalize it to unity: $A = 1$. The dynamics of the model in the baseline therefore is driven exclusively by the law of motion of the real wage (36), which plays the role of the exogenous variable a_t in section 3. Therefore, in the "long run" the real wage converges to $\bar{w} = \frac{d}{1-\rho_w}$.

We assume individual expectations are formed adaptively as shown in (45). Therefore optimal output and employment will be:

$$Q_{i,t} = \eta \zeta_t \left(\bar{x}_{i,t} + \mathbf{1}_P \Delta_{i,t}^P \right)^{\frac{1}{\delta-1}} \tag{49}$$

$$N_{i,t} = \frac{1}{w_t} \eta^\delta \zeta_t \left(\bar{x}_{i,t} + \mathbf{1}_P \Delta_{i,t}^P \right)^{\frac{\delta}{\delta-1}} \tag{50}$$

where ζ_t simplifies to

$$\zeta_t = w_t^{-\frac{1}{\delta-1}}$$

The parameter values used in simulations are gathered in Table 1. We have set $\gamma_P = 0.001$ and $\delta = 3/2$. With this calibration $\zeta_t = w_t^{-2}$, $\eta = \delta^{-2} = 4/9 \approx 0.44$ and $\eta^\delta = \delta^{-3} = 8/27 \approx 0.3$. Substituting optimal output (49)

[16] See (Palestrini and Gallegati, 2015) and (Palestrini, 2017).

Parameter	Description	Value
F	Number of firms	200
δ	Reciprocal of Cobb-Douglas exponent	3/2
γ_p	Semi-elasticity of x to excess demand	0.001
γ_τ	Sensitivity of the probability of TFP growth to the tax rate	4
g_A	Growth rate of TFP with $\tau > 0$	0.02
ρ_w	Auto-regressive parameter (law of motion of the real wage)	0.9
d	Drift (law of motion of the real wage)	0.1
σ_ε	Standard deviation of the wage shock and of the price shock	0.01
λ_0	Minimum updating coefficient	0.4

Table 1: Parameter values

and employment (50) into excess demand (41) and the latter into (40) we obtain the ALM of the Heterogeneous Agents model:

$$x_t = \exp\left\{\frac{0.001}{w_t^2}\left[0.3\sum_i\left(\bar{x}_{i,t} + \mathbf{1}_P\Delta_{i,t}^P\right)^3 - 0.44\sum_i\left(\bar{x}_{i,t} + \mathbf{1}_P\Delta_{i,t}^P\right)^2\right]\right\}\exp(\varepsilon_P) \tag{51}$$

For any value of the real wage and realization of the price shock, actual inflation is a non linear function of individual inflation expectations, which in turn are weighted averages of past inflation rates with heterogenenous exponentially decaying weights.

We run $S = 100$ Monte Carlo simulations of the Heterogeneous Agents model (with different random seeds). The duration of each simulation is $T = 40$ periods. Hence each individual expectation $x_{i,t,s}^e$ is characterized by three indices: $i = 1, 2, ...F$, $t = 1, 2, ...T$, $s = 1, 2, ...S$. For a given real wage and price shock, simulation s_0 generates in period t the distribution of $F = 200$ individual inflation expectations x_{i,t,s_0}^e, $i = 1, 2, ...F$, which in turn will generate actual inflation x_{t,s_0} through the ALM (51). Since the dynamics converge quite rapidly, after a short transient, we can define the "long run" distribution of inflation expectations as the distribution of the last period of the simulation window: x_{i,T,s_0}^e, $i = 1, 2, ...F$. The associated average expectation (i.e., inflation expected on average by the population of firms in period T) is the mean of the long run distribution

$$x^e_{T,s_0} = \frac{\sum_{i=1}^{F} x^e_{i,T,s_0}}{F} \tag{52}$$

The actual long run inflation is inflation of the last period x_{T,s_0}. The difference between actual and expected inflation (both referred to period T) is the forecast error associated to simulation s_0: $\varepsilon_{T,s_0} = x_{T,s_0} - x^e_{T,s_0}$.

We define "the bias" as the percent forecast error. The last period bias therefore is: $b_{T,s_0} = \frac{\varepsilon_{T,s_0}}{x^e_{T,s_0}} - 1$. The *bias distribution* is the distribution of $S = 100$ relative forecast errors in the last period of the simulation (one for each simulation) $b_{T,s}$ $s = 1, 2, ...S$. We evaluate the performance in forecasting of adaptive agents by means of the first and second moments of the bias distribution. The *average bias* is:

$$b = \frac{\sum_{s=1}^{S} b_{T,s}}{S} \tag{53}$$

while the degree of heterogeneity of the bias distribution is measured by the variance:

$$\sigma_b^2 = \frac{\sum_{s=1}^{S} (b_{T,s} - b)^2}{S} \tag{54}$$

By construction, when expectations are model consistent the forecasting error is $\varepsilon_t = E(x_t)[\exp(\varepsilon_P) - 1]$ and the bias is $b_t = \exp(\varepsilon_P) - 1$ so that the average error (and the average bias) is zero. In words, model consistent expectations are unbiased. Moreover, the standard deviation of the bias is close to the standard deviation of the price shock ε_P, which, in our calibration, is $\sigma_\varepsilon = 0.01$.

