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Abstract

We examine utilitarian criteria for evaluating profiles of wellbeing among infinitely many individuals. Motivated by the non-existence of a natural 1-to-1 correspondence between people when alternatives have different population structures, with a different number of people in each generation, we impose equal treatment in the form of *Strong Anonymity*. We show how a novel criterion, *Strongly Anonymous Utilitarianism*, can be characterized by combining Strong Anonymity with other regularity axioms (*Monotonicity*, *Finite Completeness*, and continuity axioms) as well as axioms of equity, sensitivity, separability, and population ethics. We relate it to other strongly anonymous utilitarian criteria and demonstrate its applicability by showing how it leads to an efficient and sustainable stream in the Ramsey model.

JEL-Codes: D630, D710, Q010.

Keywords: utilitarianism, intergenerational equity, population ethics.

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1 Introduction

Since the seminal works by Ramsey (1928), Koopmans (1960), and Diamond (1965), social evaluation criteria for distributions of wellbeing among infinitely many generations or infinitely many individuals have been analyzed by using preorders for infinite wellbeing profiles, i.e., preorders defined for countably infinite-dimensional real vectors.[1] There has been a particular interest in social evaluation criteria that are (generalized) utilitarian and anonymous, in the sense of being based on the undiscounted or unweighted sum of transformed wellbeing,[2] so that the gain of one generation or individual might make up for the loss of another, and so that the generations or individuals are treated equally. Prominent examples are the utilitarian criterion that combines the finite-horizon application of utilitarianism with the Pareto dominance for the tails of profiles (Basu and Mitra, 2007) and utilitarian overtaking or catching-up procedures with respect to time (Atsumi, 1965; von Weizsäcker, 1965; Gale, 1967).

As noted by Fleurbaey and Michel (2003, p. 795), infinite wellbeing profiles can, however, be interpreted in at least three different ways. A common interpretation is that there are representative individuals, one for each generation. That is, in wellbeing profiles $\mathbf{x}$ and $\mathbf{y}$ given by

$$\mathbf{x} = (x_1, \ldots, x_i, \ldots) \quad \text{and} \quad \mathbf{y} = (y_1, \ldots, y_i, \ldots),$$

the ith component is the wellbeing of the representative individual of ith generation, where it is normally left unspecified whether the individual in each generation i is actually identical in the two profiles (Jonsson, 2021). This interpretation is based on two underlying assumptions. One is that there is a natural 1-to-1 correspondence between the individuals of the two streams so that the bearer of wellbeing at the ith component of profile $\mathbf{x}$ can be associated with the bearer of wellbeing at the ith component of profile $\mathbf{y}$. The second assumption is that there is a natural order of counting the individuals or generations, namely according to time. Both the Basu-Mitra utilitarian criterion and the utilitarian overtaking or catching-up criteria are based on the existence of a natural 1-to-1 correspondence between the components of compared profiles. In particular, although such criteria are invariant to permutations that reorder a finite number of the components of one stream (corresponding to the axiom of *Finite Anonymity*), they are not invariant to all permutations (corresponding to the axiom of *Strong Anonymity*), implying that the natural 1-to-1 correspondence plays an essential role. Moreover, the utilitarian overtaking or catching-up criteria are sensitive for the order that the individuals are counted. In particular, by the utilitarian catching-up criterion two profiles are equally good if and only if the sum of the

[1] Other than preorders—that is, reflexive and transitive binary relations— complete and quasi-transitive relations have been also analyzed. See, for instance, Fleurbaey and Michel (2003) and Sakai (2010).

[2] Throughout this paper we will use the term *utilitarian* to refer to *generalized utilitarian* or *prioritarian* criteria, based on the sum of transformed wellbeing. Our notion of *wellbeing* need not directly correspond to the notion of *utility* as used by philosophers; if not, utility will equal a particular transformation of wellbeing. A strictly concave transformation of utility yields prioritarianism. We will allow any transformations of wellbeing that are finitely non-concave, as defined in Section 2.

difference of transformed wellbeing converges *conditionally* to zero as time goes to infinity.[3] Also, normative significance can be assigned to temporal utilitarian averages, based on an expansionist view of infinite aggregation (Vallentyne and Kagan, 1997; Van Liedekerke and Lauwers, 1997; Khan and Stinchcombe, 2018; Wilkinson, 2020; Pivato, 2021).

The second interpretation is that there exist a fixed and exogenously determined number of individuals in each generation while the number of individuals alive can vary between generations. For instance, the wellbeing profiles $\mathbf{x}$ and $\mathbf{y}$ given by

$$\mathbf{x} = (\underbrace{x_1, \ldots, x_3}_{\text{1st generation}}, \underbrace{x_4, \ldots, x_{10}}_{\text{2nd generation}}, \ldots) \quad \text{and} \quad \mathbf{y} = (\underbrace{y_1, \ldots, y_3}_{\text{1st generation}}, \underbrace{y_4, \ldots, y_{10}}_{\text{2nd generation}}, \ldots)$$

represent a situation where there exist three individuals alive in the 1st generation, seven individuals alive in the 2nd generation, and so on. In this interpretation there is no natural order for counting individuals; in particular, there is no natural order for counting individuals within each generation. This lessens the appeal of the overtaking and catching-up criteria mentioned above. On the other hand, at least if the identities of individuals alive are the same across profiles, then there exists a natural 1-to-1 correspondence between the components of two profiles. The Basu-Mitra utilitarian criterion utilizing this 1-to-1 correspondence might thus still be appealing. Moreover, we can impose that social evaluation of profiles be invariant to the order in which the individuals are counted when the same reordering is used for both streams. Such invariance, which is referred to as *Isomorphism Invariance* (Lauwers and Vallentyne, 2004), *Strong Relative Anonymity* (Asheim, d'Aspremont, and Banerjee, 2010), and the *Permutation Principle* (Askell, 2018), implies Finite Anonymity.[4] However, it rules out evaluation based on temporal averages. Furthermore, utilitarian criteria — like the *Time-Invariant Overtaking* criterion (Asheim, d'Aspremont, and Banerjee, 2010) as well as the principles of *Differential Betterness* and *Differential Indifference* (Lauwers and Vallentyne, 2004) — that satisfy Isomorphism Invariance must be based on *unconditional convergence*: the sum of differences of transformed wellbeing must converge for each possible reordering of components.

