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Fabrice Collard, Omar Licandro

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The Neoclassical Model and the Welfare Costs of Selection

Abstract

This paper embeds firm dynamics into the Neoclassical model and provides a simple framework to solve for the transitional dynamics of economies moving towards more selection. As in the Neoclassical model, markets are perfectly competitive, there is only one good and two production factors (capital and labor). At equilibrium, aggregate technology is Neoclassical, but the average quality of capital and the depreciation rate are both endogenous and positively related to selection. At steady state, output per capita and welfare both raise with selection. However, the selection process generates transitional welfare losses that may reduce in around 60% long term (consumption equivalent) welfare gains. The same property is shown to be true in a standard general equilibrium model with entry and fixed production costs.

JEL-Codes: E130, E230, D600, O400.

Keywords: firm dynamics and selection, neoclassical model, capital irreversibility, investment distortions, transitional dynamics, welfare gains.

Fabrice Collard
Toulouse School of Economics / France
fabrice.collard@gmail.com

*Omar Licandro**
University of Nottingham / United Kingdom
omar.licandro@nottingham.ac.uk

*corresponding author

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1 Introduction

Since the seminal contributions of Frank Ramsey [41] and Robert Solow [44], and their extensions by Cass[14] and Koopmans [32], the one sector Neoclassical growth model has become the cornerstone of modern macroeconomics. Its influence rests to a large extent on its ability to replicate the salient evidence on economic growth within a simple and stylized framework, spreading well beyond growth analysis by shaping the study of business cycles, optimal taxation, and economic policy, among many other fields in macroeconomics. The parsimony and the simplicity of the model makes it possible to study the dynamics of an economy in response to an exogenous perturbation —be it a structural shock or a policy reform.

The Neoclassical model builds on, among others, the assumption of a representative firm. While convenient, this assumption precludes —by construction— the analysis of firm dynamics and market behavior, at a time where the greater availability of micro-data has shifted attention towards these problems. Since the seminal contributions of Hopenhayn [24] and Jovanovic [30], a new, complementary, framework has emerged, explicitly designed to address these latter issues (See, *e.g.* Bartelsman and Doms [7] for a review of the empirical literature). The literature on firm dynamics builds upon the heterogenous behavior of firms and the very idea that selection plays a fundamental role in explaining market performance. Competition creates and destroys firms and jobs giving rise to a reallocation process whereby resources are shifted away from the less towards the more productive firms.

The aim of this paper is to offer a general framework for the study of the transitional dynamics of general equilibrium models with firm heterogeneity and endogenous selection. It is designed to easily solve for the response of the economy to exogenous perturbations affecting the allocation of resources across heterogeneous firms, facilitating the understanding of the macroeconomic implications of the selection mechanism at work in the firm dynamics literature. By doing so, we develop a modelling strategy that facilitates the quantitative evaluation of the transitional dynamics and the associated welfare cost of selection.

A key feature of the general framework suggested in this paper is that it restricts the study of the dynamics to equilibria where the economy becomes more selective over time. In fact, the key requirement is that all along the equilibrium path the productivity cutoff is non-decreasing. This type of solution corresponds to economies facing shocks or policies that promote selection. Restricting to this family of solutions has the double advantage that all along the transition there is no need to compute neither the equilibrium distribution of firms nor their value function, reducing the system to a standard rational expectations dynamic system defined in the aggregates. First, for the family of solutions carrying this property, we show that all along the equilibrium path the productivity distribution of incumbent firms is a linear combination of the initial and the entry distributions both truncated at the current productivity cutoff —each one weighted by its share in the total mass of firms. When the economy is initially at a stationary equilibrium, the equilibrium distribution is, all along the transition path, the truncated entry distribution. Second, we show that both the entry and exit conditions only depend on current profits. When the economy is expected to become more selective over time, marginal firms don't expect their profits to increase, making current profits a sufficient statistic for the marginal firm's participation (exit or remain) choices.

This paper uses the suggested framework for the study of perfectly competitive dynamic general equilibrium models with firm heterogeneity and selection. Because of its dual genesis, we dub the main model in this paper the Ramsey-Hopenhayn model (RH in the following). Like in the standard Ramsey's growth model, the dynamics of the economy are driven by optimal savings and capital accumulation, *i.e.*, an Euler type equation governing household savings and a feasibility set governing investment and capital accumulation. Like in the Hopenhayn model, firms are heterogenous and the allocation of resources is driven by a selection mechanism. The source of selection however slightly differs from the seminal Hopenhayn [24] paper. While in the Hopenhayn model, selection is the outcome of heterogeneity and the presence of a fixed cost of production, in the RH model selection originates from the interplay between firm heterogeneity and the partial irreversibility of capital.

In the RH model, as in the putty-clay literature,[1] each firm's technology makes use of a fixed —partially irreversible— heterogeneous production factor (capital) and a flexible homogeneous production factor (labor) to produce a single homogeneous final good, that —as usual— can be either consumed or invested. To make the argument simple, production is assumed to require one unit of capital. In such a setting, the price of the fixed input acts as an entry cost, which is assumed to be paid before the firm observes its quality. For the sake of tractability, as in Melitz [38], the idiosyncratic productivity or quality attached to a particular unit of capital is assumed to be time invariant, *i.e.*, firms face no other productivity shock than the initial one. When

[1] See the work of Johansen [27], Solow [45], Sheshinski [43] and Calvo [13], among many others.

capital's quality is too low —*i.e.* below some threshold that is endogenously determined— it is optimal for the firm to close down and send the capital unit to scrap. The mere existence of this opportunity cost attached to the partial reversibility of capital is at the core of the selection process making the least productive firms exit the market. Selection takes place without the need of any fixed production cost; fixed production costs being at odds with the Neoclassical model.

Section 3 first describes the behavior of households and firms and then states the key properties of the general framework. Propositions 1 and 2 characterize the productivity distribution, showing that all along the equilibrium path it is a linear combination of the initial and the entry distribution both truncated at the current cutoff productivity. Propositions 3 and 4 characterize the exit and entry conditions showing that current profits are a sufficient statistic of the value of the firm when taking market participation decisions. Section 4 studies the Ramsey-Hopenhayn model. Proposition 6 solves for the equilibrium cutoff and shows that shocks promoting selection make the cutoff productivity jump at impact to its new, higher stationary value. Propositions 2 and 6 extend to the full equilibrium path the solution in (and the solution strategy used by) Melitz [38] to solve for the steady state equilibrium. Under some general conditions, the equilibrium distribution is the truncated entry distribution and the cutoff productivity is constant and independent on the aggregates. Then, using standard aggregation theory, we show that the equilibrium aggregate technology is Neoclassical with the average quality of capital being positively related to selection. This aggregation result is in line, although different in nature, with aggregation in the vintage capital literature, as in Solow [45] and Solow et al [46], where different vintage technologies collapse into a Neoclassical production function.[2] We then fully characterize the transitional dynamics of the aggregates.[3] By means of a parametrized exercise, we proceed to illustrate the optimal response of the economy to *(i)* an increase in the degree of capital reversibility, which could be interpreted as a policy that increases the scrap value of

[2]Aggregation of vintages technologies has been extensively used in the more recent literature on embodied technical progress or investment specific technical change. See the seminal paper by Greenwood et al [23], as well as the endogenous growth extensions by Krusell [33] and Boucekkine et al [9] and [10]. Similar aggregation results are pointed out by Hopenhayn [25] when relating the literature on firm dynamics with the recent literature on misallocation (Restuccia and Rogerson [42], for example).

[3]In the limit, the RH model encompasses the Neoclassical model in the cases of homogeneous firms and fully reversible capital.

capital, and *(ii)* a policy that reduces investment price distortions. In particular, we show that shocks leading to a permanent increase in the the productivity cutoff triggers a selection effect that leads the economy to a higher steady state, but on impact partially destroys the initial stock of capital. In both cases, we evaluate the welfare costs and benefits of selection at steady state and during the transition. In our numerical exercise, long term welfare gains are large. A 5% increase in average productivity induces steady state (consumption equivalent) welfare gains between 1.22% and 2.7%. The transitional welfare costs reduce the steady state welfare gains by around 60%. In other words, while more selection makes agents better off in the long run, the short-run welfare losses due to capital destruction are also sizeable. A simplified version of the Hopenhayn model is studied in Section 5. Section 6 studies selection in the Jones and Manuelli [28] endogenous growth model. A last section concludes.

2 Related Literature

The literature on *capital irreversibility* dates back to the seminal paper by Arrow [5], who assumes that "the resale of capital goods is impossible." In Arrow [5], investment is a sunk cost. Abel and Eberly [1] adopt a more general framework allowing firms to resale capital at a price smaller that its replacement cost. Inspired in Bentolila and Bertola [8], they use option pricing to characterize the lower bound (and the upper bound) of the marginal product of capital inducing firms to disinvest (invest). In the RH model, firms are heterogenous in productivity, face no shocks, and their investment decisions reduce to a participation choice —produce or exit. In this framework, partial reversibility exogenously determines the selling price of used capital in the secondary market, and consequently the lower bound in the user cost of capital below which firms optimally exit. This condition is the equivalent to the exit condition in the Hopenhayn [24] model of firm dynamics, where a fixed production cost plays the same role as the lower bound in the user cost of capital in the RH model.[4]

Irreversibility creates a positive wedge between the user cost of capital and its rental rate as derived by Jorgenson [29]. This wedge corresponds to an irreversibility premium (see Chirinko

[4]In Hopenhayn [24] there is no hysteresis in the sense of Dixit [17], *i.e.*, "the failure of an effect of reverse itself as its underlying cause is reversed." The fundamental reason is that in Hopenhayn firms take entry decisions before observing the state of productivity, but in Dixit firms decide whether to invest after observing the state of demand.

and Schaller [16]) which appears because some new units of capital are not productive enough to be profitably used in the production process and are therefore optimally scrapped straightaway. Otherwise stated, the irreversibility premium relates to economic depreciation. Entry decisions in the RH model face a similar tradeoff, making the user cost of capital charge an irreversibility premium at equilibrium, which is decreasing in the degree of irreversibility and increasing in selection.

In models of firm dynamics, selection is costly since it requires some units of capital —the less productive ones— to be destroyed. Economic depreciation in models of partial irreversibility is in line with obsolescence in the vintage capital literature —see Malcomson [37], Caballero and Hammour [12], Boucekkine et al [11] and Gilchrist and Williams [20], among others. Selection entails a permanent process of creative destruction, in which low productive firms are substituted by more productive ones. Since the creative destruction process is efficient, at the stationary equilibrium the RH model shares most properties of the Neoclassical theory, but endogenous selection positively affects the value of capital, production and welfare. In the short-run, on the contrary, selection may require some destruction of the initial capital stock that entails transitional welfare losses partially compensating the long term gains from selection.

Veracierto [47] studies the business cycle properties of a heterogenous firm model with exogenous exit and entry, and partial capital reversibility. When triggered by standard aggregate productivity shocks, Veracierto [47] claims that "business cycles are found to be basically the same with fully flexible or completely irreversible investment." Adding endogenous selection to the picture does not change the Veracierto *irrelevance of irreversibility* result. In our framework, capital, output and consumption react to TFP shocks in similar way as in Veracierto [47] confirming the irrelevance of irreversibility to the understanding of TFP shocks. Different from Veracierto [47], we study here the reaction of the economy to permanent shocks that by promoting selection directly affect the creation and destruction process finding that they are relevant.

Using equipment-level data from aerospace plants transacted in the secondary market, Ramey and Shapiro [40] find that "even after age-related depreciation is taken into account, capital sells for a substantial discount relative to replacement cost; the more specialized the type of capital, the greater the discount." This finding gives strong support to the assumption in this paper that capital is partial irreversible. Gavazza [19] concludes that trading frictions in the secondary market for real assets shape capital irreversibility, affecting selection in a similar way as in the Hopenhayn [24] model. By looking at a large set of industries, Lanteri [34] find similar results confirming that the market for second-hand capital goods clearly shows that capital is partial irreversible. In the same direction, Gourio [22] points out that economic downturns are associated with large reallocation of capital, leading to the loss of firm specific specialized capital goods as well as intangible capital. More recently, Vinci and Licandro [48] argue that the observed switching-track of the American GDP that followed the Great Recession is a direct consequence of a boost in bankruptcies inducing a large destruction of production capacities in line with Gourio [22] and Lanteri [34] findings.

The scrapping of old equipment and machinery, and their substitution by modern ones is a fundamental pillar of policies addressed to promote the transition to clean technologies (see Acemoglu et al [2], among others). Vehicle scrapping schemes are a good example. They were introduced in the years 2000 in many countries to encourage the substitution of environmental-unfriendly cars, promoting the modernization of the stock of automobile (see Adda and Cooper [3], and Licandro and Sampayo [35], among others). These policies were quite effective in the short run, giving support to the assumption in this paper that policies addressed to increase the scrap value of capital generate an important destruction of the initial stock of capital.[5]

3 General Framework

The main objective of this paper is to develop a simple strategy for solving dynamic general equilibrium models with firm heterogeneity and selection. The suggested strategy does not require the computation of the equilibrium distribution, neither the value function. Solving for an equilibrium path then reduces to solving a standard dynamic system defined on the aggregates. In doing so, we restrict the analysis to transitional dynamics that monotonically converge towards an equilibrium with more selection. We apply this methodology to the resolution of what we call the Ramsey-Hopenhayn model, a simplified version of Hopenhayn [24] and an endogenous growth model with firm dynamics inspired in Jones and Manuelli [28].

This section describes a general framework that encompasses both the Ramsey-Hopenhayn

[5] Adda and Ottaviani [4] study the role of policies for the transition from analogue to digital television, which required a large destruction of existing equipment for the emission and reception of television signals.

and the Hopenhayn models, and proves some important properties of the equilibrium distribution and the entry and exit process. These properties will also apply to the endogenous growth model of Section 6.

3.1 Representative Household

The economy is populated by a unit measure of identical households, each offering inelastically one unit of labor at any time t. The representative household has preferences over a consumption stream represented by the discounted utility function

$$U = \int_0^\infty u(c_t)e^{-\rho t}\,\mathrm{d}t \tag{1}$$

where $c(t)$ denotes consumption at time t, and $\rho > 0$ is the constant subjective discount rate. The instantaneous utility function $u(\cdot)$ is characterized by a constant intertemporal elasticity of substitution $\sigma > 0$.

Even if firms face idiosyncratic risk, households fully diversify it by buying the market portfolio. Consequently, the optimal behavior of the representative household is given by the standard Euler equation

$$\frac{\dot{c}_t}{c_t} = \sigma(r_t - \rho), \tag{2}$$

where r_t is the riskless equilibrium interest rate.

3.2 Firms' behavior and labor market clearing

At any time t, a continuum of heterogeneous firms of endogenous mass n_t produces a single homogeneous final good —used as the numéraire— under perfect competition. For the sake of tractability, we follow Melitz [38] and assume that firms are characterized by a time invariant firm specific productivity z drawn at entry from the continuous density function $\psi(z)$, for $z \in \mathcal{Z} \subset \mathbb{R}^+$. Let us denote by $\underline{\zeta}$ and $\bar{\zeta}$ to the lower and upper bounds of the support $\mathcal{Z}$, with $0 \leq \underline{\zeta} < 1 < \bar{\zeta}$. Without loss of generality, expected productivity at entry is assumed to be one. The final good is produced by means of a fixed production factor —capital— and a flexible production factor —labor.