From simulations we get the following fundamental result:

Result 2 *When adaptive agents do not apply BC (i.e., when $\gamma_i = 0 \forall i$ so that the indicator function $\mathbf{1}_P$ takes value 0), the average bias is significantly different from zero: $b = -2.5\%$, i.e., agents overestimate inflation by a non-negligible margin. Moreover, the standard deviation of the bias distribution $\sigma_b = 0.0012$, is sligthly bigger than the standard deviation of the price shock.*[17]

[17]This result is in line with the previous literature that compares adaptive expectations to rational expectations. Evans and Honkapohja (2001) make a similar point in the case

On the contrary, when agents apply full BC (i.e., when $\gamma_i = 1 \forall i$ *so that the indicator function* $\mathbf{1}_P$ *takes value 1) the average bias with adaptive expectations is "close to zero" (i.e., close to unbiasedness)* [18] *while the standard deviation of the bias distribution* σ_b *does not change and remains slightly above the standard deviation of the price shock.*

Let's now turn to output and employment. Each simulation generates the time series of total output $Y_{t,s_0} = \sum_{i=1}^{F} Q_{i,t,s_0} = \eta\zeta_t \sum_{i=1}^{F} \left(\bar{x}_{i,t,s_0} + \mathbf{1}_P \Delta_{i,t,s_0}^{P}\right)^{\frac{1}{\delta-1}}$ and of total employment $L_{t,s_0} = \sum_{i=1}^{F} N_{i,t,s_0} = \frac{1}{w_t}\eta^{\delta}\zeta_t \sum_{i=1}^{F} \left(\bar{x}_{i,t} + \mathbf{1}_P \Delta_{i,t,s_0}^{P}\right)^{\frac{\delta}{\delta-1}}$, $t = 1, 2, ...T$. We take the final period (period 40) employment in the Heterogeneous Agents model L_{T,s_0} and compute the ratio to total employment when expectations are unbiased (Unbiased case hereafter) L_{T,s_0}^{U} which is defined in appendix D. In this way we generate the distribution of the *employment ratio* (one for each simulation) $n_s := \frac{L_{T,s}}{L_{T,s}^{U}}$, s=1,2,...100. From simulations we get the following fundamental result:

Result 3 *Absent BC the average employment ratio is* $n = \sum_{s=1}^{S} n_s/S = 1.08$ *i.e., employment in the Heterogeneous Agents model is 8% bigger than in the Unbiased case. When agents apply full BC, on the contrary, the average employment ratio is* $n = 0.997$ *i.e., the distribution is centered around 1: employment in the Heterogeneous Agents model with full BC is approximately the same as employment in the Unbiased case.*

The rationale for Result 3 is the following. In our simulations, given the initial conditions, if firms do not apply BC to expectations formation, on average inflation expectations converge to a "long run" level x_T^e that significantly overestimates actual inflation in the same period x_T (the bias is negative, as stated in Result 2). Therefore firms optimally employ a larger number of workers and produce more than in the Unbiased case: employment is 8% greater than in the Unbiased case. If, on the contrary, they apply BC, on average adaptive expectations produce the same result as model consistent expectations, hence also GDP and total employment are observationally equivalent to those of the Unbiased case.

of constant gain expectation scheme.

[18] Closeness depends on the persistence of inflation, on the type of filter used to evaluate the drift and generate the BCT and on the correlation of heterogeneous expectations.

6 A policy experiment

We are now able to assess the effects of a (permanent) policy shock in the Heterogeneous Agents model and compare them with the effects of the same shock in the the Unbiased scenario.

Suppose the Government sets the tax rate at $\tau_0 > 0$. The probability of TFP growth therefore will be $p_\tau = \gamma_\tau \tau_0$. The parameter values used in simulations are those in table 1. We set $\gamma_\tau = 4$ and $\tau_0 = 0.05$ so that the probability of TFP growth due to the policy shock is $p_\tau = 20\%$. We set TFP growth in case of success of the policy at $g_A = 0.02$. This is the Policy scenario.

We run 100 Monte Carlo simulations of the model in the policy scenario. As before, the time span of each simulation is 40 periods. Each simulation generates a distribution of F=200 inflation rates and average inflation expectations for each period (as shown in detail above). The difference between actual and average expected inflation (both referred to period 40) is the forecast error. In this way we generate the bias distribution of 100 forecast errors (one for each simulation). By construction, the mean of the bias distribution is zero in the Unbiased case. In figure 4 we show the bias distribution generated by S=100 replications of the Heterogeneous Agents model with and without BC in the Policy scenario.

We can summarize the results of these simulations as follows:

Result 4 *Absent BC, the average bias is $b = -4\%$ and the standard deviation of the bias is $\sigma_b = 0.0224$. Firms significantly overestimate inflation in the Policy scenario, more than in the baseline scenario. With BC, the average bias goes down approximately to zero (0.5%) and the standard deviation falls to 0.0127.*

We compute the employment ratio n_s for each simulation following the procedure outlined in the baseline scenario. Figure 5 shows the distribution of the employment ratio in the Policy scenario.