Finally, the third interpretation, which this paper allows for, is that there exists a variable number of individuals alive in each generation and that the identities of individuals alive are not necessarily the same across distributions. For instance, wellbeing profiles $\mathbf{x}$ and $\mathbf{y}$ given by

$$\mathbf{x} = (\underbrace{x_1, \ldots, x_3}_{\text{1st generation}}, \underbrace{x_4, \ldots, x_9, x_{10}}_{\text{2nd generation}}, \ldots) \quad \text{and} \quad \mathbf{y} = (\underbrace{y_1, \ldots, y_3, y_4}_{\text{1st generation}}, \underbrace{y_5, \ldots, y_9}_{\text{2nd generation}}, \ldots)$$

represent, respectively, a situation where there exist three individuals alive in the 1st generation, seven individuals alive in the 2nd generation, and so on, and a situation where there exist four

[3] A series is conditionally convergent if it converges, but it need not converge for all possible reorderings.

[4] This implication holds under the axiom of *Finite Completeness* (completeness of a preorder for pairs of profiles which differ only for a finite number of components), see Asheim, d'Aspremont, and Banerjee (2010, Lemma 1).

individuals alive in the 1st generation, five individuals alive in the 2nd generation, and so on. Such pairs of wellbeing profiles arise in economic growth models involving fertility choice. In this case, not only is there no natural order for counting individuals, there is also no natural 1-to-1 correspondence between the components of the two profiles since we cannot easily assume that the identities of individuals alive are the same. Even the Basu-Mitra utilitarian criterion might not seem appealing for comparing such pairs of profiles. One possibility, which we consider in this paper, is to impose that social evaluation of profiles be invariant to the order in which the individuals are counted *even if the reordering applies to one stream only*. Indeed, we will impose the axiom of Strong Anonymity by requiring that a wellbeing profile is deemed equally good as any permutation of it, permitting also permutations that reorder infinitely many components.

The axiom of Strong Anonymity implies the axioms of Finite Anonymity and Isomorphism Invariance, and it is compatible also with the third interpretation. However, this scope of applicability is not by itself a sufficient argument for imposing such a strong impartiality axiom; rather, it requires separate justification. One position, adopted by Zuber and Asheim (2012) and Asheim and Zuber (2013), is to argue that Strong Anonymity is needed in order to properly capture equal treatment.[5] Another position is investigated in our companion paper (Asheim, Kamaga, and Zuber, 2022), where we show how Strong Anonymity for future people follows if future identities are unobservable. Combining such strong impartiality for future people with equal treatment of existing and future people yields Strong Anonymity for all, as imposed in the present paper.

The purpose of this paper is to propose and axiomatize utilitarian criteria for ranking infinite wellbeing profiles satisfying the axiom of Strong Anonymity, thus being neutral with respect to the order in which we compute the utilitarian sum. This implies that the criteria are applicable when considering the third interpretation where population size within each generation might differ. We define a new criterion, the *Strongly Anonymous (Generalized) Utilitarian* social welfare relation, and establish as our main result how Strongly Anonymous Utilitarianism can be characterized by combining Strong Anonymity with other regularity axioms (*Monotonicity*, *Finite Completeness*, and continuity axioms), an equity axiom, a sensitivity axiom, a separability axiom, and a population ethics axiom. By imposing these axioms we restore a 1-to-1 correspondence between reordered streams derived from the profiles as follows: Their components are rank-ordered starting with the worst-off and, if there is only a finite number of wellbeing components that can be rank-ordered in this manner, completed with wellbeings equal to the smallest cluster point. The smallest cluster point of the set of components in a wellbeing profile is the smallest point such that, for every neighborhood, there are infinitely many components of the set within the neighborhood. The smallest cluster point equals the *limit inferior* of the stream obtained from the profile when its components are listed, independently of in which order this listing is made.

However, the axioms do not imply that the sequence of rank-order is a natural order of counting the components of the streams that are derived in this manner. Indeed, Strongly Anonymous

[5]Lauwers (1997) discusses whether Finite Anonymity is strong enough to properly capture equal treatment.

Utilitarianism satisfies the principles of Differential Betterness and Differential Indifference (Lauwers and Vallentyne, 2004) applied to the derived streams. This means that two wellbeing profiles are equally good according to Strongly Anonymous Utilitarianism if and only if the sum of the difference of transformed wellbeing of the derived rank-ordered streams converges *unconditionally* to zero. In particular, conditional convergence according to rank does not suffice.

In a setting with an infinite, but countable, number of wellbeing components, it is impossible to combine Strong Anonymity with the strong Pareto principle, as shown by Van Liedekerke (1995) and Van Liedekerke and Lauwers (1997, p. 163). For example, the stream $(1, 0, 1, 0, \ldots, 1, 0, \ldots)$ can be transformed to the stream $(1, 1, 1, 0, \ldots, 1, 0, \ldots)$ by reordering infinitely many components, implying by Strong Anonymity that the two are equally good. However, by applying the strong Pareto principle component-by-component, the latter is strictly better than the former. Strong Anonymity is even incompatible with the weak Pareto principle (Fleurbaey and Michel, 2003). Asheim, Kamaga, and Zuber (2022) show that, when Strong Anonymity and Monotonicity are combined with the continuity axioms and the equity axiom, there can be sensitivity for an increase of the wellbeing at a particular component of an infinite wellbeing profile if and only if the wellbeing at this component is below the limit inferior. Consequently, our sensitivity axiom is a version of the strong Pareto principle restricted to wellbeing levels below the limit inferior. Thus, it allows the full force of the strong Pareto principle for comparison of profiles where an earlier component never has higher wellbeing than a later.