Entry into the market, and hence production, entails acquiring one unit of capital. Consequently, the mass of operative firms n_t is equal to the stock of capital measured in physical units. Acquiring one unit of capital requires $\eta \geq 1$ units of the consumption good. The larger η, the less efficient the economy is in producing investment goods. Accordingly, we will refer to η as an investment distortion, with $\eta = 1$ corresponding to the undistorted economy. Moreover, capital is assumed to be partially irreversible, in the sense that, when a firm optimally decides to close down, it only recovers a fraction $\theta \in (0,1)$ of the consumption good embodied in this unit of capital. We will refer to θ as the scrap value of capital, which measures the degree of capital irreversibility.

A firm with productivity z uses technology

$$y(z) = F\big(z, \ell(z)\big), \tag{3}$$

where the firm specific production function $F(z,\ell)$ is $\mathcal{C}^2$, increasing in both arguments, concave and homogeneous of degree one. Variables $y(z)$ and $\ell(z)$ denote, respectively, output and employment of a firm with productivity z.[6] We interpret productivity z as being embodied in capital, with different capital units having different qualities.

At any time t, an operative firm with productivity z decides its employment level by maximizing net revenues

$$\pi_t(z) = \max_{\ell_t(z)} F\big(z, \ell_t(z)\big) - w_t \ell_t(z), \tag{4}$$

taking the wage rate, w_t, as given. From the first-order-condition for labor, the optimal labor demand reads

$$F_2\big(z, \ell_t(z)\big) = w_t \iff \ell_t(z) = G^{-1}(w_t) z, \tag{5}$$

where, by the homogeneity of degree 1 assumption, $G\left(\frac{\ell}{z}\right) \equiv F_2\left(1, \frac{\ell}{z}\right) = F_2(z,\ell)$.

The labor market clearing condition requires

$$n_t \int_{z_t^*}^{\bar{\zeta}} \ell_t(z) \phi_t(z) \mathrm{d}z = 1,$$

where z_t^* is the productivity of the marginal firm, *i.e.* the least productive firm in operation, and $\phi_t(z)$ is the productivity density function at equilibrium. After substitution of individual labor demands $\ell_t(z)$ into the labor market clearing condition above, the equilibrium wage rate reads

$$w_t = F_2(k_t, 1), \tag{6}$$

where $k_t = \bar{z}_t n_t$ measures capital per capita in quality adjusted units.[7] Since a firm requires

[6] Technology in this paper is in line with the span of control assumption in Lucas [36].

[7] This way of measuring capital is consistent with national accounts, where after a long debate following Gordon [21]'s seminal work, investment is deflated using constant quality price indexes.

one physical unit of capital to produce, and idiosyncratic productivity z measures the quality of firm's capital, the average quality of physical capital at equilibrium is given by

$$\bar{z}_t = \int_{z_t^*}^{\bar{\zeta}} z\phi_t(z)\mathrm{d}z.$$

Under the assumption that $F_{21}(\cdot) > 0$, the wage rate is positively related to capital per capita k which crucially depends on the average quality of capital $\bar{z}_t$. In other words, selection raises wages by increasing the average quality of capital.

Substituting the equilibrium wage rate into the labor demand function (5), for all active firms, equilibrium employment becomes

$$\ell_t(z) = \frac{z}{k_t} = \frac{1}{n_t} \times \frac{z}{\bar{z}_t}.$$

The term $1/n_t$ represents average labor per firm. The second term, $z/\bar{z}_t$ accounts for the fact that labor is distributed across firms according to their relative productivity. Equalization of marginal product of labor across firms implies that more productive firms hire more labor. The reallocation of input across firms operates here through the intensive margin channel, i.e., high productive firms employ more workers than low productive firms. Substituting the equilibrium labor demand above in the production technology (3) and using the definition of capital per capita, we get

$$y_t(z) = F(z, \ell_t(z)) = z\frac{f(k_t)}{k_t} = \frac{z}{\bar{z}_t} \times \frac{f(k_t)}{n_t}. \tag{7}$$

At equilibrium, $F(z, \ell_t(z)) = F(k_t, 1)z/k_t$, since $\ell_t(z) = z/k_t$ and $F(.)$ is homogeneous of degree one. The average production per firm, $f(k)/n$, is distributed across firms depending on the relative productivity $z/\bar{z}$. More productive firms employ more labor and produce more up to the point where the marginal product of labor equalizes across firms.

3.3 An Informative Guess

This paper focuses on the analysis of equilibria where the cutoff productivity z_t^* is monotonically increasing, converging to its steady state value. This type of equilibrium usually occurs when a policy is implemented to make the economy more selective. In this context and in order to solve for the equilibrium path of the Ramsey-Hopenhayn economy in Section 4 and the endogenous growth economy in Section 6, we rely on a constructive proof (guess and verify) developed in four stages. In a first stage, we guess (see Guess 1 below) that the path for the equilibrium cutoff is non-decreasing over time. In a second stage, Proposition 2 uses Guess 1 to identify a set of sufficient conditions under which the shape of the equilibrium distribution is independent from the path of the aggregates. In a third stage, Proposition 6 imposes the conditions identified in Proposition 2 to show that the path of the equilibrium cutoff is indeed non-decreasing and independent of the aggregates, hereby verifying Guess 1. More precisely, we prove that under the conditions imposed in Proposition 2 and for the family of solutions consistent with Guess 1, a solution path for the cutoff productivity z_t^* exists and is unique. Finally, under the assumption that technology is Cobb-Douglas and the entry distribution is Pareto, for the equilibrium distribution in Proposition 2 and the path of the cutoff productivity in Proposition 6, Section 4.3 uses a shooting algorithm to solve for the aggregates in the case of the Ramsey-Hopenhayn model. Similar steps are followed in Section 6 to numerically solve for the equilibrium path of the endogenous growth economy.

A similar constructive argument is used to solve for the equilibrium of the Hopenhayn economy in Section 5. We repeat the first two stages above. Then, under the assumption that technology is Cobb-Douglas and the entry distribution is Pareto, we impose the conditions in Proposition 2 to show in Proposition 7 that the equilibrium can be characterized as a path for consumption and capital. We solve for it using a shooting algorithm and find that it monotonically converges to steady state. Since we are looking for solutions with increasing selection, the paths for capital and consumption converge to the stationary solution from below. Then, we use the exit condition to solve for the equilibrium path of the cutoff productivity, which we found to be monotonically increasing, verifying Guess 1.

Guess 1 *The equilibrium path z_t^*, for $t \geq 0$, monotonically converges from some initial $z_\iota^* \geq \underline{\zeta}$ to the stationary equilibrium z^*, $z_\iota^* < z^*$.*

To summarize, the constructive argument developed in this paper guesses that the solution path of the cutoff productivity is non-decreasing —Guess 1— to identify conditions under which an equilibrium verifying this property exists for all the economies under analysis.

3.4 Productivity Distribution

This section shows some properties of the equilibrium distribution that will prove useful when studying the dynamics of all models in this paper. One of the main technical problems faced when dealing with the dynamics of models of heterogeneous firms is that the path of the equilibrium distribution generally depends on the path of the aggregates, while the latter also depends on the distribution.

In a continuous time model where firms draw their (time invariant) productivity at entry from a known continuous density function, Proposition 1 below shows that, under very general conditions, at any time t, the equilibrium density is a linear combination of the entry density and the initial density —both truncated at the current cutoff productivity. The weights of this linear combination depend on the share of surviving initial firms on the total mass of firms, which goes to zero as time goes to infinity making the equilibrium distribution converge to the entry distribution truncated at the stationary cutoff. The main condition required to prove this result is that Guess 1 holds at equilibrium, which typically occurs when the economy is moving towards a more selective steady state. The study of the dynamics of the Ramsey-Hopenhayn, the Hopenhayn and the endogenous growth models in Sections 4, 5 and 6, respectively, will make use of this property.

If, on top of that, the economy is initially at steady state, the initial distribution is the truncated entry distribution. As shown in Proposition 2 below, a shock that monotonically shifts the equilibrium cutoff to the right, makes, at any time, the equilibrium distribution be equal to the entry distribution truncated at the current cutoff productivity.

Let us develop this argument. Firms exit the market for two different reasons. First, they exit at the exogenous exit rate $\delta > 0$, which can be interpreted as the rate of physical depreciation. In which case, their capital cannot be recycled. Second, a firm may endogenously decide to exit if its productivity is smaller than some cutoff productivity $z_t^* \in \mathcal{Z}$. For all the models studied in this paper, when restricting the analysis to the set of initial conditions making the equilibrium path for z_t^* to be non-decreasing, we show in the following sections that, for any time $t \geq 0$, there exists a unique productivity cutoff $z_t^* \in \mathcal{Z}$ such that the distribution of firms at equilibrium has support $[z_t^*, \bar{\zeta})$. This typically corresponds to policies addressed to increase selection.

We follow three steps to derive the law of motion of the equilibrium productivity distribution $\phi_t(z)$. First, let us differentiate the condition

$$\Phi_t(\bar{\zeta}) = \int_{z_t^*}^{\bar{\zeta}} \phi_t(z)\mathrm{d}z = 1,$$

with respect to t to obtain

$$\int_{z_t^*}^{\bar{\zeta}} \dot{\phi}_t(z)\mathrm{d}z = \phi_t(z_t^*)\dot{z}_t^*,$$

where $\Phi_t(z)$ denotes the cumulative distribution associated to the equilibrium density $\phi_t(z)$. For $\dot{z}_t^* > 0$, the support of the equilibrium distribution reduces, which requires an increase in the density (on average) in the new (smaller) support to compensate for the fact that firms with productivity smaller than the new cutoff are exiting.

Second, for $z \in [z_t^*, \bar{\zeta})$, the equilibrium mass of firms with productivity z is given by $n_t\phi_t(z)$, where n_t is the mass of firms at t, evolving as

$$\dot{n}_t\phi_t(z) + n_t\dot{\phi}_t(z) = -\delta n_t\phi_t(z) + e_t\psi(z), \tag{8}$$

where e_t is the mass of entrants. At any time t, incumbent firms exogenously exit at rate δ, and new entrants e_t draw productivity z at rate $\psi(z)$. Integrating (8) with respect to z over the interval $[z_t^*, \bar{\zeta})$, we obtain the law of motion of the aggregate mass of firms

$$\dot{n}_t = (\lambda_t - \delta_t)n_t \tag{9}$$

where

$$\lambda_t = \frac{e_t(1-\Psi(z_t^*))}{n_t} \quad \text{and} \quad \delta_t = \delta + \phi_t(z_t^*)\dot{z}_t^*. \tag{10}$$

Surviving new entrants $\lambda_t n_t$ cumulate into the aggregate mass of firms, at the time that some firms exit because of physical depreciation δ and obsolescence as measured by $\phi_t(z_t^*)\dot{z}_t^*$. Solving the ODE (9) for some initial mass $n_0 > 0$, the mass of firms is then given by

$$n_t = n_0\Lambda_t, \tag{11}$$

where the net growth factor of the population of firms is

$$\Lambda_t = \mathrm{e}^{\int_0^t(\lambda_s - \delta_s)\mathrm{d}s},$$

measuring cumulative entry net of physical and economic depreciation.

Finally, substituting (9) into (8) for any $z \in [z_t^*, \bar{\zeta})$, the equilibrium distribution is shown to evolve following

$$\dot{\phi}_t(z) = \frac{e_t}{n_t}\psi(z) - \big(\lambda_t - \delta_t + \delta\big)\phi_t(z). \tag{12}$$

Firms exogenously exit at rate δ, and because of the reduction in the mass of firms the distribution truncates at the rate $\delta_t - \lambda_t$. The velocity with which the past distribution is substituted by the entry distribution depends on the entry rate e_t/n_t. Since the differential equation (12) is valid only for $z \geq z_t^*$, otherwise $\phi_t(z) = 0$, the path for z_t^* critically determines the equilibrium distribution $\phi_t(z)$.

The next two propositions characterize the equilibrium distribution when equilibrium is restricted to solutions verifying Guess 1, *i.e.*, those where the economy becomes more selective over time (z_t^* is non-decreasing). Proposition 1 studies the general case of an arbitrary initial distribution $\phi_\iota(z)$ while Proposition 2 specializes to the case of an economy initially at a stationary equilibrium.

Proposition 1 *Under Guess 1, if the initial distribution $\phi_\iota(z)$ is continuous in the support $z \in [z_\iota^*, \bar{\zeta})$, $z_\iota^* \in (\underline{\zeta}, z_0^*)$, then at any $t \geq 0$ the equilibrium distribution is*

$$\phi_t(z) = \omega_t \, \frac{\phi_\iota(z)}{1 - \Phi_\iota(z_t^*)} + (1 - \omega_t) \, \frac{\psi(z)}{1 - \Psi(z_t^*)}, \tag{13}$$

where

$$\omega_t = \frac{n_\iota \big(1 - \Phi_\iota(z_t^*)\big) \, e^{-\delta t}}{n_t}$$

is the ratio of surviving initial firms to total firms.

Proof: If z_t^* is non-decreasing, with $z_0^* \geq z_\iota^*$, the solution to the ODE (12), for $z \in [z_t^*, \bar{\zeta})$, is

$$\phi_t(z) = \phi_\iota(z) \, \Gamma_t + \psi(z) \underbrace{\Gamma_t \int_0^t \frac{e_h}{n_h} \, \Gamma_h^{-1} \mathrm{d}h}_{\equiv \mathcal{I}_t},$$

where

$$\Gamma_t = \mathrm{e}^{-\int_0^t \left(\lambda_s - \delta_s + \delta\right) \mathrm{d}s} = \Lambda_t^{-1} \mathrm{e}^{-\delta t}.$$

Integrating $\phi_t(z)$ with respect to z, for $z \in [z_t^*, \bar{\zeta})$, and solving for the auxiliary variable $\mathcal{I}_t$, we get

$$\mathcal{I}_t = \frac{1 - \Gamma_t \big(1 - \Phi_\iota(z_t^*)\big)}{1 - \Psi(z_t^*)}.$$

Substituting it back into the solution for $\phi_t(z)$, then using (11) to substitute for Γ_t, we obtain (13), which completes the proof. ■

Proposition 1 establishes that, at any time t, the equilibrium density distribution of productivity across operative firms is given by the weighted average of the initial density $\phi_\iota(z)$ and the density at entry $\psi(z)$, both truncated at the cutoff productivity z_t^*. The weights are given by the shares of surviving initial and new firms in the total mass of firms, *i.e.* those that enter before and after $t = 0$, respectively.

Proposition 2 *Under Guess 1, if the initial distribution is $\phi_\iota(z) = \frac{\psi(z)}{1 - \Psi(z_\iota^*)}$ in the support $z \in [z_\iota^*, \bar{\zeta})$, $z_\iota^* \in (\underline{\zeta}, z_0^*)$, then for any $t \geq 0$ the equilibrium distribution is*

$$\phi_t(z) = \frac{\psi(z)}{1 - \Psi(z_t^*)}. \tag{14}$$

Proof: Under Guess 1, z_t^* is monotonically increasing, with $z_0^* \geq z_\iota^*$. Therefore, we have

$$1 - \Phi_\iota(z_t^*) = \int_{z_t^*}^{\bar{\zeta}} \phi_\iota(z)\mathrm{d}(z) = \int_{z_t^*}^{\bar{\zeta}} \frac{\psi(z)}{1 - \Psi(z_\iota^*)}\mathrm{d}(z) = \frac{1 - \Psi(z_t^*)}{1 - \Psi(z_\iota^*)},$$

which implies

$$\frac{\phi_\iota(z)}{1 - \Phi_\iota(z_t^*)} = \frac{\psi(z)}{1 - \Psi(z_t^*)}.$$

Substituting this result into (13) we get (14), which completes the proof. ■

Proposition 2 states that if the initial equilibrium distribution corresponds to the truncated entry distribution, and the cutoff value z_t^* is in the support of the initial distribution and shifts monotonically to the right, then the equilibrium distribution, at any time t, is the entry distribution $\psi(z)$ truncated at the current cutoff productivity z_t^*. The rationale behind Proposition 2 is simple. Shall firms be initially distributed with the same profile as the entry distribution, then the productivity of the new comers is drawn from the same distribution, pilling up over a distribution of incumbents with exactly the same profile.[8] Hence, the shape of the equilibrium distribution is invariant, but the lower bound of the support may shift to the right changing the truncation point. As we will show in the following sections, Proposition 2 holds for economies that are initially in steady state and face an unexpected shock making them more selective.