Result 5 *Absent BC, the mean employment ratio is significantly greater than one ($n = 1.137$). With BC, the distribution is centered around one (the mean is 0.990). This result is in line with the result obtained above in*

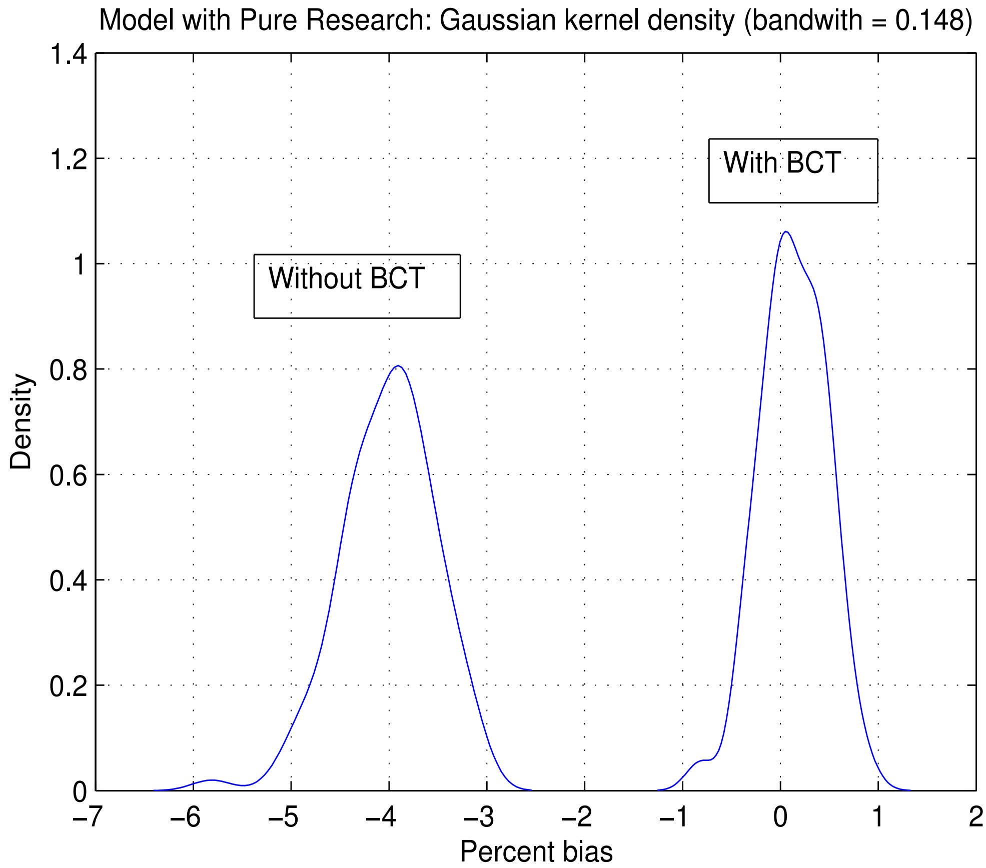


Figure 4: Distribution of the percent forecast error (kernel density estimation).

the baseline scenario. In other words, in the presence of bias employment is 14% higher than in the absence of bias. Belief correction reduces the bias almost to zero.

We are now in a position to compare the effects of the policy shock by computing the *change* in employment due to investment in fundamental research. We will denote with $\Delta^U = \frac{L_t^U(\tau=0.05)}{L_t^U(\tau=0)}$ the (gross) rate of change in total employment (in each period) due to the shock in the unbiased case. Analogously, $\Delta_t^H = \frac{L_t(\tau=0.05)}{L_t(\tau=0)}$ is the change in employment due to the shock in the Heterogeneous Agents model.

In the unbiased case we expect, $\Delta_t^U = (1-p_\tau) + p_\tau(1+g_A)^{\frac{\delta}{\delta-1}} - 1$. Since

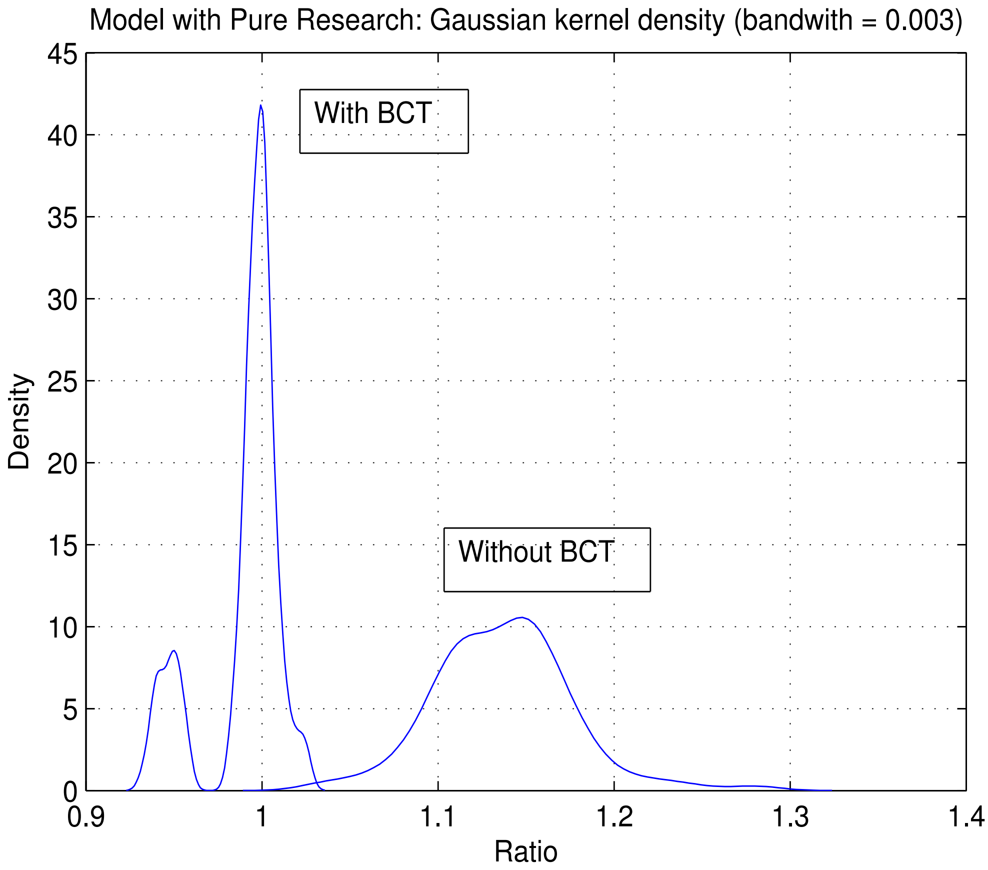


Figure 5: Distribution of the employment ratio (kernel density estimation).