We study how Strongly Anonymous Utilitarianism relates to four other social welfare relations. First, we define a *Strongly Anonymous (Generalized) Utilitarian Dominance* criterion, which is a strongly anonymous variant of the Basu-Mitra utilitarian criterion. Second, we propose a *Strongly Anonymous (Generalized) Utilitarian Catching-Up* criterion, which is a strongly anonymous variant of the utilitarian caching-up criterion in Gale (1967). Third, we take the limit of *Rank-Discounted Utilitarianism*, as proposed by Zuber and Asheim (2012), when the discount factor approaches one, leading to the *Limit of Rank-Discounted (Generalized) Utilitarianism.* Last, we consider a *Strongly Anonymous (Generalized) Utilitarian Cesàro Summation* criterion, which is a modification of the comparison of Cesàro summation of transformed wellbeings so as to satisfy the Strong Anonymity axiom.

In Section 2, we present the framework of our analysis. In Section 3, we introduce the Strongly Anonymous Utilitarian social welfare relation, as well as the Strongly Anonymous Utilitarian Dominance criterion. We present an axiomatic characterization of the former and relate it to the later (which it extends). In Section 4, we relate the Strongly Anonymous Utilitarian social welfare relation to the three other strongly anonymous utilitarian criteria by showing the links between them in terms of the notions of subrelation and weak extension. In Section 5, we demonstrate the applicability of Strongly Anonymous Utilitarianism in the Ramsey model while, in Section 6, we offer some concluding discussion. Some proofs are relegated to an appendix.

2 Framework and notation

Let $\mathbb{N}$ denote the set of natural numbers $\{1, \ldots, i, \ldots\}$. Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, and $\mathbb{R}_{++}$ the set of positive real numbers (and likewise, write $\mathbb{R}_-$ for the set of non-positive real numbers). Let $\mathbf{x} = (x_1, \ldots, x_i, \ldots)$ denote an infinite stream (or profile), where $x_i \in \mathbb{R}_+$ is a one-dimensional indicator of the wellbeing of the individual at component $i \in \mathbb{N}$. Throughout, we refer to an infinite stream as a profile since we do not commit a specific order of its components in light of what Strong Anonymity requires.

We assume that wellbeing is cardinally measurable and fully comparable. We restrict attention to profiles with bounded wellbeing, and let

$$\mathbf{X} = \left\{ \mathbf{x} = (x_1, \ldots, x_i, \ldots) \in \mathbb{R}^{\mathbb{N}} : \sup_{i \in \mathbb{N}} |x_i| < +\infty \right\}$$

denote the set of possible profiles. For $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, write $\mathbf{x} \geq \mathbf{y}$ whenever $x_i \geq y_i$ for all $i \in \mathbb{N}$; write $\mathbf{x} > \mathbf{y}$ if $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$; and write $\mathbf{x} \gg \mathbf{y}$ whenever $x_i > y_i$ for all $i \in \mathbb{N}$.

For any $n \in \mathbb{N}$ and any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, let $\mathbf{x}_n\mathbf{y}$ denote the profile $\mathbf{z}$ such that $z_i = x_i$ for all $i \leq n$ and $z_i = y_{i-n}$ for all $i > n$, that is, $\mathbf{x}_n\mathbf{y} = (x_1, \ldots, x_n, y_1, y_2, \ldots)$. For any $z \in \mathbb{R}$, write $z\mathbb{1}_{\mathbb{N}} = (z, z, \ldots) \in \mathbf{X}$ for a constant profile. For any $z \in \mathbb{R}$ and $\mathbf{x} \in \mathbf{X}$, let $(z, \mathbf{x})$ denote the profile $(z, x_1, x_2, \ldots)$ and write $(z\mathbb{1}_{\mathbb{N}}, \mathbf{x}) = (z, x_1, z, x_2 \ldots)$. For any $n \in \mathbb{N}$ and $\ell \in \mathbb{R}_+$, write

$$\mathbf{X}_n^\ell = \{\mathbf{x} \in \mathbf{X} : x_i \leq \ell \text{ for all } i \leq n \text{ and } x_j = \ell \text{ for all } j \geq n+1\} \ .$$

A *permutation* π of $\mathbb{N}$ is a bijection on $\mathbb{N}$. Let Π denote the set of all permutations of $\mathbb{N}$. For any $\mathbf{x} \in \mathbf{X}$ and any $\pi \in \Pi$, write $\mathbf{x}_\pi = (x_{\pi(1)}, x_{\pi(2)}, \ldots) \in \mathbf{X}$. Two subsets of $\mathbf{X}$ will be of particular interest. First, we introduce the set $\mathbf{X}^+$ of non-decreasing streams in $\mathbf{X}$. Formally,

$$\mathbf{X}^+ = \{\mathbf{x} \in \mathbf{X} : x_i \leq x_{i+1} \text{ for all } i \in \mathbb{N}\} \, .$$

The other subset of $\mathbf{X}$ playing a key role in the remainder of the paper is the set of profiles which can be reordered into non-decreasing streams. This set is denoted $\mathbf{X}^\uparrow$ and is defined by

$$\mathbf{X}^\uparrow = \{\mathbf{x} \in \mathbf{X} : \text{there exists } \pi \in \Pi \text{ such that } \mathbf{x}_\pi \in \mathbf{X}^+\} \, .$$

By way of illustration, note that $\mathbf{x} = (1, 0, \ldots, 0, \ldots)$ is not non-decreasing and cannot be reordered into a non-decreasing stream, while $\mathbf{y} = (1, 0, 1, \ldots, 1, \ldots)$ is not non-decreasing but can be reordered into the non-decreasing stream $\mathbf{y}_\pi = (0, 1, 1, \ldots, 1, \ldots)$ by letting π permute the first and second components. Hence, of these two profiles, only $\mathbf{y}$ is in $\mathbf{X}^\uparrow$.