Average Productivity. Under the conditions of Proposition 2, the average productivity level in the economy is given by

$$\bar{z}_t = \frac{1}{1 - \Psi(z_t^*)} \int_{z_t^*}^{\bar{\zeta}} z\psi(z)\mathrm{d}z, \tag{15}$$

which only depends on z_t^*. This property will be used in the following sections.

Pareto Distribution. In the sequel, in accordance with the empirical evidence, the entry distribution will be assumed to be Pareto with tail parameter κ, $\kappa > 1$, and unbounded support $\bar{\zeta} = \infty$. The assumption that the expected productivity at entry is one, implies that $\underline{\zeta} = \frac{\kappa-1}{\kappa}$. Under the conditions of Proposition 2, it is easy to show that the equilibrium distribution is Pareto with tail parameter κ, lower bound $z_t^* > 1$ and average productivity $\bar{z}_t = \frac{\kappa}{\kappa-1} z_t^* > 1$.

[8]Note that this property does not hold if z_t^* decreases over time, since the support of the time t distribution will include values of z that were out of the support of the past distributions.

3.5 Selection and free entry

Profits and firm's value. Since $F(z, \ell_t)$ is homogeneous of degree one, it is easy to show that net revenues $\pi(z)$ of operative firms, as defined in (4), are linear in z at equilibrium, *i.e.*,

$$\pi_t(z) = zF_1(k_t, 1) = zf'(k_t). \tag{16}$$

This can be shown by substituting the first order condition for labor (5) and equilibrium wages (6) in the profit function (4). Since net revenues correspond to the return to capital, they are equal to the marginal product of aggregate capital per worker, $f'(k_t)$, weighted by the productivity of the firm, z, measuring the quality of firm's capital.

At any time t, let us denote by $\tau(z)$, $\tau(z) \geq 0$, the time at which an incumbent firm with productivity z, $z \geq z_t^*$, will endogenously exit. If $z \geq z_t^*$ for all $t \geq 0$, $\tau(z) = \infty$. Otherwise, it is implicitly defined by the condition $z = z_\tau^*$, which is time invariant. Consequently, under Guess 1, the expected value at time t of a firm with productivity z, for $z \geq z_t^*$, is given by

$$v_t(z) = \int_t^{\tau(z)} \Big(zf'(k_s) - \mu\Big)\, \mathrm{e}^{-\int_t^s (r_h+\delta)\mathrm{d}h}\, \mathrm{d}s \geq \theta, \tag{17}$$

where $\mu \geq 0$ is the fixed cost of production. Profits are discounted by the discount factor $\mathrm{e}^{-\int_t^s r_h \mathrm{d}h}$, where r_t is the equilibrium interest rate, and multiplied by the survival probability $\mathrm{e}^{\delta(t-s)}$, where $\delta > 0$ is the exogenous exit rate.

Exit condition. Firms with $z \leq z_t^*$ exit and recover the scrap value θ, implying $v_t(z_t^*) = \theta$. The following proposition characterizes the exit condition.

Proposition 3 *Under Guess 1, the exit condition is*

$$z_t^* f'(k_t) - \mu = (r_t + \delta)\theta. \tag{EC}$$

Proof: Differentiating (17) with respect to time, for $t < \tau(z)$, we get

$$\dot{v}_t(z) = (r_t + \delta)v_t(z) - \big(zf'(k_t) - \mu\big). \tag{18}$$

For the marginal firm, $v(z_t^*) = \theta$. Since z_t^* is monotonically increasing, $\dot{v}(z_t^*) = 0$. Consequently

$$z_t^* f'(k_t) - \mu = (r_t + \delta)\theta,$$

which completes the proof. ■

The (EC) condition states that firms exit when current profits $zf'(k_t) - \mu$ are not large enough to cover for the opportunity cost of the unit of capital owned by the firm, *i.e.*, the

Jorgenson [29] user cost of capital $(r_t + \delta)\theta$. In which case, the firm closes down and recovers θ. Since we are restricting the analysis to solutions where z_t^* cannot decrease, the marginal firm has no prospect of making any capital gain, implying that the condition $v_t(z_t^*) = \theta$ collapses to (EC).

Under partial reversibility, the degree of capital reversibility as measured by θ, affects the incentives to exit by raising the user cost of capital, becoming a key determinant of the cutoff productivity z^*. A larger degree of capital reversibility promotes exit by allowing firms to extract a higher value from their scrapped capital. Equation (EC) is of particular relevance for the arguments developed in Section 4 in the case of the Ramsey-Hopenhayn model, in which we will assume μ equal to zero. In that case, condition (EC) reads

$$r_t + \delta = \frac{z_t^*}{\theta} f'(k_t).$$

The return to capital when measured at the extensive margin is equal to the marginal product of aggregate capital $f'(k)$ weighted by the ratio of the marginal quality of capital, z^*, to its scrap value θ.

Free entry condition. With probability $\Psi(z_t^*)$, a newly created firm draws a productivity smaller than z_t^*. In this case, not being able to generate enough profits to cover for the opportunity cost of capital, the firm exits and recovers θ by sending its capital to scrap. With probability $1 - \Psi(z_t^*)$ a newly created firm draws a productivity larger than z_t^* getting as return the expected value of an operative firm, denoted by $\mathbb{E}_{z \geq z_t^*}(v_t(z))$. A firm is therefore indifferent between entering and staying idle when the expected return equals the entry cost η. Consequently, the free entry condition reads

$$\Psi(z_t^*)\theta + (1 - \Psi(z_t^*)) \underbrace{\int_{z_t^*}^{\bar{\zeta}} v_t(z)\phi_t(z)\mathrm{d}z}_{\mathbb{E}_{z \geq z_t^*}(v_t(z))} = \eta. \tag{19}$$

As can be seen from equation (19), both a reduction in investment distortions, as measured by η, and an increase in the degree of capital reversibility θ promote entry, the effect of θ being shaded by the probability of failure $\Psi(z^*)$. The following proposition uses Proposition 2 to simplify the expected value of firms in this framework, unraveling the free entry condition (FE).

Proposition 4 *If the initial distribution $\phi_\iota(z) = \frac{\psi(z)}{1-\Psi(z_\iota^*)}$, for $z \in [z_\iota^*, \bar{\zeta})$, $z_\iota^* \geq \underline{\zeta}$, under Guess*

1, the free entry condition reads

$$\Psi(z_t^*)\theta + \big(1-\Psi(z_t^*)\big)\ \frac{\bar{z}_t f'(k_t)-\mu}{r_t+\delta} = \eta. \tag{FE}$$

Proof: From Proposition 2, equation (FE) becomes

$$\int_{z_t^*}^{\bar{\zeta}} v_t(z)\psi(z)\mathrm{d}z = \eta - \theta\Psi(z_t^*).$$

Differentiating it with respect to time

$$-v_t(z_t^*)\psi(z_t^*)\dot{z}_t^* + \int_{z_t^*}^{\bar{\zeta}} \dot{v}_t(z)\psi(z)\mathrm{d}z = -\theta\psi(z_t^*)\dot{z}_t^*.$$

and using the fact that $v_t(z_t^*) = \theta$ and (18), we get

$$\int_{z_t^*}^{\bar{\zeta}} \dot{v}_t(z)\psi(z)\mathrm{d}z = \int_{z_t^*}^{\bar{\zeta}} \Big((r_t+\delta)v_t(z) - \big(zf'(k_t)-\mu\big)\Big)\psi(z)\mathrm{d}z = 0,$$

or equivalently

$$(r_t+\delta)\underbrace{\int_{z_t^*}^{\bar{\zeta}} v_t(z)\psi(z)\mathrm{d}z}_{\eta-\theta\Psi(z_t^*)} - (1-\Psi(z_t^*))(\bar{z}_t f'(k_t)-\mu) = 0,$$

which completes the proof. ■

From the exit condition, the value of being marginally successful $v(z^*)$ is equal to the opportunity cost θ of being unsuccessful. Consequently, over time, the expected change in the value of successful firms $\int_{z_t^*}^{\bar{\zeta}} \dot{v}_t(z)\psi(z)\mathrm{d}z$ is zero, implying that the expected value of being a successful firm at time t, $\mathbb{E}_{z\geq z_t^*}\big(v_t(z)\big)$, is equal to the flow of profits of a firm making forever the current average profits, $\bar{z}_t f'(k_t)-\mu$, discounted at the current interest rate r_t. The user cost of capital becomes

$$(r_t+\delta)\frac{\eta-\theta\Psi(z^*)}{1-\Psi(z^*)}.$$

Notice that the cost of creating a successful firm is $\eta-\theta\Psi(z^*)$, and $1-\Psi(z^*)$ is the probability of being successful. In economies without investment distortions, the term multiplying $r+\delta$ is larger than one, measuring the irreversibility premium suggested by Chirinko and Schaller [16].

Investment returns. The following proposition characterizes the returns to investment r_t in equilibrium.

Proposition 5 *If the initial distribution* $\phi_\iota(z) = \frac{\psi(z)}{1-\Psi(z_\iota^*)}$, *for* $z \in [z_\iota^*, \bar{\zeta})$, $z_\iota^* \geq \underline{\zeta}$, *under Guess 1, and* $z_0^* \geq z_\iota^*$ *the equilibrium interest rate solves*

$$\eta - \theta = \Big(1-\Psi(z_t^*)\Big)\frac{f'(k_t)(\bar{z}_t - z_t^*)}{r_t+\delta}. \tag{R}$$

Proof: Substitute the (EC) condition into the (FE) condition to get

$$(r_t + \delta)(\eta - \theta) = (1 - \Psi(z_t^*))(\bar{z}_t - z_t^*)f'(k_t),$$

which completes the proof. ■

Equation (R) epitomizes the fundamental tradeoff faced by the marginal incumbent firm —the one with productivity z_t^*. By paying the entry cost η and recovering the scrap value θ, the marginal firm can exit and enter again. It will succeed in creating a profitable firm with probability $1 - \Psi(z^*)$. In this case, it expects to get an additional return from moving from z^* to $\bar{z}$. The equilibrium condition (R), which combines the free entry condition (FE) and the exit condition (EC), can be interpreted as an arbitrage condition defining the return to investment, encompassing firm's creation and destruction. What really matters for selection is the ratio θ/η, as we clearly see in Proposition 6.

Summary. One of the main technical problems faced when dealing with the dynamics of heterogeneous firms models is that the equilibrium path critically depends on the value of firms, and computing value functions requires information on the full equilibrium path. This paper shows that by restricting the analysis to equilibria moving towards more selection, value functions are not needed to compute the equilibrium path, drastically reducing the dimensionality of the problem. The fundamental reason is the following. When the economy moves towards more selection, the marginal firm knows that its profits cannot improve: when making zero profits, future profits will never become positive again. Propositions 3 and 4 formalise this statement by proving that for this family of solutions the conditions determining entry and exit only depend on current profits.

4 Ramsey-Hopenhayn Model

This section analyzes the dynamic properties of the Ramsey-Hopenhayn model. Throughout, we will assume strictly positive scrap values and zero fixed production costs, *i.e.*, $\theta > 0$ and $\mu = 0$.

4.1 Equilibrium cutoff

When the initial distribution is the truncated entry distribution, truncated at a small enough initial cutoff, Proposition 6 below shows that a time invariant equilibrium cutoff $z_t^* = z^*$, $\forall t \geq 0$, verifying Guess 1 exists and is unique. A direct implication of Proposition 6 is that, following a permanent shock, the cutoff productivity jumps up to the new, higher steady state at the time of the shock, and then remains constant forever. This situation typically arises when the economy is initially at steady state and an unexpected shock induces more selection. These are the type of condition that we will impose in Section 4.3 for the study of the transitional dynamics.

Proposition 6 *If $\mu = 0$, $\hat{\theta} = \theta/\eta > \underline{\zeta}$, and $\phi_\iota(z) = \frac{\psi(z)}{1-\Psi(z_\iota^*)}$ for $z \in [z_\iota^*, \bar{\zeta})$, $z_\iota^* \in (\underline{\zeta}, z^*)$, there exists a constant equilibrium cutoff $z_t^* = z^*$, $\forall t \geq 0$, that solves*

$$\frac{z^*}{\hat{\theta}} = \int_{z^*}^{\bar{\zeta}} z\psi(z)dz + z^*\Psi(z^*). \tag{20}$$

Proof: Under Guess 1 and $\mu = 0$, equation (FE) in Proposition 4 becomes

$$(r_t + \delta)\big(\eta - \theta\Psi(z_t^*)\big) = \big(1 - \Psi(z_t^*)\big)f'(k_t)\bar{z}_t.$$

Use the (EC) condition to substitute $Af'(k_t)$ and multiply both sides by z_t^*/η to get

$$z_t^*\big(1 - \hat{\theta}\Psi(z_t^*)\big) = \big(1 - \Psi(z_t^*)\big)\hat{\theta}\bar{z}_t, \tag{21}$$

where $\hat{\theta} = \theta/\eta$. Add the term $\hat{\theta}z_t^*\Psi(z_t^*)$ to both sides, divide both sides by $\hat{\theta}$, and use the definition of $\bar{z}_t$ to rewrite the equilibrium condition as

$$\frac{z_t^*}{\hat{\theta}} = \int_{z_t^*}^{\bar{\zeta}} z\psi(z)\mathrm{d}z + z_t^*\Psi(z_t^*) \equiv \mathcal{A}(z_t^*). \tag{22}$$

The left-hand-side of (22) is linear on z_t^*, crosses the origin and has slope $1/\hat{\theta} > 1$. Since the entry distribution has unit mean, $\underline{\zeta} < 1$ and $\mathcal{A}(\underline{\zeta}) = 1$. Moreover, the $\lim_{z^* \to \bar{\zeta}} \mathcal{A}(z^*)/z^* = 1$, implying that $\mathcal{A}(z^*)$ converges to the diagonal as z^* goes to $\bar{\zeta}$. It is easy to see that $\mathcal{A}(z)$ crosses $z/\hat{\theta}$ at least once in the interior of $\mathcal{Z}$, which proves existence.
The first derivative of $\mathcal{A}(z)$ is

$$\mathcal{A}'(z) = \Psi(z) \in (0, 1).$$

Since the slope of $\mathcal{A}(z)$ is smaller than the slope of $z/\hat{\theta}$ for all $z \in \mathcal{Z}$, $\mathcal{A}(z)$ can only crosses $z/\hat{\theta}$ once, showing that a unique solution verifying Guess 1 exists, which completes the proof. ∎

At equilibrium, z^* depends only on the parameters of the entry distribution $\psi(z)$, and the ratio of the degree of capital reversibility to the investment distortion, *i.e.* $\hat{\theta} = \theta/\eta$. Figure 1 provides a graphical representation of the existence and uniqueness of z^*.[9] Since $\mathcal{A}(\underline{\zeta}) = 1$, it is easy to see that at equilibrium $z^* > \hat{\theta}$. More importantly, as shown in Corollary 1 below,

[9]It is interesting to see that, differently from Melitz [38], no restriction is imposed on the entry distribution, a part from continuity, to prove existence and unicity of the cutoff productivity.