we have set $p_\tau = 0.2; g_A = 0.02$ and $\delta = 1.5$ in the simulation, $\Delta^U = 0.0122$. In other words, the unbiased case implies an increase of employment due to the shock approximately equal to 1.2% per period.

Figure 6 plots the time series of the change in employment Δ^H obtained from the simulation of the Heterogeneous Agents model. The series shows a jump process due to the TFP evolution over time. Note that initial values are below 1 because the introduction of taxation depresses economic activity but then it grows steadily over time. It goes from 0.8576 in $t_0 = 1$ to 1.608 in period $T = 40$. This implies an average rate of employment growth equal to $(1.608/0.8676)^{1/36} = 1.0173$. The fact that the differential rate of growth is close to the unbiased solution is robust across simulations. The Monte Carlo analysis shows a mean rate of growth equal to 1.2085% with standard deviation 13.5782%. This leads to the final

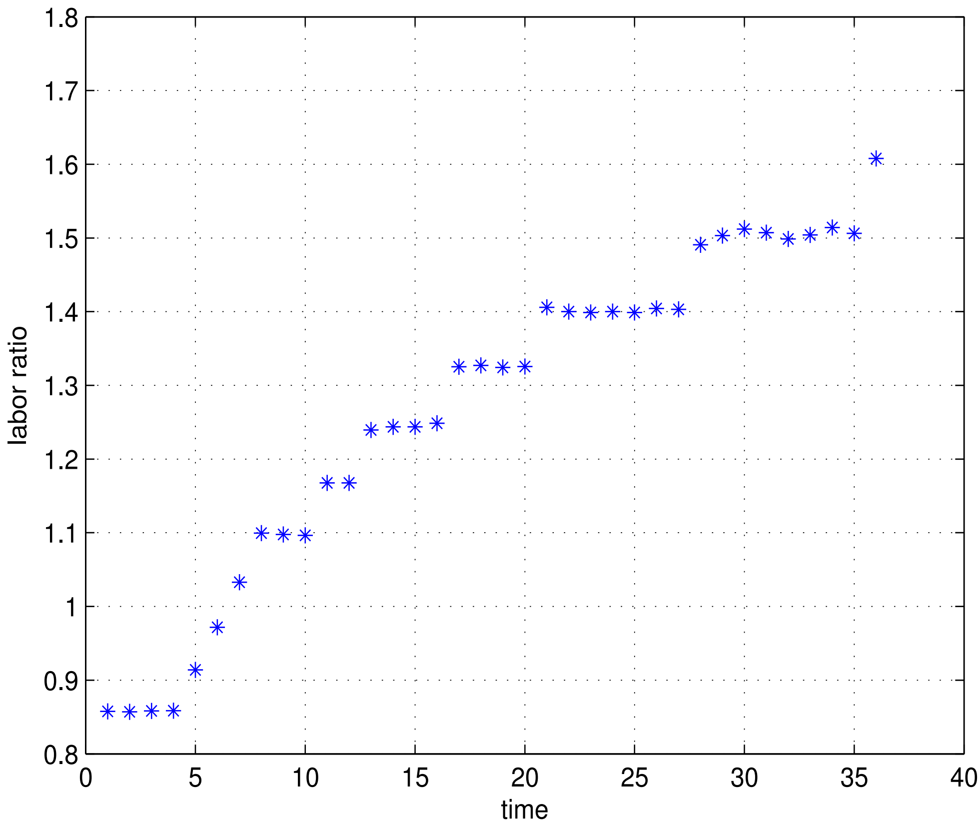


Figure 6: The change in employment Δ^H in Heterogeneous Agents model.

Result 6 *The long run change in employment due to the policy shock estimated with the Heterogeneous Agents model has the same order of magnitude of the long run change in employment due to the policy shock in the unbiased solution.*

This is an interesting and a puzzling observation. Why this result? The intuition goes as follows. Unbiased agents know (in expectation) the profit function whereas adaptive agents have imperfect information due to uncertainty of the future sales price. This implies that initial levels of economic variables may be different in the unbiased case and in the Heterogeneous Agents model. But when TFP growth shifts the profit function, adaptive agents for which the bias does not change tend to follow the movement of the distribution profit more or less in line with rational agents. Put differently, as to levels of macrovariables the Heterogeneous agents model (without BC) and the unbiased solution produce different results. But the results of the policy shocks in terms of change in employment are, given the bias, observationally equivalent.