For any profile $\mathbf{x} \in \mathbf{X}$, write $\ell(\mathbf{x}) := \liminf_{i \to +\infty} x_i$ for the limit inferior of $\mathbf{x}$, implying that $\ell(\mathbf{x})$ is the smallest cluster point of $\mathbf{x}$. Define the set $L(\mathbf{x})$ by $L(\mathbf{x}) := \{i \in \mathbb{N} : x_i < \ell(\mathbf{x})\}$.

We say that a transformation function $u : \mathbb{R} \to \mathbb{R}$ is *finitely non-concave* whenever[6]

[6] G_u is an index of non-concavity introduced by Chateauneuf, Cohen, and Meilijson (2005).

$$G_u = \sup_{x_1<x_2\leq x_3<x_4} \left[\tfrac{u(x_4)-u(x_3)}{x_4-x_3} \Big/ \tfrac{u(x_2)-u(x_1)}{x_2-x_1} \right] < +\infty.$$

Let U denote the set of continuous, increasing, and finitely non-concave transformation functions. For fixed $u \in U$, define the sequence of functions $v_n^u : \mathbf{X} \to \mathbb{R}$, for $n \in \mathbb{N}$, by, for all $\mathbf{x} \in \mathbf{X}$,

$$v_n^u(\mathbf{x}) = \sum_{i=1}^{n} u(x_i)\,.$$

Hence, v_n^u determines the sum of the transformed wellbeing of the first n components of $\mathbf{x}$. For fixed $u \in U$, define the sequence of functions $w_n^u : \mathbf{X} \to \mathbb{R}$, for $n \in \mathbb{N}$, by, for all $\mathbf{x} \in \mathbf{X}$,

$$w_n^u(\mathbf{x}) = \inf_{\pi\in\Pi} \sum_{i=1}^{n} u(x_{\pi(i)})\,.$$

It is actually the case that w_n^u is the sum of the transformed wellbeing of the n smallest components of $\mathbf{x}$ if $|L(\mathbf{x})| \geq n$, and the sum of the transformed wellbeing of the $|L(\mathbf{x})|$ smallest components of $\mathbf{x}$ plus $(n - |L(\mathbf{x})|)u(\ell(\mathbf{x}))$ otherwise.

For any profile $\mathbf{x} \in \mathbf{X}$, define the derived rank-ordered stream $\mathbf{x}_{[\,]} = (x_{[1]}, \ldots, x_{[r]}, \ldots) \in \mathbf{X}^+$ as follows: Fix $u \in U$ and set $w_0^u(\mathbf{x}) = 0$. Let, for all $r \in \mathbb{N}$,

$$x_{[r]} = u^{-1}\big(w_r^u(\mathbf{x}) - w_{r-1}^u(\mathbf{x})\big)$$

be the wellbeing level of the individual at rank r.[7] Applied to $\mathbf{x} = (1, 0, \ldots, 0, \ldots)$ and $\mathbf{y} = (1, 0, 1, \ldots, 1, \ldots)$, we obtain $\mathbf{x}_{[\,]} = (0, 0, \ldots, 0, \ldots)$ since $L(\mathbf{x})$ is empty, while $\mathbf{y}_{[\,]} = (0, 1, 1, \ldots, 1, \ldots)$. For any $\mathbf{x} \in \mathbf{X}$ and any $\pi \in \Pi$, write $\mathbf{x}_{[\pi]} = (x_{[\pi(1)]}, x_{[\pi(2)]}, \ldots) \in \mathbf{X}^{\uparrow}$; i.e., $\mathbf{x}_{[\pi]}$ is the reordering of $\mathbf{x}_{[\,]}$ obtained by using the permutation π.

Component $[r]$ of the derived stream $\mathbf{x}_{[\,]}$ can be interpreted as individual with 'rank' r when wellbeing is ordered from the lowest level to the highest (below the limit inferior). Hence, $\mathbf{x}_{[\,]}$ is a reordered stream of wellbeing, but where people above the limit inferior are discarded, as these have 'infinite' rank (in the sense that there are infinitely many people with lower wellbeing). Note that, for fixed $u \in U$ and any $\mathbf{x} \in \mathbf{X}$, we have $w_n^u(\mathbf{x}) = v_n^u(\mathbf{x}_{[\,]}) = \sum_{r=1}^{n} u(x_{[r]})$ for all $n \in \mathbb{N}$.

A *social welfare relation* (SWR) on a set $\mathbf{X}$ is a preorder (a reflexive and transitive binary relation) $\succsim$ where, for any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ entails $\mathbf{x}$ is deemed socially at least as good as $\mathbf{y}$. The symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively. An SWR $\succsim'$ *weakly extends* an SWR $\succsim$ if, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim \mathbf{y}$ implies $\mathbf{x} \succsim' \mathbf{y}$. An SWR $\succsim$ is a *subrelation* to an SWR $\succsim'$ if $\succsim'$ weakly extends $\succsim$ and, for all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succ \mathbf{y}$ implies $\mathbf{x} \succ' \mathbf{y}$.

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[7] Equivalently, we can derive $\mathbf{x}_{[\,]}$ as the non-decreasing permuted infinite subsequence of the elements in $L(\mathbf{x})$ if $L(\mathbf{x})$ is infinite, and the non-decreasing permuted finite subsequence of the elements in $L(\mathbf{x})$, followed by $x_{[r]} = \ell(\mathbf{x})$ for $r > |L(\mathbf{x})|$, if $L(\mathbf{x})$ is finite (Zuber and Asheim, 2012, p. 1576).

3 Strongly Anonymous Utilitarianism

As mentioned in the introduction, there are settings with infinitely many people where there is no natural 1-to-1 correspondence between the component of different wellbeing profiles. This is true in particular if population structure depends on the choice of alternatives. Then there is no natural way of computing a utilitarian sum, which is a problem for extending utilitarian criteria to that context. We seek ways of computing the sum that are neutral with respect to how we compute the sum, leading to the novel criterion of Strongly Anonymous Utilitarianism.