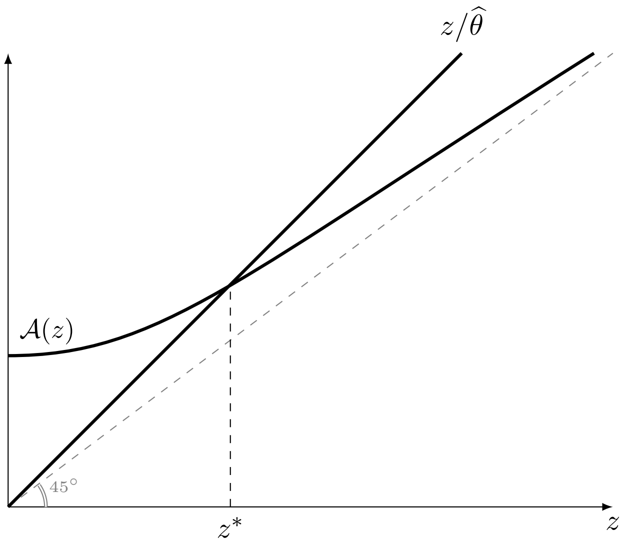


Figure 1: Determination of the cutoff productivity z^*

under the conditions of Proposition 6, when $\hat{\theta}$ increases z^* grows proportionally more than $\hat{\theta}$. Economies with a higher degree of capital reversibility θ or lower investment distortions η, on top of being more efficient, are more selective, the effects of θ and η on z^* being the mirror image of each other.

Corollary 1 *Under the conditions of Proposition 6,*

$$\frac{dz^*}{d\hat{\theta}}\frac{\hat{\theta}}{z^*} > 1.$$

Proof: Take logs and totally differentiate the equilibrium condition $x/\hat{\theta} = \mathcal{A}(x)$, use the result above that $\mathcal{A}'(x) = \Psi(x)$ and reorganize terms to get

$$\frac{\mathrm{d}x}{\mathrm{d}\hat{\theta}}\frac{\hat{\theta}}{x} = \left(1 - \frac{\Psi(x)x}{\mathcal{A}(x)}\right)^{-1}.$$

which, by definition of $\mathcal{A}(x)$, is larger than 1. This completes the proof. ∎

4.2 Aggregate Equilibrium

Feasible Allocations. With population normalized to one, per capita production results from the aggregation of firms' production in (7), *i.e.*,

$$y_t = n_t \int_{z_t^*}^{\bar{\zeta}} y_t(z)\phi_t(z)\mathrm{d}z = F(k_t, 1) \equiv f(k_t). \tag{23}$$

Aggregate technology is therefore Neoclassical and has the same functional form as the individual firm's technology. In addition, from equations (6) and (16), labor and capital are paid their marginal productivities, since $w_t = F_2(k_t, 1)$ and $\pi_t(z) = F_1(k_t, 1)z$. The only relevant difference with respect to a Neoclassical technology is that the average quality of capital, as measured by $\bar{z}_t$, instead of being (normalized to) one is endogenous and increasing in selection.[10] For the same reason, capital measured at constant quality and consumption are different goods.[11]

On top of producing y_t, the Ramsey-Hopenhayn economy recycles discarded capital. Since capital is partially reversible, meaning that a fraction of its intrinsic content can be reverted to the production process. Let us denote by s_t the stock of recycled capital available for consumption or investment at time t. We model the recycling technology in line with the modeling of the investment technology. In the continuous time Neoclassical growth model, capital goods produced at time t become operative at time t+dt. This avoids the undesirable possibility of producing capital that produce capital again and again at the same moment in time. Similarly, in this paper, scrapped capital at time t is assumed to be recycled and used for consumption or investment at time $t + \mathrm{d}t$. Otherwise, if it were used at time t scrapped capital could be recycled and recycled again and again until it becomes productive enough to be used in production.

How does scrapped capital cumulate into the stock of recycled capital s_t? Equation (24) summarizes the recycling process under Guess 1:

$$\dot{s}_t = \theta\Big(\Psi(z_t^*)e_t + \phi_t(z_t^*)n_t\dot{z}_t^*\Big) - s_t. \tag{24}$$

Changes in recycled capital s_t is the outcome of three components. The first component pertains to entries. In each period t, a fraction $\Psi(z_t^*)$ of the e_t new firms draw a productivity level z smaller than z_t^*, and therefore decide to exit, sending their capital to scrap. The second component relates to the evolution of the productivity cutoff z_t^* which, under Guess 1, shifts to the right. Therefore incumbent firms which productivity becomes smaller than the new z_t^* optimally close down sending their capital to scrap too. Accordingly, $\phi_t(z_t^*)n_t$ incumbent firms endogenously exit at the rate $\dot{z}_t^*$, $\dot{z}_t^* \geq 0$. The last term reflects the assumption that scrapped capital recycled at time t reverts to the production process at time $t + \mathrm{d}t$.

[10]By differentiating $\bar{z}$ in (15) with respect to z^*, it is easy to see that $\bar{z}$ is increasing in z^*, implying that average productivity increases with selection.

[11]It is important to note that y_t measures per capita production in consumption units, which differs from GDP as measured in National Accounts. However, when z_t^* is time invariant, as it is the case in the Ramsey-Hopenhayn model studied in this paper, the average quality of capital is time invariant, making the relative price of capital and consumption become time invariant and easing aggregation.

At any time t, the economy has a stock of capital $k_t = n_t\bar{z}_t$, producing $f(k_t)$, and a stock of recycled capital s_t. Produced and recycled goods are consumed or invested, such that[12]

$$f(k_t) + s_t = c_t + \eta e_t. \tag{25}$$

The last term corresponds to investment. Each of the e_t new entrants acquires η consumption goods to build the unit of capital required to start producing.

Finally, differentiating $k_t = \bar{z}_t n_t$ with respect to t, substituting e_t from (25) and $\dot{n}_t/n_t$ from (9), denoting $\bar{\gamma}_t = \dot{\bar{z}}_t/\bar{z}_t$, and using the definition of λ_t in (10), we obtain the following law of motion for quality adjusted capital

$$\dot{k}_t = q_t\big(f(k_t) + s_t - c_t\big) + (\bar{\gamma}_t - \delta_t)k_t, \tag{26}$$

where δ_t is defined in (10) and

$$q_t = \bar{z}_t\Big(1 - \Psi(z_t^*)\Big)/\eta. \tag{27}$$

As in Greenwood et al [23], q_t in equation (27) measures the state of technology in the investment sector, *i.e.*, the rate at which one unit of the consumption good is turned into quality adjusted capital.[13] One unit of capital requires η units of the consumption good to be produced, becoming operative with probability $1 - \Psi(z_t^*)$, and having expected quality $\bar{z}_t$.

Under the conditions of Proposition 6, z_t^* is constant at equilibrium. Let us denote this constant by z^*. A direct implication of Proposition 6 is then $\delta_t = \delta$ and $\bar{\gamma} = 0$, such that (FC_{rh}) and (24) become

$$\dot{k}_t = q\big(f(k_t) + s_t - c_t\big) - \delta k_t, \tag{FC_{rh}}$$

$$\dot{s}_t = \hat{\theta}\Psi(z^*)\big(f(k_t) - c_t\big) - \big(1 - \hat{\theta}\Psi(z^*)\big)s_t. \tag{SC}$$

with $q = \bar{z}\Big(1 - \Psi(z^*)\Big)/\eta$ being time invariant. Given a stock of capital k_t and a stock of recycled capital s_t, (FC_{rh}) and (SC) describe the feasible set of consumption, c_t, net (quality adjusted) investment, $\dot{k}_t$, and the creation of recycled capital $\dot{s}_t$.

[12] We define gross investment as $\eta e - s$ following the *Measuring Capital OECD Manual* (`https://www.oecd.org/sdd/productivity-stats/43734711.pdf`). The manual states that "GFCF is defined as the acquisition, less disposals, of fixed assets... The assets acquired may be new or they may be used assets that are traded on second-hand markets. The assets disposed of may be sold for continued use by another economic unit, they may be simply abandoned by the owner or they may be sold as scrap and be broken down into reusable components, recoverable materials, or waste products." (p. 124)

[13] An economy where the equilibrium distribution of capital quality were permanently moving to the right at a constant rate will feature investment specific technical change as in Greenwood et al [23].

Equilibrium Aggregates. Under the conditions of Proposition 6, for a given equilibrium cutoff productivity z^*, a given equilibrium distribution $\phi(z) = \psi(z)/(1-\Psi(z^*)$, and given initial conditions $k_0 > 0$ and $s_0 > 0$, an aggregate equilibrium is a path (c_t, k_t, s_t), $t \geq 0$, that solves[14]

$$\begin{aligned} \dot{c}_t/c_t &= \sigma\left(\frac{z^*}{\theta}f'(k_t) - \rho - \delta\right) \\ \dot{k}_t &= q\left(f(k_t) + s_t - c_t\right) - \delta k_t \\ \dot{s}_t &= \hat{\theta}\Psi(z^*)\left(f(k_t) + s_t - c_t\right) - s_t. \end{aligned} \tag{RH}$$

The first condition results from substituting the equilibrium interest rate r_t from (EC) into the household's Euler equation (2). This is therefore a standard Euler equation where the interest rate corresponds to the return of the marginal firm z^*. The other two conditions were derived just above. As it will become clear in Section 4.3, selection affects the initial conditions k_0 and s_0, since a jump in z^* at the initial time suddenly destroys capital, cutting the capital stock down and increasing the stock of recycled capital.

Stationary Equilibrium. In a steady state equilibrium, the path for $\{c_t, k_t, s_t\}$ is time invariant. Imposing stationarity to the equilibrium conditions in (RH) above, the stationary values of c and k are given by

$$\begin{aligned} f'(k) &= \frac{\theta}{z^*}(\rho + \delta) \\ c &= f(k) - \frac{\theta}{z^*}\delta k. \end{aligned} \tag{RH$_{ss}$}$$

Since more selective economies have a lower θ/z^*, they also have a larger steady state stock of capital. Moreover, the effective depreciation rate also decreases with selection, meaning that the gains in quality more than compensate the cost of recycling capital, making the effective depreciation rate smaller than δ —the physical depreciation rate.

At steady state the stock of recycled capital is

$$s = \frac{\delta\theta\Psi(z^*)}{1-\Psi(z^*)}\underbrace{\frac{k}{\bar{z}}}_{n}.$$

At any time t, a fraction δ of firms is replaced and a fraction $\Psi(z^*)$ of their physical capital is sent to scrap and recycled at the rate θ. The multiplier $\left(1-\Psi(z^*)\right)^{-1}$ reflects the fact that, in period $t + \mathrm{d}t$, recycled capital is also subject to selection, scrapping and recycling.

In the following section, we implement separate, permanent shocks to the degree of capital reversibility θ and the investment distortions η, both shocks delivering exactly the same increase

[14] A standard transversality condition must be added to the definition of equilibrium.

in the cutoff z^*. As can be seen by inspecting the stationary equilibrium system (RH_{ss}), the effect on the aggregates of these shocks differ substantially. Investment distortions have no direct effect on the stationary value of the aggregates, the effect operating through z^* only. By contrast, increasing capital reversibility directly raises the marginal product of capital and the depreciation rate at the stationary equilibrium, even if, as shown in Corollary 1, the indirect effect through selection dominates. The additional negative effect on capital and consumption partially compensates the indirect positive effect of selection through z^*. The differential effect of θ and η is fundamentally due to the fact that a reduction in investment distortions promote entry, whereas an increase in capital reversibility promotes exit. Both hence have opposite effects on the equilibrium mass of firms, which increases following a decline in η and reduces following an increase in θ.

4.3 Transitional Dynamics

In this section, we consider a baseline economy facing both investment distortions and partial capital irreversibility. We analyze the effects of reducing these distortions on the equilibrium system (RH).

Let us assume the entry distribution is Pareto with tail parameter $\kappa > 1$ and lower bound $\zeta = \frac{\kappa-1}{\kappa}$, the production technology is Cobb-Douglas, $f(k) = Ak^\alpha$ with $A > 0$ and $\alpha \in (0,1)$. The economy was in steady state before $t = 0$, with investment distortion $\eta_\iota > 1$ and degree of partial reversibility $\theta_\iota < 1$, where index ι refers to the past steady state equilibrium. There exists a cutoff productivity, z_ι^*, solving (20), a productivity distribution $\phi_\iota(z) = \psi(z)/(1 - \Psi(z_\iota^*)) = \kappa(z_\iota^*)^\kappa z^{-\kappa-1}$, for $z \geq z_\iota^*$, a stock of capital k_ι solving

$$\alpha A k_\iota^{\alpha-1} = \frac{\theta_\iota}{z_\iota^*}(\rho + \delta),$$

and a mass of recycled capital

$$s_\iota = \theta_\iota \, \frac{\Psi(z_\iota^*)}{1 - \Psi(z_\iota^*)} \, \delta n_\iota$$

with $n_\iota = k_\iota/\bar{z}_\iota$, and

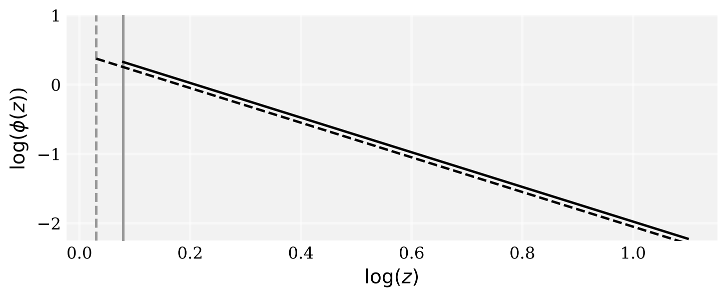
$$\bar{z}_\iota = \frac{\kappa}{\kappa - 1} z_\iota^*.$$

It is easy to see that a permanent productivity shock raising A will have the same effect as in the baseline Neoclassical growth model irrespective of the degree of capital irreversibility and innovation distortions.[15] Since z^* does not depend on any parameter of the production technology, changes in A have no effect on selection.[16]

Let us now analyze the effects of taming down the bite of the distortions in this economy. More precisely, let us assume that at time $t = 0$ capital becomes more reversible or investment distortions abridge. More formally, from $t = 0$ the degree of capital reversibility becomes $\theta > \theta_\iota$ or, alternatively, investment distortions become $\eta < \eta_\iota$. The size of the shock is such that the new θ and η produce the same z^*.

From (20), for all $t \geq 0$, $z_t^* = z^* > z_\iota^*$. Consequently, the equilibrium distribution is $\phi_t(z) = \phi(z) = \psi(z)/(1 - \Psi(z^*)) = \kappa(z^*)^\kappa z^{-\kappa-1}$, for $z \geq z^*$ and the average productivity is $\bar{z}_t = \bar{z} = \frac{\kappa}{1-\kappa} z^*$. The change in the equilibrium distribution is depicted in Figure 2, where the cutoff productivity is seen to shift to the right (right shift in the vertical gray line).

Figure 2: Distribution of Firms: $\log(\phi(z))$



- - - - At initial steady state, ——— At final steady state.