7 Conclusions

Agents adopting a Friedman-Nerlove adaptive algorithm to form expectations are bound to make systematic errors. This consequence of the canonical adaptive expectations scheme is patently unrealistic and has doomed the idea in the macroeconomic literature, paving the way to rational expectations. In this paper we reconsider the behavior of the Adaptive agent in search of a more satisfactory characterization. We notice first that agents holding heterogeneous Adaptive Expectations can be collectively right if the individual forecast errors wash out in the aggregate, i.e., if the mean forecast error across agents is zero. In this case adaptive expectations are unbiased. In the canonical adaptive expectations scheme unbiasedness occurs only when the economy is in a stationary state. If the macroeconomic variable of interest has a drift, adaptive expectations are wildly off the mark. We have shown that heterogeneous adaptive agents who correct their expectation updating rule with a Belief Correction term – a proxy of the drift – can substantially reduce the average forecast error and (under some parameterizations) may avoid the trap of systematic forecast errors.

We have then used this notion to assess the effects of biasedness on macroeconomic performance in a streamlined agent based setting characterized by heterogeneous expectations. This feature heavily affects the results of policy experiments. The introduction of a sales tax to finance fundamental research affects TFP and is a source of drift of GDP. Absent belief correction, the mean forecast error is negative and significant: purely adaptive firms significantly overestimate inflation. Since optimal production and employment are increasing with expected inflation, in this setting production and employment are significantly bigger than they would be if agents had model-consistent unbiased expectations. With belief correction, the average error goes down approximately to zero and the standard deviation shrinks dramatically: adaptive firms that follow a belief correction strategy are collectively unbiased.

Further experimentation is required to assess the applicability of these ideas to larger and more complex macroeconomic agent based models. The wide range of variables over which agents must form expectations in larger models adds layers of complexity to the design of the belief correction (and bias mitigation) mechanism. We are convinced, however, that this terrain is worth exploring to provide a convincing benchmark for this class of models.

A The rational solution under adaptive learning

On the basis of the PLM – see (2) – the rational expectation of x_t is $E(x_t) = b_0 + b_1\bar{a}$. Setting $x_t^e = E(x_t)$ (model-consistent expectations) and plugging this expression into the ALM – see (1) – we get the implied actual law of motion:

$$x_t = a_t + \alpha(b_0 + b_1\bar{a}) \tag{55}$$

The Rational Expectations (RE) solution is a fixed point of the mapping T from the PLM (2) to the implied ALM (55) and must satisfy:

$$T(b_0, b_1) = (\alpha b_0 + \alpha b_1\bar{a}, 1) \tag{56}$$

Hence the RE solution is $b_0 = \frac{\alpha}{1-\alpha}\bar{a},\ b_1 = 1$ and the state variable is indeed a linear function of the shock:

$$x_t = a_t + \frac{\alpha}{1-\alpha}\bar{a} \tag{57}$$

In the absence of the white noise disturbance ($\tilde{a}_t = 0\ \forall t$), $a_t = \bar{a}$ and

$$x^* = \bar{x} := \frac{\bar{a}}{1-\alpha} \tag{58}$$

From (57) follows that the forecasting mistake is $\varepsilon_t := x_t - E(x_t) = \tilde{a}_t$. Since $\tilde{a}_t$ is white noise, $E(\varepsilon_t) = 0$: rational expectations are unbiased. This means that, over a given time window, the rational agent may over-forecast the state variable in some intervals and underforecast it in some other intervals within the time window.

In an “adaptive learning setting” à la (Evans and Honkapohja, 2001), the agent learns the parameters b_0 and b_1 by means of standard econometric techniques. The adaptive learning process is governed by a system of ordinary linear differential equations:

$$S_{DE}: \begin{cases} \frac{db_0}{dt} = \alpha(b_0 + b_1\bar{a}) - b_0 \\ \frac{db_1}{dt} = 1 - b_1 \end{cases} \tag{59}$$

The RE solution is the steady state of S_{DE}. A well known result of this literature is the *E-stability Principle*: the RE solution is stable under adaptive learning (i.e., the agent is indeed capable of learning the RE values of the parameters) if it is also a locally stable steady state of the associated S_{DE}. In our simple example the E-stability principle is satisfied for $\alpha < 1$. In other words, the learning process tends to the RE solution (rational learning) only if $\alpha < 1$.

B System S_2

Let's consider system S_2 with $\gamma \in (0, 1)$. The system yields the second order difference equation (27) whose characteristic equation is

$$\theta^2 - \alpha(1+\gamma)\theta + \alpha\gamma = 0 \tag{60}$$

The roots of this equation are

$$\begin{cases} \theta_1 = \frac{\alpha(1+\gamma)}{2} - \frac{\sqrt{\alpha[\alpha(1+\gamma)^2 - 4\gamma]}}{2} \\ \theta_2 = \frac{\alpha(1+\gamma)}{2} + \frac{\sqrt{\alpha[\alpha(1+\gamma)^2 - 4\gamma]}}{2} \end{cases} \tag{61}$$

There are two cases. The first one, in turn, has three sub-cases.

Case 1 The first the case is characterized by $\alpha > 0$. There are three sub-cases, depending on the sign of the discriminant of the characteristic equation.

Case 1.1 The discriminant is positive if

$$\alpha > \alpha_2 := \frac{4\gamma}{(1+\gamma)^2} \tag{62}$$

where $\alpha_2 < 1$. In this case the characteristic equation has two distinct roots. Standard algebra shows that (62) is also the condition for $0 < \theta_1 < 1$. As to θ_2 it turns out that

$$\begin{cases} 0 < \theta_2 < 1 \text{ if } \alpha_2 < \alpha < 1 \\ \theta_2 > 1 \text{ if } \alpha > 1 \end{cases} \tag{63}$$

Therefore, if $\alpha_2 < \alpha < 1$ both roots are positive and smaller than one and the dynamics is monotonically converging to the steady state. If $\alpha > 1$ one root is higher than one and the other is lower than one. The steady state therefore is a saddle point. Out of the steady state the dynamics is diverging.