3.1 The Strongly Anonymous (Generalized) Utilitarian SWR

To motivate Strongly Anonymous Utilitarianism, consider first the class of *(Generalized) Utilitarian Catching-Up* SWRs (Gale, 1967) (where every $u \in U$ corresponds to a different SWR):[8] There exists $u \in U$ such that, for any profiles $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x}$ is at least as good as $\mathbf{y}$ if and only if

$$\liminf_{n \to +\infty} \big(v_n^u(\mathbf{x}) - v_n^u(\mathbf{y}) \big) \geq 0 \,.$$

The utilitarian cathching-up criterion is not invariant to the order in which the individuals are counted. For a version of the utilitarian catching-up criterion that satisfies Isomorphism Invariance, consider the following definition of the class of *Weak (Generalized) Utilitarian Catching-Up* SWRs (Lauwers and Vallentyne, 2004, p. 322): There exists $u \in U$ such that, for any profiles $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x}$ is at least as good as $\mathbf{y}$ if and only if there exists $\Delta \in \mathbb{R}_+ \cup \{+\infty\}$ such that,

$$\text{for all } \pi \in \Pi \,, \ \lim_{n \to +\infty} \big(v_n^u(\mathbf{x}_\pi) - v_n^u(\mathbf{y}_\pi) \big) = \Delta \,.$$

Through its unconditional convergence, the Weak Utilitarian Catching-Up criterion satisfies the principles of Differential Betterness and Differential Indifference (Lauwers and Vallentyne, 2004). The *Differential Betterness* principle states that, when the sum of the positive differences of transformed wellbeing $\sum_{i \leq n: x_i > y_i} \big(u(x_i) - u(y_i) \big)$ converges to some finite number $\Delta_+ > 0$ and the sum of the negative differences of transformed wellbeing $\sum_{i \leq n: y_i > x_i} \big(u(x_i) - u(y_i) \big)$ converges to some finite number $\Delta_- < 0$, then we should deem $\mathbf{x}$ better than $\mathbf{y}$ whenever $\Delta_+ + \Delta_- > 0$. We should also deem $\mathbf{x}$ better than $\mathbf{y}$ when the sum of positive difference is infinite and the sum of negative difference converges to some finite number. Similarly, the *Differential Indifference* principle states that, when Δ_+ and $|\Delta_-|$ are both finite and equal to each other, we should be indifferent between $\mathbf{x}$ and $\mathbf{y}$. The criterion satisfies these principles since, by Riemann's series theorem, unconditional and absolute convergence are equivalent.

To illustrate the effect of Isomorphism Invariance in this setting, consider $\mathbf{x} = (1, 0, 1, 0, \dots)$

[8] Throughout we consider transformation functions u in the set U, implying that u is not only continuous and increasing, but also finitely non-concave. Finite non-concaveness follows from our axiomatization of Strongly Anonymous Utilitarianism, defined in Definition 2. However, we might also define the various utilitarian criteria without insisting on u being finitely non-concave.

and $\mathbf{y} = (0, 1, 0, 1, \dots)$. Then $\liminf_{n\to+\infty} \big(v_n^u(\mathbf{x}) - v_n^u(\mathbf{y})\big) = 0$ and $\liminf_{n\to+\infty} \big(v_n^u(\mathbf{y}) - v_n^u(\mathbf{x})\big) = u(0) - u(1) < 0$, implying that $\mathbf{x}$ better than $\mathbf{y}$ according to the Utilitarian Catching-Up criterion. However, there exists $\pi \in \Pi$ such that $\mathbf{x}_\pi = (1, 1, 0, 1, 1, 0, \dots)$ and $\mathbf{y}_\pi = (0, 0, 1, 0, 0, 1, \dots)$ so that $v_n^u(\mathbf{x}_\pi) - v_n^u(\mathbf{y}_\pi) \to +\infty$ as $t \to +\infty$. Also, there exists $\pi' \in \Pi$ such that $\mathbf{x}_{\pi'} = (0, 0, 1, 0, 0, 1, \dots)$ and $\mathbf{y}_{\pi'} = (1, 1, 0, 1, 1, 0, \dots)$ so that $v_n^u(\mathbf{x}_{\pi'}) - v_n^u(\mathbf{y}_{\pi'}) \to -\infty$ as $t \to +\infty$. Hence, $\mathbf{x}$ and $\mathbf{y}$ are not comparable according to the Weak Utilitarian Catching-Up criterion. The mathematical reason for why Isomorphism Invariance matters is that the sum of difference between the transformed wellbeing of $\mathbf{x}$ and $\mathbf{y}$ does not unconditionally converge (nor diverge to one of $+\infty$ or $-\infty$). Conceptually, this means that the comparison of $\mathbf{x}$ and $\mathbf{y}$ made by the Utilitarian Catching-Up criterion depends on the order in which the individuals are counted.

Not even Weak Utilitarian Catching-Up allows for the third way of interpreting profiles, where there is no natural 1-to-1 correspondence between the components of different profiles. This interpretation can be permitted by considering strongly anonymous criteria. Asheim, Kamaga, and Zuber (2022) show that Strong Anonymity, when combined with rather weak auxiliary axioms, leads us to disregard wellbeing levels above the limit inferior. This motivates criteria defined on the derived rank-ordered profiles $\mathbf{x}_{[\,]}$ and $\mathbf{y}_{[\,]}$, as exemplified by the following class of SWRs.