At time $t = 0$, selection induces a reduction in the stock of capital, with

$$k_0 = n_\iota \Big(1 - \Phi_\iota(z^*)\Big)\bar{z} < n_\iota \bar{z}_\iota = k_\iota.$$

The increase in the stock of recycled capital depends on type of shock, with $\tilde{\theta} \in \{\theta, \theta_\iota\}$,

$$s_0 = s_\iota + \tilde{\theta}\, n_\iota \Phi_\iota(z^*) > s_\iota.$$

Selection hence operates as a negative shock on the initial stock of capital. Even if the average productivity of capital goes up, it does so at the price of destroying a fraction of the existing capital stock. The associated obsolescence cost results from firms closing down and sending their machines to scrap, as witnessed by the increase in s_0 with respect to s_ι.

[15] See King and Rebelo [31].

[16] This result is in line with Veracierto (2002) showing "that investment irreversibilities do not play a significant role in an otherwise standard real business-cycle model."

The solution for the aggregates is a path $\{c_t, k_t, s_t\}$, for $t \geq 0$, that solves (RH) given the initial conditions k_0 and s_0 above, and the equilibrium solution for z^*.

Figure 3 illustrates these dynamics. Our benchmark parametrization,[17] as reported in Table 1, assumes, without loss of generality, a unit scaling parameter A. Households are assumed to discount the future with a discount rate, ρ, of 5% and have preferences characterized by a unit elasticity of intertemporal substitution ($\sigma = 1$). The Pareto tail parameter is set to 1.5, which lies in the range of estimated values in the empirical literature (see [**?**], [39], [26] among others). Investment distorsions are such that it takes 1.2 units of the consumption good to build a unit of capital, ($\eta = 1.2$). The parameters θ, α and δ are then set such that the model generates a consumption share of 0.8, a capital to output ratio of 3, and a capital income share of 0.35. We then engineer a one shot 2.66% (resp. 2.6%) permanent increase (resp. decrease) in θ (resp. η) to generate the 5% permanent increase in z^* reported in Figure 2. A 5% increase in the average quality of capital, with $\alpha = 0.35$ is equivalent to a 1.7% (permanent) TFP shock.

Table 1: Parameters

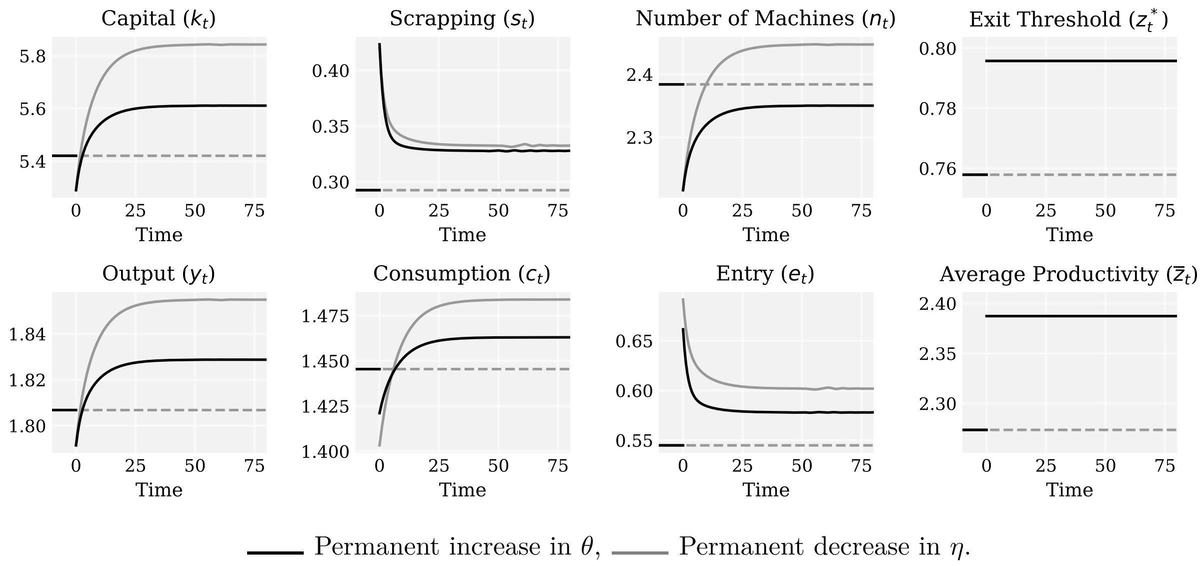
Preferences		Technology					
ρ	σ	A	α	δ	κ	η	θ
0.050	1.000	1.000	0.350	0.066	1.500	1.200	0.758

As can be observed from Figure 3, the increase in selection induced by both shocks improves the steady state average quality of capital, raising production and consumption in the long run. An increase in capital reversibility promotes additional exit, reducing the mass of firms at the new stationary equilibrium. As shown in Corollary 1, the increase in average productivity more than compensates the decline in the mass of firm, increasing (quality adjusted) capital at steady state, but increasing it by less than the 5% gain in average productivity. A reduction in investment distortions, by contrast, by promoting entry increases the mass of firms, making the increase in capital, output and consumption be twice as large as the one resulting from the equivalent increase in capital reversibility.

Both shocks, by increasing selection, exerts an identical initial negative effect on the capital

[17] Although this exercise does not pretend to constitute a full fledged calibration of the model, the values of the parameters were set in lines with previous studies that undertook a proper calibration of a model of firm dynamics.

Figure 3: Transition Dynamics (Ramsey-Hopenhayn Model)



stock, hereby reducing output. Since the economy becomes more selective, the less productive firms exit the market, thereby reducing the initial stock of capital both in physical and quality adjusted units, as measured by n_t and k_t, by around 7% and 2%, respectively. A reduction in η has the additional effect of reducing the cost of creating new firms, which remains unaffected by an increase in θ.

Even if the incentives are there for new, more productive firms to be created, the creation process takes time and requires new costly investments. Consequently, consumption initially reduces even more than capital to allow for a larger entry of firms. This effect is stronger in the case of a reduction in investment distortions —since it promotes entry— relative to an improvement in capital reversibility —since it induces more exit. The effects on scrapping are, however, quite similar.

In order to evaluate the consumption equivalent welfare gains and losses from selection, let us define the welfare gain of an increase in θ or a reduction in η as the extra permanent percentage consumption, Δ_c, the representative individual should be given in order to be indifferent between living in the initial and the post-shock economy. This gain is measured as

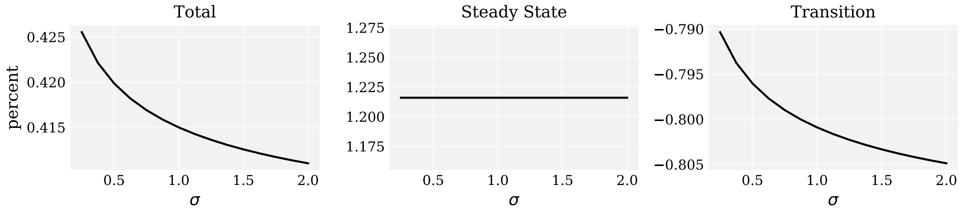
$$\int_0^\infty u\Big((1+\Delta_c)c_\iota\Big)\mathrm{e}^{-\rho t}\mathrm{d}t = \int_0^\infty u(c_t)\mathrm{e}^{-\rho t}\mathrm{d}t \Longleftrightarrow 1+\Delta_c = \frac{1}{c_\iota}u^{-1}\left(\rho\int_0^\infty u(c_t)\mathrm{e}^{-\rho t}\mathrm{d}t\right),$$

where c_ι represents steady state consumption before the change in θ or η, and c_t is the post-change consumption path.

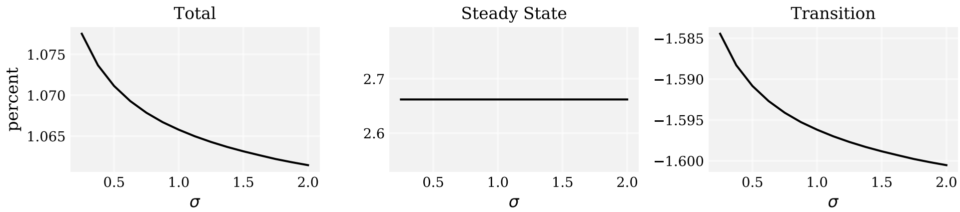
As can be observed from Figure 4, total and steady state welfare gains from mitigating

Figure 4: Welfare Gains (Ramsey-Hopenhayn Model)

(a) Permanent increase in θ



(b) Permanent decrease in η



investment distortions and alleviating capital irreversibilities are of first order. Mitigating investment distortions generate welfare gains that are twice as large as the gains from alleviating capital irreversibilities. A 5% increase in selection, that permanently raises TFP by 1.7%, generates total welfare gains of around 1.07% or 0.42%, depending on selection being enhanced through shocks that directly affect creation/entry or destruction/exit, respectively. Steady state welfare gains are between three and two and a half times as larger as total gains, with losses produced by the initial capital destruction accounting for 60% to 65% of the long term welfare gains from selection.

The intertemporal elasticity of substitution σ does not affect the steady state equilibrium for $\{c, k, z^*\}$ and $\phi(z)$, neither the initial drop on capital from k_ι to k_0, but the initial drop in consumption and the velocity of convergence from k_0 to the new steady state. For this motive, it is interesting to see how welfare gains depend on σ. As expected, steady state welfare gains are positive and invariant with respect to σ, as illustrated by the middle panel of Figure 4, since the steady state consumption path does not depend on it. Moreover, the lower σ is, the closer the initial consumption will be to its steady state value, reducing the welfare cost of the initial capital destruction emerging from selection. As shown in Figure 4, for our particular example, the additional losses associated to an increase in σ are small.

4.4 Instantaneous Recycling

In Section 4.2, we assume there is a recycling technology transforming one unit of capital discarded at time t into θ units of the final good, $\theta < 1$, which can be consumed or invested at $t+\mathrm{d}t$. In this section, we explore the alternative assumption that discarded capital is recycled and used again at period t, which is the standard assumption regarding the working of secondary markets. Al along this subsection, Proposition 6 holds, implying that an equilibrium exists with constant cutoff productivity z^*. Let us also assume the economy was at a steady state up to the initial time, implying that Proposition 2 also holds and the equilibrium distribution $\phi(z)$ is the entry distribution truncated at the constant equilibrium cutoff z^*.

Feasibility condition. In this context, condition 25 becomes

$$f(k_t) = c_t + \big(\eta - \theta\Psi(z^*)\big)e_t, \tag{28}$$

where the last term represents gross investment net of recycled capital, which as assumed just above instantaneously revert to the production process.

From conditions (9) and (10), using $k_t = \bar{z}n_t$, the feasibility condition reads

$$\dot{k}_t = q\big(f(k_t) - c_t\big) - \delta k_t, \tag{29}$$

where

$$q = \frac{1 - \Psi(z^*)}{\eta - \theta\Psi(z^*)}\ \bar{z}.$$

The rate at which one unit of the consumption good is turned into quality adjusted capital positively depend on the average quality of capital $\bar{z}$ and on the scarp value of capital θ, but negatively on the investment distortion η and on the cutoff productivity z^*.

Equilibrium. Under the conditions of Proposition 6, for a given equilibrium cutoff z^*, a given equilibrium distribution $\phi(z) = \psi(z)/(1 - \Psi(z^*)$, and given initial conditions $k_0 > 0$ and $s_0 > 0$, an aggregate equilibrium is a path (c_t, k_t), $t \geq 0$, solving[18]

$$\begin{aligned} \dot{c}_t/c_t &= \sigma\left(\frac{z^*}{\theta}f'(k_t) - \rho - \delta\right) \\ \dot{k}_t &= q\left(f(k_t) - c_t\right) - \delta k_t. \end{aligned} \tag{IR}$$

[18]A standard transversality condition must be added to the definition of equilibrium.

The first condition results as before from substituting the equilibrium interest rate r_t from (EC) into the household's Euler equation (2).

The stationary solution for $\{c, k\}$ is identical to the solution in Section 4.2, *i.e.*,

$$\begin{aligned} f'(k) &= \frac{\theta}{z^*}(\rho + \delta) \\ c &= f(k) - \frac{\theta}{z^*}\delta k. \end{aligned} \tag{RH$_{ss}$}$$

At time $t = 0$ then, capital is

$$k_0 = n_\iota \left(\Big(1 - \Phi_\iota(z^*)\Big) + \Phi_\iota(z^*)\hat{\theta}\Big(1 - \Psi(z^*)\Big) \underbrace{\sum_{i=0}^{\infty} \Big(\hat{\theta}\,\Psi(z^*)\Big)^i}_{\left(1-\hat{\theta}\Psi(z^*)\right)^{-1}} \right) \bar{z}. \tag{30}$$

The second term in parenthesis represents the instantaneous recycling of the scrapped initial capital. Differently from Section 4.3, if the shock moves $\hat{\theta}$ close enough to one, the initial reduction in the mass of firms maybe more than compensated by the increase in the average productivity $\bar{z}$, in which case selection will generate an initial creation of capital instead of an initial destruction. It is easy to see in equation (30) that this is the case when $\hat{\theta}$ approaches one, since in this case k_0 approaches $n_\iota \bar{z}$ with $\bar{z} > \bar{z}_\iota$.[19]

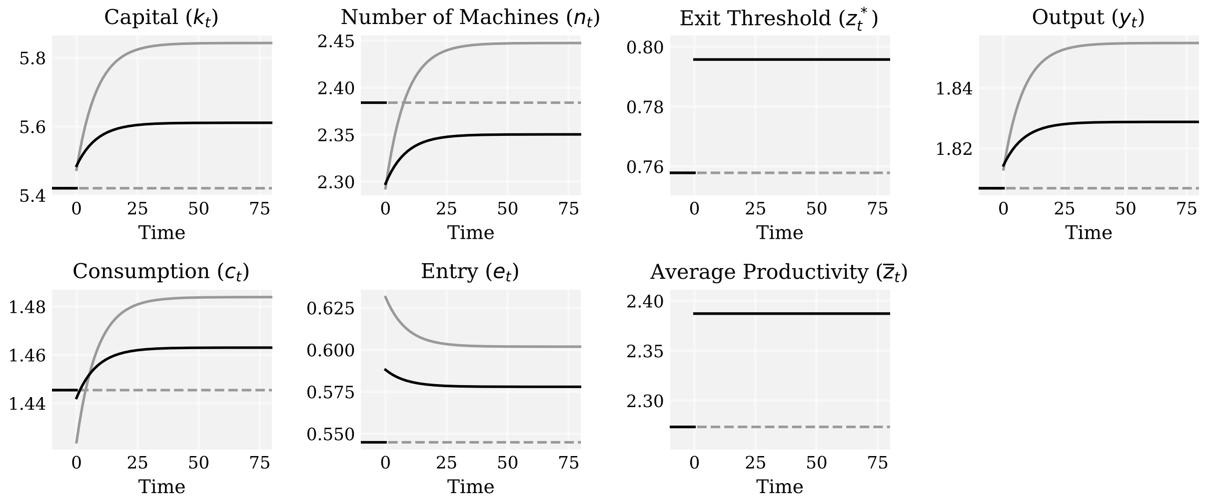
The dynamic properties of the RH economy with instantaneous delays are the same as the properties of the Neoclassical model. It is interesting to notice that for $\eta = 1$ and $\theta = \underline{\zeta}$, there is no selection since $z^* = \underline{\zeta}$. In this case $z^*/\theta = q = 1$, making the equilibrium of the Ramsey-Hopenhayn economy under instantaneous recycling, as represented by (IR), be the Neoclassical model with initial condition k_0. If $\eta > 1$, $\theta > \underline{\zeta}$, and $\theta/\eta > \underline{\zeta}$, then $\theta/z^* < 1$ and $q > 1$. Selection makes capital and consumption to be larger in the RH economy with instantaneous recycling relative to the Neoclassical economy.