Case 1.2 The discriminant is zero if $\alpha = \alpha_2$, and the characteristic equation has repeated (and identical) roots: $\theta_1 = \theta_2 = \frac{\alpha_2(1+\gamma)}{2} = \frac{2\gamma}{1+\gamma} < 1$. The steady state is stable and the dynamics is monotonically converging.

Case 1.3 The discriminant is negative if $0 < \alpha < \alpha_2$. In this case, the characteristic equation has two complex roots. The dynamic path is oscillatory. Oscillations are dampening over time because the growth factor $r = \sqrt{\alpha\gamma}$ is smaller than one. In fact, in this subcase $\alpha < 1$. The transitional dynamics is characterized by oscillatory convergence to the steady state.

Case 2 The second case is characterized by $\alpha < 0$. In this case, the discriminant is positive. Hence there will be two distinct roots. The smaller one, θ_1, is negative. Simple algebra shows that

$$\begin{cases} \theta_1 < -1 & \text{if } \alpha < -\alpha_1 \\ -1 < \theta_1 < 0 & \text{if } -\alpha_1 < \alpha < 0 \end{cases} \tag{64}$$

where

$$\alpha_1 := \frac{1}{1+2\gamma} \tag{65}$$

The higher root, θ_2, is positive and it is easy to ascertain that it is smaller than one: $0 < \theta_2 < 1$. If $-\alpha_1 < \alpha < 0$ the roots have opposite sign and they are both smaller than one (in absolute value): the dynamics is oscillatory and converging to the steady state. If $\alpha < -\alpha_1$ one root is higher than one (in absolute value) and the other is lower than one. The steady state therefore is unstable. Out of the steady state the dynamics is diverging. To study S_2' it is sufficient to set $\gamma = 1$. In this case the cut off values are $\alpha_1 = 1/3$ and $\alpha_2 = 1$. We summarize these conclusions in the following table, which proves Proposition 2.

Table 2: Dynamic properties

System	divergence	oscillatory conv.	monotonic conv.	divergence
S_2	$\alpha < -\alpha_1$	$-\alpha_1 < \alpha_2$	$\alpha_2 < \alpha < 1$	$\alpha > 1$
S_2'	$\alpha < -\frac{1}{3}$	$-\frac{1}{3} < \alpha < 1$		$\alpha > 1$

C System S_3 with incomplete BC

Let's consider system S_3 (see (33)). We rewrite it here for the reader's convenience:

$$\begin{cases} x_t = a_t + \alpha x_t^e \\ x_t^e = \lambda x_{t-1} + (1-\lambda)x_{t-1}^e + \gamma N^{-1}\sum_{i=1}^{N}\Delta x_{t-i} \end{cases}$$

It may be written in matrix form as

$$\begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}\begin{pmatrix} x_t \\ x_t^e \end{pmatrix} = \begin{pmatrix} a_t \\ 0 \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ \lambda + \frac{\gamma}{N} & 1-\lambda \end{bmatrix}\begin{pmatrix} x_{t-1} \\ x_{t-1}^e \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{\gamma}{N} & 0 \end{bmatrix}\begin{pmatrix} x_{t-N-1} \\ x_{t-N-1}^e \end{pmatrix} \tag{66}$$

and represented in companion form

$$Az_t = c + Bz_{t-1}$$

where the augmented vector z_t is

$$z_t = \begin{pmatrix} x_t \\ x_t^e \\ x_{1t} \\ \vdots \\ x_{N,t} \end{pmatrix}$$

where, as usual, the auxiliary variables are equal to lagged values; i.e., $x_{it} = x_{t-i}$. The vector $c = (a_t, 0_{N+1})^T$.

Matrix A is

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tag{67}$$

with

$$A_1 = \begin{bmatrix} 1 & -\alpha & 0_N^T \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0_{N+1} & I_{N+1} \end{bmatrix}.$$

Matrix B can be decomposed in 4 blocks

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix} \tag{68}$$

where

$$B_1 = \begin{bmatrix} 0_{N+2} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \lambda + \frac{\gamma}{N}, 1 - \lambda, 0_{N-1}^T, -\frac{\gamma}{N} \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & 0_{N+1}^T \end{bmatrix}$$

$$B_4 = \begin{bmatrix} 0_{(N+2)\times 2}, I_{N-1}, 0_{N-1}. \end{bmatrix}$$

The invertibility of matrix A allows us to write the system in the form

$$z_t = A^{-1}c + Cz_{t-1}$$

where matrix C is computed by pre-multiplying the inverse of matrix A to matrix B, $C = A^{-1}B$. In the case $N = 4$ the matrix is

$$C = \begin{bmatrix} (\alpha(\gamma + 4\lambda))/4 & -\alpha(\lambda - 1) & 0 & 0 & 0 & -(\alpha\gamma)/4 \\ \gamma/4 + \lambda & 1 - \lambda & 0 & 0 & 0 & -\gamma/4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The grid search solves numerically the problem of the maximum eigenvalues, say $\theta(\alpha, \gamma, \lambda)$ of the matrix C

$$\max_{\alpha,\gamma,\lambda} |\theta(\alpha, \gamma, \lambda)|$$

The search step used for the 3 parameters is $\delta = 0.01$; i.e., $\alpha \in (-1+\delta, \ldots, 1-\delta)$, $\lambda \in (\delta, 2\delta, \ldots, 1)$,$\gamma \in (\delta, 2\delta, \ldots, 1)$. This two millions iterations produced mainly complex eigenvalues less then 1 in absolute value with maximum value of 0.9999.