Definition 1. *The Strongly Anonymous (Generalized) Utilitarian Catching-Up* SWR $\succsim^C$*.* There exists $u \in U$ such that, for any profiles $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^C \mathbf{y}$ if and only if

$$\liminf_{n\to+\infty} \big(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\big) = \liminf_{n\to+\infty} \big(v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]})\big) \geq 0\,.$$

By the definition of the sequence of functions $w_n^u : \mathbf{X} \to \mathbb{R}$, for $n \in \mathbb{N}$, $\succsim^C$ satisfies the axiom of Strong Anonymity as $\mathbf{x}_{[\,]}$ is not only the rank-ordered profile derived from $\mathbf{x}$ but also the rank-ordered profile derived from $\mathbf{x}_\pi$ for any $\pi \in \Pi$. If $\mathbf{x} = (1, 0, 1, 0, \dots)$ and $\mathbf{y} = (0, 1, 0, 1, \dots)$, which are equally good by Strong Anonymity, then we obtain $\mathbf{x} \sim^C \mathbf{y}$ as $\mathbf{x}_{[\,]} = \mathbf{y}_{[\,]} = (0, 0, 0, 0, \dots)$.

However, our axiomatization will lead us to a criterion that is not only strongly anonymous, but which does not even depend on the order of ranks. In particular, we should not give priority to component [1] by always listing this component first, as $\succsim^C$ does. In analogy to the difference between Utilitarian Catching-Up and Weak Utilitarian Catching-Up, this leads us naturally to the following class of SWRs which add Isomorphism Invariance to the 1-to-1 correspondence between the ranks of the derived streams $\mathbf{x}_{[\,]}$ and $\mathbf{y}_{[\,]}$.

Definition 2. *The Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$*.* There exists $u \in U$ such that, for any profiles $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^U \mathbf{y}$ if and only if there exists $\Delta \in \mathbb{R}_+ \cup \{+\infty\}$ such that

$$\text{for all } \pi \in \Pi\,, \lim_{n\to+\infty} \big(v_n^u(\mathbf{x}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\big) = \Delta\,.$$

If, for all $\pi \in \Pi$, $\lim_{n\to+\infty} \big(v_n^u(\mathbf{x}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\big) = \Delta$, then $\lim_{n\to+\infty} \big(v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]})\big) = \Delta$. Hence, the following result is an immediate consequence of Definitions 1 and 2.

Observation 1. *For any $u \in U$, the Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$ *associated with u is a subrelation to the Strongly Anonymous (Generalized) Utilitarian Catching-Up* SWR $\succsim^C$ *associated with u.*

Let $\mathbf{D} = \{(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{X} : \text{there exists } \delta \in \mathbb{R}_+ \text{ such that } \sum_{i:y_i > x_i}(y_i - x_i) = \delta\}$. Then, as we prove in Appendix A.2, we can characterize Strongly Anonymous Utilitarianism as follows.

Lemma 1. *Let the Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$ *be associated with the transformation function $u \in U$. For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^U \mathbf{y}$ if and only if $(\mathbf{x}_{[\,]}, \mathbf{y}_{[\,]}) \in \mathbf{D}$ and*

$$\lim_{n\to\infty} \left(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\right) \geq 0\,.$$

Strongly Anonymous Utilitarianism can thus be viewed as an application of the principles of Differential Betterness and Differential Indifference, but to the derived rank-ordered profiles $\mathbf{x}_{[\,]}$ and $\mathbf{y}_{[\,]}$.

To illustrate why Strongly Anonymous Utilitarian Catching-Up needs to be modified so as to satisfy Isomorphism Invariance—which we add when moving from this criterion to Strongly Anonymous Utilitarianism—we use an example that is a bit more complicated. Consider the SWRs $\succsim^C$ and $\succsim^U$ defined for the identity transformation function $u(z) = z$ and let the profiles $\mathbf{x}$ and $\mathbf{y}$ be given by (see Example 1 of subsection 4.3 for a formal definition):

$$\begin{aligned}
\mathbf{x} &= (z, \quad \underbrace{\tfrac{2}{3}, \ \tfrac{2}{3}}_{\text{2 components}}, \quad \underbrace{\tfrac{2}{3}, \dots, \tfrac{2}{3}}_{\text{3 components}}, \quad \underbrace{\tfrac{4}{5}, \dots, \tfrac{4}{5}}_{\text{4 components}}, \quad \underbrace{\tfrac{4}{5}, \dots, \tfrac{4}{5}}_{\text{5 components}}, \quad \dots) \\
\mathbf{y} &= (\tfrac{1}{2}, \quad \overbrace{\tfrac{1}{2}, \ \tfrac{1}{2}}, \quad \overbrace{\tfrac{3}{4}, \dots, \tfrac{3}{4}}, \quad \overbrace{\tfrac{3}{4}, \dots, \tfrac{3}{4}}, \quad \overbrace{\tfrac{5}{6}, \dots, \tfrac{5}{6}}, \quad \dots)\,.
\end{aligned}$$

Note that

$$\mathbf{x} - \mathbf{y} = (z - \tfrac{1}{2}, \quad \underbrace{\tfrac{1}{6}, \ \tfrac{1}{6}}_{\text{2 components}} \quad \underbrace{-\tfrac{1}{12}, \dots, -\tfrac{1}{12}}_{\text{3 components}}, \quad \underbrace{\tfrac{1}{20}, \dots, \tfrac{1}{20}}_{\text{4 components}}, \quad \underbrace{-\tfrac{1}{30}, \dots, -\tfrac{1}{30}}_{\text{5 components}}, \quad \dots)\,.$$

Hence, the first component of $\mathbf{x} - \mathbf{y}$ equals $z - \frac{1}{2}$, the two next $\frac{1}{6}$ summing to $\frac{1}{3}$, the three next $-\frac{1}{12}$ summing to $-\frac{1}{4}$, the four next $\frac{1}{20}$ summing to $\frac{1}{5}$, and so on. Hence,

$$\sum_{i=1}^{\infty} \left(x_i - y_i\right) = z - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots = z - 1 + \ln 2\,,$$

implying that $\mathbf{x} \succsim^C \mathbf{y}$ if and only if $z \geq 1 - \ln 2 \approx 0.30685$, since both $\mathbf{x}$ and $\mathbf{y}$ are non-decreasing so that $\mathbf{x} = \mathbf{x}_{[\,]}$ and $\mathbf{y} = \mathbf{y}_{[\,]}$. However, such an alternating harmonic series is not absolute convergent, as each of the series $1, \frac{1}{3}, \frac{1}{5}, \dots$ and $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots$ diverges, and thus not unconditionally convergent by the Riemann series theorem. Hence, $\succsim^U$ deems $\mathbf{x}$ and $\mathbf{y}$ incomparable for any z.