Transitional dynamics. Let us set the same parameters as in Table 1, and introduce the same shocks as in Section 4.3. The transitional dynamics is represented in Figure 5. By comparing Figures 3 and 5, we can see that the RH economy with instantaneous recycling moves from the same initial to the same final steady state equilibrium as in the RH economy (with delayed recycling) of Section 4.3. The fundamental difference is, as we can see in Figure 5, that capital destruction flips to capital creation since instantaneous recycling allows for an increase of capital

[19]In the degenerate case $\hat{\theta} = 1$, the cost of recycling is zero, then firms will instantaneously recycle until they get the maximum productivity $\overline{\zeta}$.

quality that overpass the initial reduction in the number of firms. Consequently, capital, output, consumption and gross investment (as represented by firms' entry) converge to the same steady state but starting from a higher level.

Figure 5: Transition Dynamics (Instantaneous Recycling Model)



Permanent increase in θ, Permanent decrease in η.

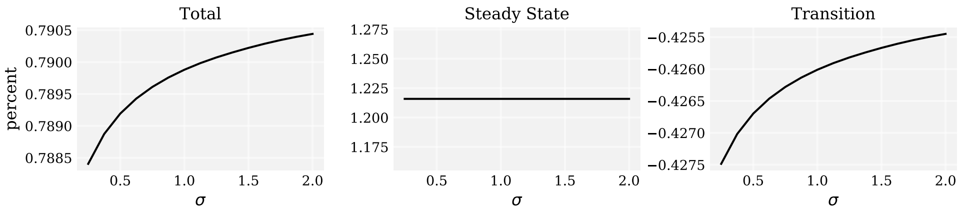
By comparing Figures 4 with 6, it can be observed that stationary welfare gains are the same, irrespective of recycling being delayed or instantaneous. The cost of the transition are much smaller when recycling is instantaneous representing, representing around 35% of steady state welfare gains. Since there is capital creation, the role of σ on shaping transitional flips compared to the model with delayed recycling. Smaller σ is, longer it takes to reach the new steady state, generating larger transitional loses. Notice that this relation was broken in Figure 4 due to initial capital destruction.

5 Hopenhayn Model

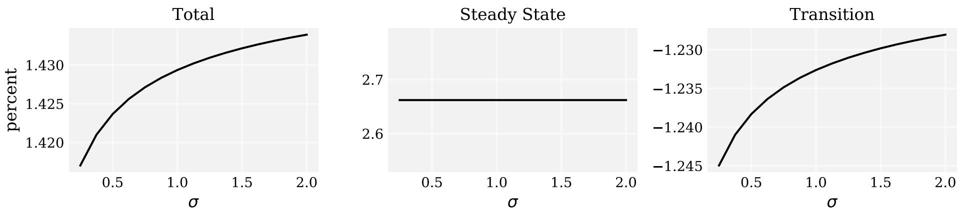
This paper studies a simplified version of Hopenhayn [24] where firm's productivity, in the spirit of Melitz [38], is time invariant. It corresponds to the general framework in Section 3, by setting $\theta = 0$ and $\mu > 0$. As we show in this section, differently from the Ramsey-Hopenhayn model, when the Hopenhayn economy faces a permanent decline in the investment distortion η, the cutoff productivity does not jump upward but monotonically converges to the new steady state with more selection. As in the previous section, and for comparison purposes, we restrict the analysis of the Hopenhayn model to the particular case of a Pareto entry distribution and a

Figure 6: Welfare Gains (Instantaneous Recycling Model)

(a) Permanent increase in θ



(b) Permanent decrease in η



Cobb-Douglas production function.

Let us interpret k_t here as the aggregate stock of intangible capital. The creation of a firm involves some fully irreversible intangible investment, normalized here to one. This investment makes technology $F(z, \ell)$ available to the firm, which value depends on productivity z. At any time t, n_t of these technologies are operative with average productivity $\bar{z}_t$, implying an aggregate stock of intangibles equal to $k_t = n_t \bar{z}_t$.

Selection. Under the full irreversibility assumption, *i.e.* when $\theta = 0$, the (EC) condition becomes

$$\alpha z_t^* A k_t^{\alpha-1} = \mu, \tag{EC$_h$}$$

relating the cutoff productivity to the marginal product of intangible capital. Under Guess 1 the marginal firm has indeed no prospect of generating any capital gain.

The entry and exit process, as represented by equation (R), becomes

$$\Big(1 - \Psi(z_t^*)\Big) \frac{\left(\bar{z}_t - z_t^*\right)\, \alpha A k_t^{\alpha-1}}{r_t + \delta} = \eta.$$

From (EC$_h$) and the additional assumption that the entry distribution is Pareto, the equilibrium interest rate as described by (R) becomes

$$r_t + \delta \; = \frac{\mu^{1-\kappa}\, \zeta^\kappa}{\eta(\kappa - 1)} \left(\alpha A k_t^{\alpha-1}\right)^\kappa \; = \frac{1 - \Psi(z_t^*)}{\eta(\kappa - 1)}\, \mu. \tag{R$_h$}$$

Decreasing returns to capital are consequently at work. In addition, and differently from the RH model, the equilibrium cutoff moves one-to-one with the real interest rate. An increase in

z^*, *Ceteris Paribus*, reduces the probability $1-\Psi(z_t^*)$ of being successful, lowering the return to entry.

Equilibrium Aggregates. At any time t, the economy has a stock of intangible capital $k_t = n_t\bar{z}_t$, which produces Ak_t^α. Some production is allocated to cover the fixed production cost, the remaining is consumed or invested according to

$$\underbrace{Ak_t^\alpha - \mu n_t}_{\text{GDP}} = c_t + \eta e_t \tag{31}$$

where e_t is entry, measuring investment in physical units. Fixed production costs need to be subtracted from production to get a value added measure of GDP, which then is allocated to consumption and investment. Differently from the Ramsey-Hopenhayn model, the stock of recycled capital is zero, since intangible capital is fully irreversible.

The following proposition characterizes the dynamics of the Hopenhayn model, which differently from the Ramsey-Hopenhayn model.

Proposition 7 *Under Guess 1, an initial distribution* $\phi_\iota(z) = \frac{\psi(z)}{1-\Psi(z_\iota^*)}$*, for* $z \in [z_\iota^*, \bar{\zeta})$*,* $z_\iota^* > \underline{\zeta}$*, a Pareto entry distribution* $\Psi(z) = 1 - \left(\frac{\underline{\zeta}}{z}\right)^\kappa$*,* $\kappa > 1$ *and* $\underline{\zeta} > 0$*, and a Cobb-Douglas production function* $f(k) = k^\alpha$*,* $\alpha \in (0,1)$*, the feasibility condition becomes*

$$\dot{k}_t = \hat{q}\ k_t^{(\alpha-1)(\kappa-1)}\big(Ak_t^\alpha - c_t\big) - \left(\hat{\delta}_0 + \hat{\delta}_1 k_t^{\kappa(\alpha-1)}\right) k_t, \tag{FC$_h$}$$

with

$$\hat{q} = \frac{\kappa\underline{\zeta}^\kappa\left(\frac{\mu}{\alpha A}\right)^{1-\kappa}}{\eta(\kappa-1)(\kappa+\alpha-\kappa\alpha)}, \qquad \hat{\delta}_0 = \frac{\delta}{\kappa+\alpha-\kappa\alpha} \quad \textit{and} \quad \hat{\delta}_1 = \frac{\underline{\zeta}^\kappa(\alpha A)^\kappa \mu^{1-\kappa}}{\eta(\kappa+\alpha-\kappa\alpha)}.$$

Proof: From the definition of k_t,

$$\dot{k}_t = \dot{n}_t\bar{z}_t + n_t\dot{\bar{z}}_t.$$

Under Guess 1, $\phi_t(z) = \frac{\psi(z)}{1-\Psi(z_t^*)}$. Then, from equation (9) and the definition of $\bar{z}_t$

$$\dot{n}_t = e_t\big(1-\Psi(z_t^*)\big) - \big(\delta + \phi_t(z_t^*)\dot{z}_t^*\big)n_t \quad \text{and} \quad \dot{\bar{z}}_t = \phi_t(z_t^*)\big(\bar{z}_t - z_t^*\big)\dot{z}_t^*.$$

Combine the three equations to get

$$\dot{k}_t = \bar{z}_t e_t\big(1-\Psi(z_t^*)\big) - \delta k_t - n_t\phi_t(z_t^*)z_t^*\dot{z}_t^*.$$

Differentiating the (EC$_h$) condition above and substituting $f(k) = Ak^\alpha$ we get

$$\dot{z}_t^* = -\frac{f''(k_t)}{f'(k_t)}z_t^*\dot{k}_t \quad \Rightarrow \quad \frac{\dot{z}_t^*}{z_t^*} = (1-\alpha)\frac{\dot{k}_t}{k_t}.$$

Substitute the expression for $\dot{z}_t^*$ into the expression for $\dot{k}_t$ above, and use (31) to get

$$\dot{k}_t = \frac{\bar{z}_t\big(1-\Psi(z_t^*)\big)\big(Ak_t^\alpha - c_t\big)/\eta - \big(\delta + (1-\Psi(z_t^*))\mu/\eta\big)k_t}{1+(1-\alpha)\frac{z_t^*}{\bar{z}_t}\ \phi_t(z_t^*)z_t^*}$$

Under the assumption that the entry distribution is Pareto with tail parameter κ and after substitution of the (EC_h) condition above, the last equation becomes (FC_h), which completes the proof. ∎

At equilibrium, the feasibility condition FC_h represents the evolution law of the stock of intangible capital. The intangible investment technology transforms one unit of the consumption into $\hat{q}\ k_t^{(\alpha-1)(\kappa-1)}$ units of intangible capital, which, due to destruction and obsolescence, depreciates at rate $d_t = \hat{\delta}_0 + \hat{\delta}_1 k_t^{\kappa(\alpha-1)}$.

Under the assumption that the entry distribution is Pareto and the production technology is Cobb-Douglas, given the initial condition $k_0 > 0$, the equilibrium path for (c_t, k_t), $t \geq 0$, solves[20]

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \sigma\Big(ak_t^{(\alpha-1)\kappa} - \delta - \rho\Big) \\ \dot{k}_t &= \hat{q}\ k_t^{(\alpha-1)(\kappa-1)}\big(Ak_t^\alpha - c_t\big) - \Big(\hat{\delta}_0 + \hat{\delta}_1 k_t^{\kappa(\alpha-1)}\Big)\,k_t, \end{aligned} \tag{H}$$

where

$$a = \frac{(\alpha A)^\kappa \mu^{1-\kappa}\ \underline{\zeta}^\kappa}{\eta(\kappa-1)},$$

and $\hat{q}$, $\hat{\delta}_0$ and $\hat{\delta}_1$ as defined above.

The first condition results from substituting the equilibrium interest rate r_t from Lemma 5 into the Euler equation (2). The second condition was derived in Proposition 7. Saddle path stability is shown in the Appendix A.

Finally, given the equilibrium path for k_t, the (EC_h) condition determines the equilibrium path for z_t^*.

Steady State. At a steady state of the Hopenhayn model

$$\begin{aligned} k &= \left(\tfrac{a}{\delta+\rho}\right)^{\frac{1}{\kappa(1-\alpha)}} \\ c &= \tfrac{\beta A}{\kappa}k^\alpha - \tilde{\delta}_0\ k^\beta \\ z^* &= \left(\tfrac{\mu}{(\delta+\rho)\eta(\kappa-1)}\right)^{\frac{1}{\kappa}}\underline{\zeta}, \end{aligned} \tag{H$_{ss}$}$$

where $\tilde{\delta}_0 = \hat{\delta}_0/\hat{q} = \frac{\kappa-1}{\kappa}\underline{\zeta}^{-\kappa}\left(\frac{\mu}{\alpha A}\right)^{\kappa-1}\eta\delta$, and $\beta = \alpha+\kappa-\alpha\kappa$. Note that $z^* > \underline{\zeta}$ at steady state iff $\mu > (\delta+\rho)\eta(\kappa-1)$. A decline in innovation distortions η increases selection, the stock of capital (when measured in quality units) and consumption per capita at steady state since producing capital is cheaper.

[20] A transversality condition has to be added to the definition of equilibrium.

Transitional Dynamics. In this case, we set $\theta = 0$ and maintain all parameter values used in the previous section. The fixed cost parameter μ is set such the Hopenhayn model generates the same threshold z^* as in the Ramsey-Hopenhayn model, leading to set $\mu = 0.24$. We then engineer a 7.05% permanent reduction in η such that the threshold z^* increase permanently by 5%. Figure 7 reports the implied transition dynamics, Figure 8 reports the welfare gains.

Figure 7: Transition Dynamics (Hopenhayn Model)

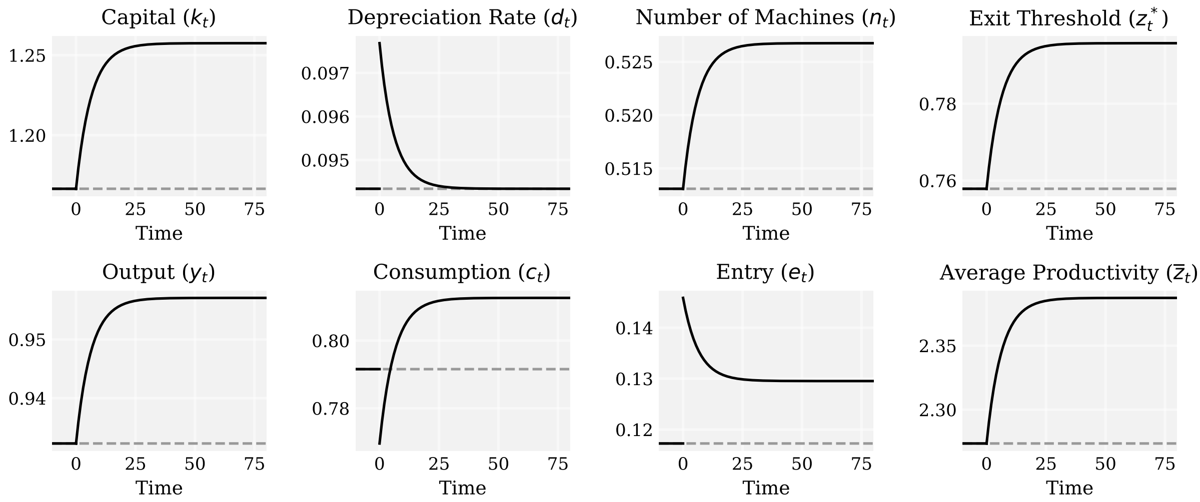
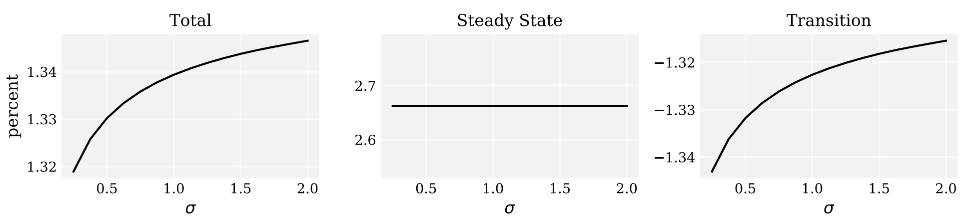


Figure 8: Welfare Gains (Hopenhayn Model)



Differently from the RH model, the cutoff and average productivities do not jump at $t = 0$, but monotonically converge to the new steady state, implying that there is no destruction of intangible capital at the time of the shock. An increase in selection raises the steady state value of capital, which as in the Neoclassical growth model, converges monotonically to its new long run value starting from its past value. As can be observed from Figure 7, output and consumption also follow a standard pattern. Intangible investments as represented by entry follow a standard pattern too. Figure 7 also depicts the behavior of the depreciation rate of intangible capital d_t. In the transition, the cutoff productivity z_t^* monotonically increases, making the marginal firm exit, which destroys its intangible capital.