This numerical result shows that the system remains stable, and the dominant complex eigenvalues imply an oscillatory convergence to the steady-state.

D Price adjustment and the unbiased solution(s)

Consider the representative agent case. The production function of the representative firm is

$$Q_t = A_t N_t^{\frac{1}{\delta}} \tag{69}$$

The firm knows (at the beginning of t) the costs she must incur $TC_t = P_t w_t N_t$ but is uncertain on the sale price $P_{s,t}$ which will be revealed only at the end of the period. Therefore the firm has to form expectations on the sale price. We will denote the rational expectation (based on the information set available at the beginning of t) of the sale price with $E(P_{s,t})$. In this setting, the firm chooses at the beginning of t the optimal quantity by maximizing the expectation of profits to be realized at the end of the period:

$$E(\Pi_t) = (1-\tau)E(P_{s,t})Q_t - P_t w_t {A_t}^{-\delta} Q_t^{\delta}$$

Dividing both sides by P_t we get:

$$\frac{E(\Pi_t)}{P_t} = (1-\tau)E(x_t)Q_t - w_t {A_t}^{-\delta} Q_t^{\delta}$$

where the state variable $x_t := \frac{P_{s,t}}{P_t}$ is the ratio of the end of period price to the beginning of period price (i.e., the intra-period gross inflation rate). The term $E(x_t) := \frac{E(P_{s,t})}{P_t}$ is the expected inflation rate based on the information set available in t (which does not include x_t). From the FOC we determine optimal output:

$$Q_t = \eta \zeta_t E(x_t)^{\frac{1}{\delta-1}} \tag{70}$$

where

$$\zeta_t := A_t^{\frac{\delta}{\delta-1}} {w_t}^{-\frac{1}{\delta-1}}$$

$$\eta := \left(\frac{1-\tau}{\delta}\right)^{\frac{1}{\delta-1}}$$

While η is a given and constant parameter, the determinants of ζ_t – i.e., the wage rate and TFP – are governed by stochastic processes spelled out in the text (see section 5). Both ζ_t and η are elements of the information set available at the beginning of t.

Therefore optimal employment is

$$N_t = \frac{1}{w_t}\eta^{\delta}\zeta_t E(x_t)^{\frac{\delta}{\delta-1}} \tag{71}$$

When the market opens, for each firm actual sales (S_t) may be different from output planned ad the beginning of period t and brought to the market at the end of the period (Q_t). The demand accruing to the firm, in fact, is $1/F$ of the aggregate wage bill $w_t F N_t$. Hence $S_t = \min(Q_t, w_t N_t)$. Excess demand will be $ED_t = F(w_t N_t - Q_t)$. We assume that the market price evolves according to the stochastic adjustment process (40) which we rewrite here for the reader's convenience:

$$x_t = \exp\left(\gamma_p ED_t\right)\exp(\varepsilon_P)$$

Using the equations for the optimal output and employment just derived, we can rewrite excess demand as follows:

$$ED_t = F\left[w_t\left(\frac{Q_t}{A_t}\right)^{\delta} - Q_t\right] = \zeta_t F\left[\eta^{\delta}E(x_t)^{\frac{\delta}{\delta-1}} - \eta E(x_t)^{\frac{1}{\delta-1}}\right] \tag{72}$$

Substituting (72) into (40) we obtain the ALM

$$x_t = \exp\left\{\gamma_p\zeta_t F\left[\eta^{\delta}E(x_t)^{\frac{\delta}{\delta-1}} - \eta E(x_t)^{\frac{1}{\delta-1}}\right]\right\}\exp(\varepsilon_P) \tag{73}$$

According to the deterministic skeleton of the ALM (i.e., the expression in curly braces), x_t is a *non linear* function of $E(x_t)$, for any given combination of A_t and w_t. As already pointed out in the text, non linearity is due exclusively to decreasing returns.

It is easy to see that market clearing occurs (and the price is stationary) when the expression in brackets is zero, i.e., when output is $Q_t^M = \zeta_t$ (and employment is $N_t^M = (\zeta_t/A_t)^{\delta}$. This particular level of output is produced when expected inflation is $E(x_t)^M = \eta^{1-\delta}$.

In this context, we can compute model-consistent (rational for short) expectations. Expectations are *model consistent* if agents know the "true model" (i.e., the ALM) of the economy. In order to determine model-consistent expectations we must take the expected value of (73). We get:

$$E(x_t) = \exp\left\{\gamma_p \zeta_t F \left[\eta^\delta E(x_t)^{\frac{\delta}{\delta-1}} - \eta E(x_t)^{\frac{1}{\delta-1}}\right]\right\} \tag{74}$$

The rational expectation $E(x_t)^U$ is the solution for $E(x_t)$ of (74), i.e., the fixed point of the map defined by the deterministic skeleton of the ALM. When expectations are formed rationally, by construction the forecasting error is $\varepsilon_t = E(x_t)[\exp(\varepsilon_P) - 1]$ so that $E(\varepsilon_t) = 0$. In words, rational expectations are unbiased, hence the superscript. Given the unbiased solution $E(x_t)^U$, it is immediate to compute output and employment, using (70) and (71) respectively. Hence total output and employment in the unbiased solution are $Y_t^U = FQ_t = F\eta\zeta_t(E(x_t)^U)^{\frac{1}{\delta-1}}$ and $L_t^U = FN_t = \frac{F}{w_t}\eta^\delta\zeta_t E(x_t)^{\frac{\delta}{\delta-1}}$.