Next, consider the following class of SWRs, which combines a finite application of w_n^u and dominance for the tails of the derived rank-ordered profiles.

Definition 3. *The Strongly Anonymous (Generalized) Utilitarian Dominance* SWR $\succsim^D$. There exists $u \in U$ such that, for any profiles $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^D \mathbf{y}$ if and only there exists $n \in \mathbb{N}$ such that

$$w_n^u(\mathbf{x}) - w_n^u(\mathbf{y}) \geq 0 \quad \text{and} \quad x_{[r]} \geq y_{[r]} \text{ for all } r > n\,.$$

Strongly Anonymous Utilitarian Dominance is related to the Basu-Mitra utilitarian criterion for infinite wellbeing profiles. Basu and Mitra (2007) define their criterion by combining a finite application of classical utilitarianism and dominance for the tails of the profiles $\mathbf{x}$ and $\mathbf{y}$.[9] The main difference between Strongly Anonymous Utilitarian Dominance and the Basu-Mitra utilitarian criterion is that the former satisfies Strong Anonymity as a strong impartiality requirement.

Lemma 1 implies that when, for some $n \in \mathbb{N}$, $w_n^u(\mathbf{x}) \geq w_n^u(\mathbf{y})$ ($w_n^u(\mathbf{x}) > w_n^u(\mathbf{y})$ respectively) and $x_{[r]} \geq y_{[r]}$ for all $r > n$, we have $\mathbf{x} \succsim^U \mathbf{y}$ ($\mathbf{x} \succ^U \mathbf{y}$ respectively) for $\succsim^U$ associated with $u \in U$. We therefore obtain the following result.

Observation 2. *For any $u \in U$, the Strongly Anonymous (Generalized) Utilitarian Dominance* SWR $\succsim^D$ *associated with u is a subrelation to the Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$ *associated with u.*

However, Strongly Anonymous Utilitarian Dominance does not satisfy an intuitive minimal equity condition. In particular, if $\mathbf{x} = \big(0, 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\big)$ and $\mathbf{y} = \big(z, 1, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\big)$, with $z < -1$, and with u being the identity function, we do not have $\mathbf{x} \succsim^D \mathbf{y}$ even though z can be arbitrarily low.[10] This is so because $\mathbf{x}_{[\,]} = \big(0, 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\big)$ and $\mathbf{y}_{[\,]} = \big(z, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \dots\big)$, so that $w_n^u(\mathbf{x}) > w_n^u(\mathbf{y})$ for all $n \in \mathbb{N}$ but $y_{[r]} > x_{[r]}$ for all $r > 1$. This problem does not arise with Strongly Anonymous Utilitarianism; indeed, with u being the identity function, $\mathbf{x} \succ^U \mathbf{y}$ since

$$\text{for all } \pi \in \Pi\,, \lim_{n \to +\infty} \big(v_n^u(\mathbf{x}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\big) = 0 - z - \sum\nolimits_{r=2}^{\infty} \big(\tfrac{1}{2}\big)^{r-1} = -z - 1 > 0\,.$$

It is useful to remark that, independently of u, if $\ell(\mathbf{x}) > \ell(\mathbf{y})$, then $\mathbf{x} \succ^D \mathbf{y}$ and, therefore, $\mathbf{x} \succ^U \mathbf{y}$ because $\succsim^D$ is a subrelation to $\succsim^U$. Indeed, if $\ell(\mathbf{x}) > \ell(\mathbf{y})$, then there exists $n \in \mathbb{N}$ such that, for all $r \geq n$ $x_{[r]} > \ell(\mathbf{y}) \geq y_{[r]}$ so that, for some $m \geq n$ $w_m^u(\mathbf{x}) \geq w_m^u(\mathbf{y})$ and $x_{[r]} > y_{[r]}$ for all $r \geq m$. We also have that $\mathbf{x} \succsim^U \mathbf{y}$ implies $\ell(\mathbf{x}) \geq \ell(\mathbf{y})$.

3.2 Axioms

In this section we provide an axiomatic characterization of the Strongly Anonymous (Generalized) Utilitarian SWR.

[9] See also the transfer-sensitive criteria proposed by Bossert, Sprumont, and Suzumura (2007) and a generalized representation presented by d'Aspremont (2007).

[10] The worst-off component can thus face a very large loss when going from $\mathbf{x}$ to $\mathbf{y}$ while the second component has a small gain, and everyone else is indifferent.

3.2.1 Regularity axioms

The first axiom asserts that an SWR must be complete for profiles that have the same tail.

Axiom (Finite Completeness). For any $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbf{X}$ and any $n \in \mathbb{N}$, either $\mathbf{x}_n\mathbf{z} \succsim \mathbf{y}_n\mathbf{z}$, or $\mathbf{y}_n\mathbf{z} \succsim \mathbf{x}_n\mathbf{z}$, or both.

Next, to present a continuity axiom, we let $d_1 \colon \mathbf{X} \times \mathbf{X} \to \mathbb{R}_+$ be the distance function given by, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$d_1(\mathbf{x}, \mathbf{y}) = \min \left\{ 1, \sum\nolimits_{i=1}^{\infty} |x_i - y_i| \right\}.$$

Using the distance function d_1, we define weak continuity as follows; see Svensson (1980).

Axiom (Weak Continuity). For any $\mathbf{x} \in \mathbf{X}$, the sets $\{\mathbf{y} \in \mathbf{X} : \mathbf{y} \succsim \mathbf{x}\}$ and $\{\mathbf{y} \in \mathbf{X} : \mathbf{x} \succsim \mathbf{y}\}$ are closed in $(\mathbf{X}, d_1)$.