Steady state welfare gains of raising selection by 5% through a reduction in investment distortions are very similar in both the Hopenhayn and the Ramsey-Hopenhayn model. However, the cost of transition is 20% larger in the Ramsey-Hopenhayn model due to the initial intangible capital destruction.

6 Endogenous Growth

Following Jones and Manuelli [28], this section adds to technology $F(z,\ell)$ in (3) an AK term in line with the literature on learning-by-doing (LBD) —see Arrow [6]. To simplify the argument, let us assume that the concave part of the Jones and Manuelli [28]'s technology is Cobb-Douglas, *i.e.*

$$y(z) = z^{\alpha}\ell(z)^{1-\alpha}(1 + Ak^{1-\alpha}), \tag{32}$$

$A > 0$ and $\alpha \in (0,1)$. The new term $Ak^{1-\alpha}$ is a LBD knowledge spillover. Under $\mu = 0$, all previous results for z^* in Section 4 hold, implying that all along the equilibrium path $z_t^* = z^*$ independent of the aggregates, and $\phi_t(z)$ is the entry distribution truncated at z^*. Following similar steps as in Section 4, the equilibrium conditions for the aggregates read (see Appendix B)

$$\begin{aligned} \dot{c}_t/c_t &= \sigma\left(\frac{z^*}{\theta}\alpha(k_t^{\alpha-1} + A) - \rho - \delta\right) \\ \dot{k}_t &= q(k_t^{\alpha} + s_t - c_t) - (\delta - qA)k_t \\ \dot{s}_t &= \hat{\theta}\Psi(z^*)(k_t^{\alpha} + Ak_t + s_t - c_t) - s_t. \end{aligned} \tag{EG}$$

At a balanced growth path, consumption, capital and recycled capital all grow at the same endogenous rate

$$g = \sigma\left(\frac{z^*\alpha A}{\theta} - \rho - \delta\right), \tag{33}$$

which is increasing in selection. The consumption to capital ratio and the recycled capital to capital ratio are

$$\frac{c}{k} = A - \left(1 - \frac{\hat{\theta}\Psi(z^*)}{1+g}\right)\frac{\delta + g}{q}$$

and

$$\frac{s}{k} = \frac{\hat{\theta}\Psi(z^*)}{1+g}\,\frac{\delta + g}{q},$$

where $q = \bar{z}\Big(1 - \Psi(z^*)\Big)/\eta$ is defined as in Section 4.2. Notice that if there were no selection, $z^*/\theta = q = 1$, and no recycling, $s = 0$, then $g = \sigma\left(\alpha A - \rho - \delta\right)$ and $c = (A - \delta - g)k$ as in the standard LBD model of endogenous growth.

As in Section 4.3, let us assume the entry distribution is Pareto with tail parameter $\kappa > 1$ and lower bound $\zeta = \frac{\kappa-1}{\kappa}$. Let the economy be in a balanced growth path before $t = 0$, with investment distortion $\eta_\iota > 1$ and degree of partial reversibility $\theta_\iota < 1$. The past cutoff productivity z_ι^* solves (20), and the past productivity distribution is $\phi_\iota(z) = \psi(z)/(1-\Psi(z_\iota^*)) = \kappa(z_\iota^*)^\kappa z^{-\kappa-1}$, for $z \geq z_\iota^*$. The past number of firms is n_ι, $n_\iota > 0$, the corresponding past stock of capital is $k_\iota = n_\iota \bar{z}_\iota > 0$,[21] and the past mass of recycled capital is

$$s_\iota = \theta_\iota \, \frac{\Psi(z_\iota^*)}{1-\Psi(z_\iota^*)} \, \delta n_\iota.$$

As we did in Section 4.3, let us assume that from $t = 0$ the degree of capital reversibility becomes $\theta > \theta_\iota$ or, alternatively, investment distortions become $\eta < \eta_\iota$. The size of the shock is such that the new θ and η produce the same z^*. From (20), for all $t \geq 0$, $z_t^* = z^* > z_\iota^*$. Consequently, the equilibrium distribution is $\phi_t(z) = \phi(z) = \psi(z)/(1-\Psi(z^*)) = \kappa(z^*)^\kappa z^{-\kappa-1}$, for $z \geq z^*$ and the average productivity is $\bar{z}_t = \bar{z} = \frac{\kappa}{1-\kappa} z^*$.

Like in Section 4.3, at time $t = 0$ selection induces a reduction in the stock of capital, with

$$k_0 = n_\iota \Big(1 - \Phi_\iota(z^*)\Big) \bar{z} < n_\iota \bar{z}_\iota = k_\iota.$$

The increase in the stock of recycled capital depends on type of shock, with $\tilde{\theta} \in \{\theta, \theta_\iota\}$,

$$s_0 = s_\iota + \tilde{\theta} \, n_\iota \Phi_\iota(z^*) > s_\iota.$$

As in the Ramsey-Hopenhayn model, selection operates as a negative shock on the initial stock of capital.

Table 2: Parameters (Endogenous Growth)

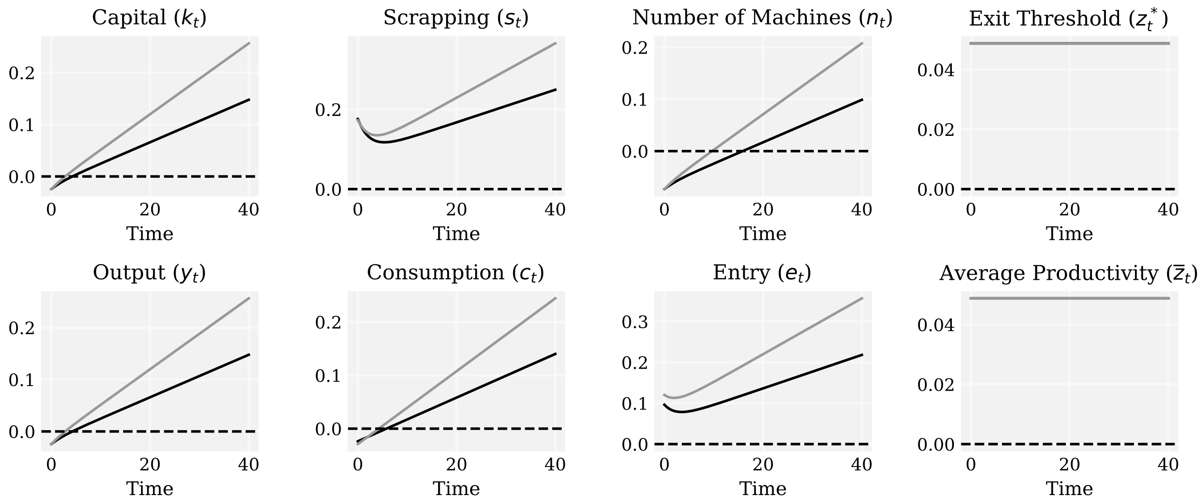
Preferences		Technology					
ρ	σ	A	α	δ	κ	η	θ
0.050	1.000	0.333	0.350	0.066	1.500	1.200	0.876

For the simulation exercise, calibrated parameters are in Table 2. They are set equal to the corresponding parameters in Table 1 with the exception of θ and A that are recalibrated to the growth rate be 2% and the capital to output ratio be three, respectively. As before, the permanent shocks on θ and η are designed to increase z^* by 5%. Selection, by increasing the marginal product of capital, makes the economy to grow faster. From equation (33), it becomes

[21] As usual in endogenous growth models, the stock of capital at the balanced growth path is indeterminate, meaning that k_ι may take any arbitrary value.

clear that the η shock has a larger growth effect than the corresponding θ shock. This differential effect can be easily observed in Figure 9, which represents deviation (in logs) to the initial balanced growth path. The simulated permanent reduction in investment distortions increases annual growth by around 0.685 percent points, but the equivalent increase in reversibility raises it by 0.333. Initial capital destruction, both in physical (number of machines) and quality units (k_t), reduces the stock of capital at the time of the shock, but the increase in the growth rate makes both measures of capital to overpass the previous trend on a few years. Consumption and output follow.

Figure 9: Transition Dynamics (Endogenous Growth)



Steady state welfare gains, as measured in consumption equivalent, are of 5.99% and 11.37% for the θ and η shocks, respectively. When compared to the welfare gains in the RH model, the growth effect of selection multiplies welfare gains by the order of four. The cost of the transition is similar than before, but in relative terms much smaller for all values of σ in (0,2) (for this reason, we decided not to include a Figure).

7 Conclusions

This paper proposes a simple modelling strategy allowing for easily solving the equilibrium path of dynamic general equilibrium economies with heterogenous firms *à la* Melitz [38]. This modelling strategy applies to economies transiting towards a more selective equilibrium. The easiness of the approach relies on the key result that, an economy transiting towards a more selective steady state equilibrium, at any time the equilibrium distribution is the entry distribution truncated at the current productivity cutoff. One of the main advantages of the approach is that it allows for an easy evaluation of the welfare costs of the transition, in particular when selection requires an initial destruction of the capital stock. We find that welfare gains from selection are of first order but the transition generates welfare losses of around 60% of the steady state welfare gains.

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A Stability Hopenhayn Model

An equilibrium is a path $\{c_t, k_t\}$, given $k_0 > 0$ (and a Transversality condition), s.t.

$$\begin{aligned} \frac{\dot{c}_t}{c_t} &= \sigma\Big(ak_t^{(\alpha-1)\kappa} - \delta - \rho\Big) \\ \dot{k}_t &= \hat{q}\; k_t^{(\alpha-1)(\kappa-1)}(Ak_t^\alpha - c_t) - \Big(\hat{\delta}_0 + \hat{\delta}_1 k_t^{\kappa(\alpha-1)}\Big)\, k_t, \end{aligned} \tag{H}$$

where a, $\hat{q}$, $\hat{\delta}_0$ and $\hat{\delta}_1$ as defined in Section 5.

The $\dot{c}_t = 0$ locus corresponds to the steady state value

$$k^h = \left(\frac{a}{\delta+\rho}\right)^{\frac{1}{\kappa(1-\alpha)}} = \left(\frac{\alpha\beta A}{\kappa(\delta+\rho)}\widehat{q}\right)^{\frac{1}{\kappa(1-\alpha)}}.$$

The $\dot{k}_t = 0$ locus is

$$c(k) = \frac{\beta A}{\kappa}k^\alpha - \tilde{\delta}_0\, k^\beta,$$

with $\alpha \in (0,1)$ and $\beta > 1$. Properties of the $\dot{k}_t = 0$ locus:

- $c(0) = 0$. There exist $k_{\max} > 0$ such that $c(k_{\max}) = 0$ with $k_{\max} = \left(\frac{\beta A}{\kappa\tilde{\delta}_0}\right)^{\frac{1}{\kappa(1-\alpha)}}$. Let us assume $k_\iota < k_{\max}$.

- The first derivative is

$$c'(k) = \frac{\alpha\beta A}{\kappa}k^{\alpha-1} - \beta\tilde{\delta}_0\, k^{\beta-1}.$$

  s.t.

  - $\lim c'(k)_{k\to 0^+} = +\infty$.
  - Golden rule: there exits k_g such that $c'(k_g) = 0$, with $k^g = (\alpha/\beta)^{\frac{1}{\kappa(1-\alpha)}} g_{\max} < g_{\max}$.
  - Moreover, $k^h < k^g$. We indeed have

$$k^g = \left(\frac{\alpha A}{\kappa\widetilde{\delta}_0}\right)^{\frac{1}{\kappa(1-\alpha)}}$$

    Using the fact that $\widetilde{\delta}_0 = \widehat{\delta}_0/\widehat{q}$,

$$k^g = \left(\frac{\alpha A\widehat{q}}{\kappa\widehat{\delta}_0}\right)^{\frac{1}{\kappa(1-\alpha)}}$$

    We then have

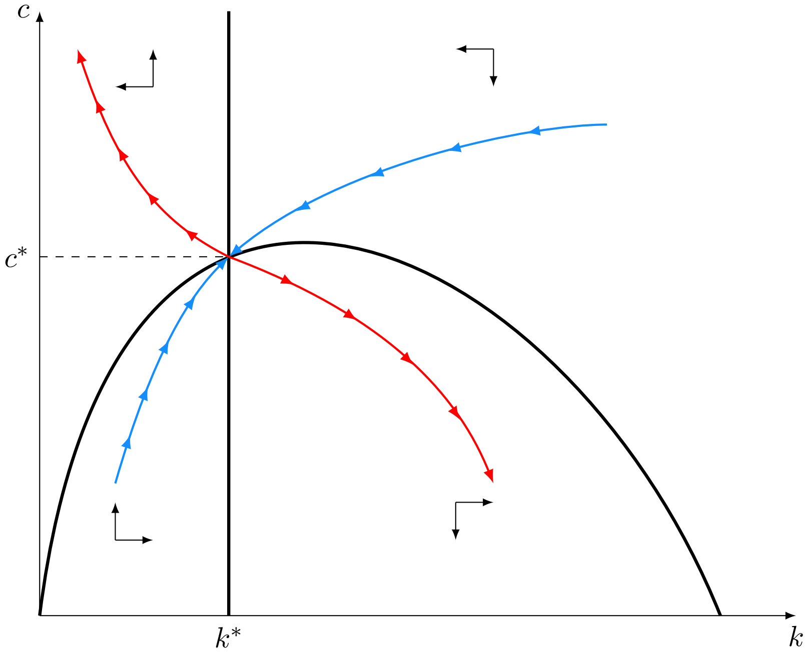
$$k^g > k^h \iff \left(\frac{\alpha A\widehat{q}}{\kappa\widehat{\delta}_0}\right)^{\frac{1}{\kappa(1-\alpha)}} > \left(\frac{\alpha\beta A}{\kappa(\delta+\rho)}\widehat{q}\right)^{\frac{1}{\kappa(1-\alpha)}} \iff \rho > 0$$

- The second derivative is

$$c''(k) = \frac{(\alpha-1)\alpha\beta A}{\kappa} k^{\alpha-2} - (\beta-1)\beta\tilde{\delta}_0\, k^{\beta-2} < 0.$$

Figure 10 depicts the phase diagram associated to the dynamic system, which exhibits saddle-path stability.