Since the deterministic skeleton of the ALM is non linear, there could be multiple fixed points for any combination of parameter values. In the simulations (see table 1) we set $\delta = 3/2$. To illustrate the determination of rational solutions, let's consider the baseline parametrization $\tau = 0$ so that $A = 1$. Let's consider the long run value of the real wage $\bar{w} = 1$ so that we can pin down the numerical value of ζ_t which turns out to be unity. In this case $\eta \approx 0.44$ and $\eta^\delta \approx 0.3$. Moreover $\gamma_P = 0.001$ and $F = 200$. Therefore the ALM specializes to

$$x_t = \exp\left\{0.2\left[0.3E(x_t)^3 - 0.44E(x_t)^2\right]\right\}\exp(\varepsilon_P) \tag{75}$$

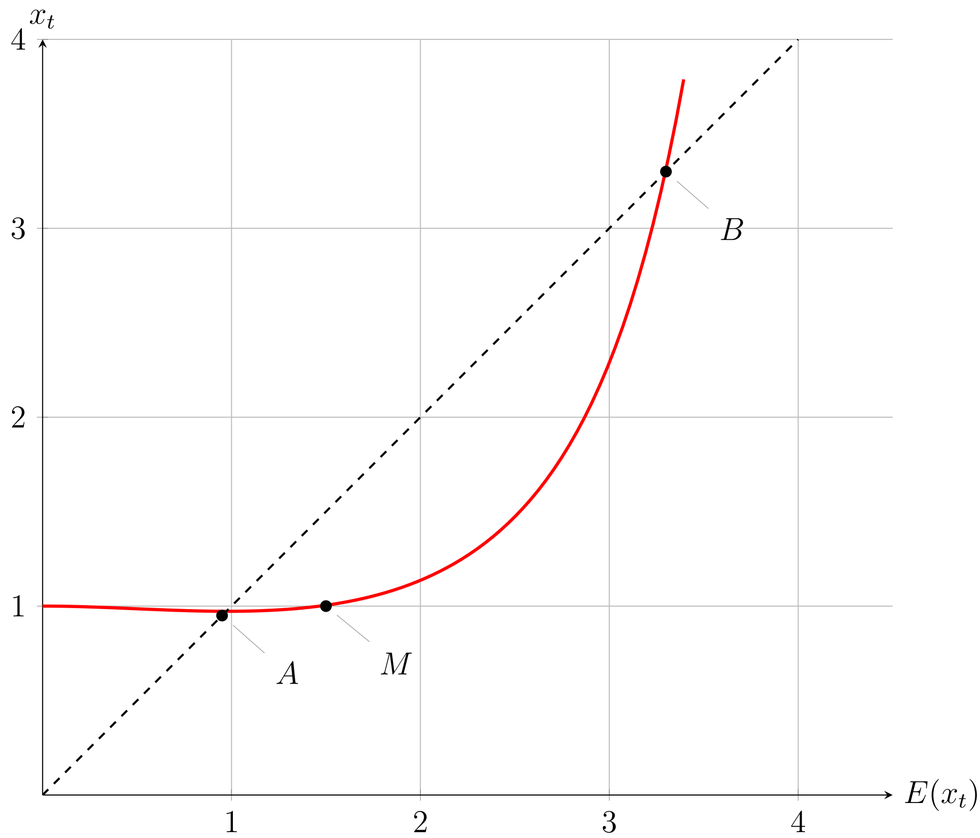
Market clearing occurs (and the price is stationary) when $Q_t^M = \zeta_t = 1$ (so that $N_t^M = 1$) and $E(x_t)^M = \delta = 1.5$, i.e., when agents expect the sale price to be 50% greater than the initial price. With these expectations, however, $x_t = 1$, i.e., the end-of-period price is the same as the initial price.[19] In other words the market clearing solution is not the unbiased solution: if

[19] To be precise, the expression in brackets is zero also when $E(x_t) = 0$, a root which we discard for obvious reasons.

expectations are such that market clearing occurs, then expectations are incorrect.

In figure 7 we represent the ALM with the baseline parametrization. Point $M = (1.5, 1)$ is the market clearing solution. All the points of the ALM between the intercept and M are characterized by $E(x_t) < 1.5$ and excess supply so that $x_t < 1$. Of course the opposite occurs for points of the ALM to the right of M. Point M does not lie on the 45 degree line, hence it is not an unbiased solution. The unbiased solutions are the fixed points, one of which characterized by excess supply and deflation (see point A) and the other by excess demand and inflation (B). With the chosen parametrixation the unbiased solutions are $E(x_t)_1^U = 0.972$ and $E(x_t)_2^U = 3.3$. Given the initial conditions, in our simulation the unbiased solution is the smaller one, which is less than but very close to 1. Given this unbiased solution, in the baseline with model consistent expectations output and employment per firm are $Q_1^U = 0.54$ and $N_1^U = 0.40$ so that total GDP and employment are $Y_1^U = 108$ and $N_1^U = 80$.

Figure 7: **Model consistent expectations and unbiased solutions**



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