Figure 10: Hopenhayn Model: Phase Diagram



To study the local stability of the dynamics, let's repeat the system under study

$$\begin{aligned} \dot{c}_t &= \sigma c_t \Big(a k_t^{(\alpha-1)\kappa} - \delta - \rho \Big) \\ \dot{k}_t &= \hat{q}\; k_t^{(\alpha-1)(\kappa-1)} (A k_t^{\alpha} - c_t) - \Big(\hat{\delta}_0 + \hat{\delta}_1 k_t^{\kappa(\alpha-1)} \Big)\, k_t. \end{aligned} \tag{H'}$$

Linearizing the system around the steady state, and using the steady state restrictions, we get

$$\begin{pmatrix} \dot{\hat{k}}_t \\ \dot{\hat{c}}_t \end{pmatrix} = \begin{pmatrix} \delta_0 + \rho + \delta & -\frac{\kappa(\rho+\delta)}{\alpha(\kappa+\alpha-\alpha\kappa)} \frac{k^*}{y^*} \\ \sigma(\alpha-1)\kappa(\rho+\delta)\frac{c^*}{k^*} & 0 \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix} = J \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}$$

where $\hat{x}_t \equiv x_t - x^*$. The determinant of the matrix J is given by

$$|J| = -\frac{\sigma(1-\alpha)(\kappa(\rho+\delta))^2}{\alpha(\kappa+\alpha-\alpha\kappa)} \frac{c^*}{y^*} < 0$$

such that the product of the 2 eigenvalues is negative. The steady state is therefore (locally) saddle path stable.

B Endogenous Growth Model

At any time t, an operative firm with productivity z solves

$$\pi_t(z) = \max_{\ell_t(z)} z^\alpha \ell_t(z)^{1-\alpha}(1 + Ak^{1-\alpha}) - w_t \ell_t(z),$$

taking w_t as given. From the first-order-condition for labor, following the same steps as in the main text, the labor market clearing condition implies an equilibrium wage $w_t = (1-\alpha)(k^\alpha + Ak)$ and an equilibrium labor demand $\ell_t(z) = \frac{z}{k_t}$. Substituting the equilibrium labor demand in the production technology (32), we get

$$y_t(z) = (k^{\alpha-1} + A)z.$$

Aggregating over all firms, we get the Jones and Manuelli [28] aggregate technology

$$y_t = k_t^\alpha + Ak_t.$$

Substituting equilibrium production, labor demand and wages into the profit function, profits become

$$\pi_t(z) = \alpha(k_t^{\alpha-1} + A)z.$$

It does imply that the exit condition

$$\alpha(k_t^{\alpha-1} + A)z^* = (r_t + \delta)\theta \tag{EC}$$

Recycled capital follows

$$\dot{s}_t = \theta\Psi(z^*)e_t - s_t.$$

with

$$f(k_t) + s_t = c_t + \eta e_t.$$

Finally, following similar steps as in the main text

$$\dot{k}_t = q(k_t^\alpha + Ak_t + s_t - c_t) - \delta k_t, \tag{34}$$

where $q = \bar{z}(1 - \Psi(z^*))/\eta$. Operating them, we obtain the equilibrium system (EG).

Online Appendix
Not for Publication

C Calibration

Ramsey-Hopenhayn Model. Parameters κ and ρ are set to standard values. η is also set. The calibration aims at selecting α, δ and θ such that at steady state the capital to output ratio, the share of consumption in GDP and the share of capital on total income are equal to ko, co and rko, respectively.

The capital to output ratio and the share of consumption are

$$k^{1-\alpha} = \text{ko} \quad \text{and} \quad ck^{-\alpha} = \text{co}.$$

From the Euler equation (2), at steady state $r_t = \rho$. The capital income share allows then solve for δ, *i.e.*,

$$\frac{(r+\delta)k}{k^{\alpha}} = (\rho+\delta)\underbrace{k^{1-\alpha}}_{\text{ko}} = \text{rko}, \quad \Rightarrow \quad \delta = \frac{\text{rko}}{\text{ko}} - \rho.$$

Use the steady state feasibility condition in (RH_{ss}) to solve for z^*

$$\underbrace{ck^{-\alpha}}_{\text{co}} = 1 - \frac{\delta\theta}{z^*}\underbrace{k^{1-\alpha}}_{\text{ko}} \quad \Rightarrow \quad z^* = \frac{\delta\,\text{ko}}{1-\text{co}}\,\theta.$$

Substitute z^* in the (EC) condition, assuming $\mu = 0$, to solve for α

$$z^*\alpha\underbrace{k^{\alpha-1}}_{\text{ko}^{-1}} = (\rho+\delta)\theta \quad \Rightarrow \quad \alpha = (1-\text{co})\,\frac{\rho+\delta}{\delta}$$

Finally, use $\bar{z} = \frac{\kappa}{\kappa-1}z^*$ and substitute the expression above for z^* into (R) and solve the equation below for θ

$$\rho+\delta = \frac{\left(\frac{\zeta(1-\text{co})}{\delta\,\text{ko}\;\theta}\right)^{\kappa}}{\eta-\theta}\;\alpha\underbrace{k^{\alpha-1}}_{\text{ko}^{-1}}\,\frac{\frac{\delta\,\text{ko}}{1-\text{co}}\;\theta}{\kappa-1}. \tag{R}$$

Hopenhayn Model. Parameters κ and ρ are set to standard values. There is full irreversibility, *i.e.*, $\theta = 0$. The calibration aims at selecting α, δ, μ and η such that at steady state the capital to output ratio, the share of consumption in GDP , the share of capital on total income and the cutoff productivity z^* are equal to ko, co, ro and Z^*, respectively.

Parameter δ is set using the same procedure as in the RH model. The stationary equilibrium condition, which results from combining (9) and (10), reads

$$\delta n = \Big(1 - \Psi(z^*)\Big)e.$$

Use the previous condition to substitute for e in the feasibility condition, which then reads

$$c = k^{\alpha} - \mu n - \eta e \quad \Rightarrow \quad \underbrace{\frac{c}{k^{\alpha} - \mu n}}_{\text{co}} = 1 - \frac{\frac{\eta\delta}{1-\Psi(z^*)}\, n}{k^{\alpha} - \mu n} = 1 - \frac{\eta\delta\ \bar{z}}{1-\Psi(z^*)}\ \text{ko}.$$

Consequently

$$\eta = \frac{1-\text{co}}{\delta\ \text{ko}}\ \frac{1-\Psi(z^*)}{\bar{z}}.$$

The (EC) condition give the ratio

$$\frac{\mu}{\alpha} = z^* \underbrace{\left(\frac{1}{\text{ko}} - \frac{\mu}{\bar{z}}\right)}_{k^{\alpha-1}} \quad \Rightarrow \quad \mu\left(\frac{1}{\alpha} + \frac{\kappa-1}{\kappa}\right) = \frac{z^*}{\text{ko}}. \tag{35}$$

Substitute (EC) into the (R) condition to get the ratio

$$\frac{\mu}{\eta} = \frac{(\kappa-1)(\rho+\delta)}{1-\Psi(z^*)}. \tag{36}$$

Use the stationary equilibrium condition, which results from combining (9) and (10),

$$\delta n = \Big(1 - \Psi(z^*)\Big)e$$

to substitute e in the feasibility condition. Then, multiply both sides by $k^{-\alpha}$ to get

$$\underbrace{ck^{-\alpha}}_{\text{co}} = 1 - \left(\frac{\eta\delta}{1-\Psi(z^*)} + \mu\right)\underbrace{nk^{\alpha}}_{\bar{z}\text{ko}^{-1}}.$$

Finally, use $k = \bar{z}n$ and operate to get

$$\delta\eta\left(\frac{z^*}{\zeta}\right)^{\kappa} + \mu = \frac{1-\text{co}}{\bar{z}\ \text{ko}^{-1}} \tag{37}$$

Combine (35), (36) and (37) to solve for μ, η and α.

D Scrapping

The stock of recycled capital moves from any time t to $t+h$, $h > 0$, according to (we exlude $(\delta_t - \delta)n_t$ to simplify the presentation of the main argument)

$$s_{t+h} - s_t = \Big(\underbrace{\theta\Psi(z_t^*)(f(k_t) - c_t)}_{\text{new created capital}} - \underbrace{(1 - \theta\Psi(z_t^*))s_t}_{\text{recycled capital}}\Big)h$$

which is equivalent to

$$k_{t+h} - k_t = \Big(q_t(f(k_t) + s_t - c_t) - \delta k_t\Big)h,$$

where $1 - \theta\psi(z_t^*)$ is the rate at which the stock s_t of recycled capital transform into new capital, exiting from s_t. Notice that $1 - \theta\psi(z_t^*)$ is equivalent to depreciation rate δ in the law of motion for capital.

Then, divide by h and take the limit when $h \to 0$, to get

$$\dot{s}_t = \theta\Psi(z_t^*)e_t - s_t. \tag{SC}$$

E Full Reversibility

Let us assume $\theta = \eta$, $\delta = 0$, and the upper-bound of the entry distribution $\bar{\zeta} < \infty$.

1. From Proposition 3, $z^* = \bar{\zeta}$. It will be equal to infinity is the entry distribution were unbounded.

2. From Proposition 2, the equilibrium distribution degenerates with all the mass in $\bar{\zeta}$. If the entry distribution were unbounded, the equilibrium distribution will polarize, with mass moving to the right.

3. At SS, from Section 3.4,
$$f'(k) = \frac{\rho}{\bar{\zeta}},$$
implying that the stationary capital reaches is maximum possible value, which is finite. In the case the entry distribution is unbounded, k_t will go to infinity.

4. Indeed, from Section 3.4, the initial value of capital is $k_0 = 0$. Then, the economy will jump to the stationary equilibrium with zero capital remaining trapped in the zero equilibrium forever. This cannot be optimal!!! (Notice, on top of it, that from Proposition 4, $\dot{k}_t = 0$)

Converging to full reversibility. The interesting equilibrium is the one where there is no jump on θ, but it moves continuously from θ_ι to η, $\eta = 1$.

1. Use Proposition 3 to solve for z_t^*. Remind that the equilibrium distribution is given by Proposition 2.

2. Since there is no jump on θ at $t = 0$, there is no jump on z^*, which implies that $k_0 = k_\iota$ and $s_0 = s_\iota$.

3. Use (EE-FC-SC) to solve for c_t, k_t and s_t.

In this case, since θ monotonically converges to $\eta = 1$, z^* monotonically converges to $\bar{\zeta}$. Since there is no depreciation, the equilibrium distribution monotonically converges to the degenerate distribution (all the mass concentrated in $\bar{\zeta}$). I expect to see that k_t monotonically converges to the new steady state instead of converging to zero.

Permanent growth. Let us assume that $\bar{\zeta} = \infty$. Concerning θ_t, notice the following. Totally differentiate (EC-FE") to get

$$\frac{\mathrm{d}z_t^*}{z_t^*} = \frac{1}{1 - \theta_t \Psi(z_t^*)} \frac{\mathrm{d}\theta_t}{\theta_t}.$$

Then, in order to z_t^* grow at a constant rate γ, it has to be that

$$\frac{\mathrm{d}\theta_t}{\theta_t} = \gamma\Big(1 - \theta_t \Psi(z_t^*)\Big).$$

Notice that when θ_t converges to one, since z_t^* converges to infinity

Use the Pareto distribution and the assumption that z_t^* grow at the rate γ to rewrite it as a first order differential equation with given θ_0, *i.e.*,

$$\mathrm{d}\theta_t = \left(1 - \theta_t \left(1 - \underline{\zeta}^{\kappa} \left(z_0^* \mathrm{e}^{\gamma t}\right)^{-\kappa}\right)\right) \gamma \theta_t.$$

Of course, a similar exercise could be performed in an economy with $\theta = 1$ and $\eta_\iota > 1$ with η monotonically converging to 1.

F RH with Instantaneous Recycling

We could have alternatively assumed that scrapped capital at t is recycled at t, instead of $t+\mathrm{d}t$. In which case, under Proposition 6, the efficiency condition (25) becomes

$$f(k_t) = c_t + \big(\eta - \theta\Psi(z^*)\big)e_t. \tag{38}$$

It does imply that the feasibility condition reads now

$$\dot{k}_t = q_h\big(f(k_t) - c_t\big) - \delta k_t, \tag{FC$_{rh*}$}$$

with

$$q_h = \frac{\bar{z}\big(1 - \Psi(z^*)\big)}{\eta - \theta\Psi(z^*)}.$$

Notice that under this alternative assumption, the feasibility condition is identical to the feasibility condition of the Neoclassical model apart from q_h being different from one. Indeed, under this assumption $s_t = 0$ for all t.

Under the alternative assumption that recycled capital is currently used for consumption or investment, for a given equilibrium cutoff productivity z^*, a given equilibrium distribution $\phi(z) = \psi(z)/(1 - \Psi(z^*)$ and a given initial conditions $k_0 > 0$, the system becomes

$$\begin{aligned} \dot{c}_t/c_t = & \quad \sigma\left(q_h f'(k_t) - \rho - \delta\right) \\ \dot{k}_t = & \quad q_h\left(f(k_t) - c_t\right) - \delta k_t, \end{aligned} \tag{Rh}$$

with $q_h = \frac{\bar{z}\left(1-\Psi(z^*)\right)}{\eta-\theta\Psi(z^*)} = z^*/\theta > 1$. The last equality results from operating in equation (21). This economy has the same properties as the Neoclassical model, but the marginal product of capital and the rate transforming consumption goods into investment endogenously depend on selection, being both equal to z^*/θ. Indeed, an economy with selection is more efficient than the Neoclassical economy. First, as stated in Proposition 6 and Corollary 1, z^*/θ is larger than one and increasing in selection, meaning that capital is more productive in more selective economies, raising the equilibrium interest rate and the growth rate of consumption. Second, even when consumption transform into physical capital at the rate $1/\eta < 1$, it transforms into adjusted capital at a rate larger than one, increasing with selection. Remind that in the Neoclassical model, the interest rate is equal to the marginal product of capital (minus the depreciation rate) and consumption transform into capital at rate $1/\eta$.

The equilibrium path in (Rh) has the same structure than in Greenwood et al [23]. In both models consumption and capital are different goods, consumption transforming into capital at rate q_h. Indeed, in Greenwood et al [23], q_h is growing at the investment specific or embodied rate of technical progress. In our model, q_h is constant but endogenously determined by selection.

As in the the Ramsey-Hopenhayn model, the stationary values of c and k become

$$\begin{aligned} f'(k) &= \frac{\theta}{z^*}(\rho + \delta) \\ c &= f(k) - \frac{\theta}{z^*}\delta k. \end{aligned}$$

Notice that a shock to selection could be modeled in the framework of the Neoclassical growth model as an investment specific technical shock. The only difference between the two models comes from the effect of selection on initial capital that we study just below. As we show below, under the assumptions in this appendix, selection in facts increases the initial stock of capital making the economy even more efficient.

In order to simulate the dynamic system (RH) under the same shocks as in Section 4.3, the initial stock of quality adjusted capital has to adjust to

$$k_0 = n_\iota \left(\Big(1 - \Phi_\iota(z^*)\Big) + \Phi_\iota(z^*)\hat{\theta}\Big(1 - \Psi(z^*)\Big) \underbrace{\sum_{i=0}^{\infty} \Big(\hat{\theta}\, \Psi(z^*)\Big)^i}_{\left(1-\hat{\theta}\Psi(z^*)\right)^{-1}} \right) \bar{z}$$

implying that, after using the free entry and exit conditions as combined in (21),

$$k_0 = n_\iota \left(\Big(1 - \Phi_\iota(z^*)\Big)\ \bar{z} + \Phi_\iota(z^*) z^* \right) > n_\iota \bar{z}_\iota = k_\iota.$$

Given that scrapped capital may be recycled within the period again and again, more selection makes $k_0 > k_\iota$. The interaction between the entry and exit conditions makes that the expected productivity of scrapped capital becomes z^* at equilibrium. The larger productivity of the recycled capital units more than compensate the destruction associated to scrapping.