An additional weak continuity axiom applies when combining two profiles with a constant profile. It requires that indifference between two such profiles when combined with constant profiles in the neighborhood of this particular constant profile should be preserved in the limit.

Axiom (Restricted Continuity). For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, any $\ell \in \mathbb{R}$, and any $\underline{\ell} < \bar{\ell}$ such that $\underline{\ell} \le \ell \le \bar{\ell}$, if $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$ for all $\ell' \in (\underline{\ell}, \bar{\ell}) \setminus \{\ell\}$, then $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell \mathbb{1}_{\mathbb{N}}, \mathbf{y})$.

Lastly, we impose that an SWR satisfy the conjunction of the following two axioms when there is no natural 1-to-1 correspondence between the components of different wellbeing profiles.

Axiom (Monotonicity). For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if $\mathbf{x} > \mathbf{y}$, then $\mathbf{x} \succsim \mathbf{y}$.

Axiom (Strong Anonymity). For any $\pi \in \Pi$ and $\mathbf{x} \in \mathbf{X}$, $\mathbf{x} \sim \mathbf{x}_\pi$.

Jointly, they imply that a profile is at least as good as another if some reordering of the one profile makes its wellbeings as high as the other when the comparison is made component-by-component.

These first five axioms are our basic regularity conditions.

Definition 4. An SWR $\succsim$ is *regular* whenever it satisfies Finite Completeness, Weak Continuity, Restricted Continuity, Monotonicity and Strong Anonymity.

3.2.2 An equity axiom

For an SWR $\succsim$, let the set $\mathcal{E}$ be defined as

$$\mathcal{E} = \{\varepsilon \in \mathbb{R}_{++} : \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbf{X} \text{ with } y_i < x_i \le x_j < y_j \text{ for some } i, j \in \mathbb{N}, \text{ and } x_k = y_k \text{ for all } k \in \mathbb{N} \setminus \{i, j\}, \mathbf{x} \succsim \mathbf{y} \text{ whenever } x_j - y_j = \varepsilon\,(y_i - x_i)\},$$

The set $\mathcal{E}$ considers conditions under which a transfer from rich to poor people is acceptable. *Hammond equity* (Hammond, 1976) claims that any loss for the rich is acceptable, so that $\mathcal{E} =$

$\mathbb{R}_{++}$. On the other hand, an SWR $\succsim$ satisfies the *Pigou-Dalton Transfer* principle (in its weak version) only if $1 \in \mathcal{E}$. Indeed, non-leaky transfers are always welfare improving according to the Pigou-Dalton Transfer principle. Last, utilitarian SWRs (with u being the identity function) are such that $(0, 1] \in \mathcal{E}$. Hence, in all these cases, $\mathcal{E}$ is nonempty. Here we suggest a much weaker axiom than both Hammond equity and Pigou-Dalton, and which generalized utilitarianism (in the case of finite profiles) satisfies for any transformation function $u \in U$: We require that transfers from rich to poor people are acceptable if the wellbeing loss for the rich is sufficiently small.

Axiom (Limited Inequity). $\mathcal{E} \neq \emptyset$.

As discussed at the end of subsection 3.1, the Strongly Anonymous Utilitarian Dominance SWR does not satisfy this minimal equity condition. It is thus excluded from our discussion although it is a subrelation to the other criteria that we will study.

3.2.3 A sensitivity axiom

If increased wellbeing for some is deemed better provided that no-one's wellbeing is decreased, then we adopt the Pareto principle. However, for choices that affect who will exist or not, so that different people exist in different wellbeing profiles, we cannot apply the Pareto principle *stricto-sensu*. We need to resort to sensitivity principles where the bearer of wellbeing at one component in one profile is different from the bearer of wellbeing at the same component in another profile.

Asheim, Kamaga, and Zuber (2022, Proposition 5) show that, if we impose the regularity and equity axioms mentioned above, sensitivity is necessarily limited. Actually, the maximal extent of sensitivity when increasing one component of the profile is given by the following axiom.

Axiom (Liminf-Restricted Dominance). For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ with $x_i > y_i$ for some $i \in \mathbb{N}$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$, $\mathbf{x} \succ \mathbf{y}$ whenever $\ell(\mathbf{x}) > y_i$.

Liminf-Restricted Dominance requires that the strong Pareto principle be satisfied when increasing the wellbeing of people below the limit inferior.

When an SWR satisfies Monotonicity, Strong Anonymity, and Liminf-Restricted Dominance, it clearly satisfies a strong Pareto principle on the domain $\mathbf{X}^{\uparrow}$ of all profiles that can be permuted into non-decreasing streams. This is the principle that Zuber and Asheim (2012) named *Restricted Strong Pareto*.

3.2.4 A separability axiom

Separability principles guarantee that only affected people matter when comparing two profiles, implying that we can ignore the wellbeing of people not affected by a choice. This is particularly convenient in an intertemporal context where the available alternatives never affect the wellbeing of past people. The following separability axiom asserts that evaluation must be independent of unconcerned people, provided the wellbeing of affected people is below the limit inferior; in order words, when people that matter for the choice are at finite rank.

Axiom (Liminf-Restricted Finite Separability). For any $\mathbf{x}$, $\mathbf{y}$, $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}} \in \mathbf{X}$, if there exists a finite subset $N \subset \mathbb{N}$ such that:

(i) $x_i = \tilde{x}_i < \min\{\ell(\mathbf{x}), \ell(\tilde{\mathbf{x}})\}$ for all $i \in N$,

(ii) $y_i = \tilde{y}_i < \min\{\ell(\mathbf{y}), \ell(\tilde{\mathbf{y}})\}$ for all $i \in N$,

(iii) $x_j = y_j$ and $\tilde{x}_j = \tilde{y}_j$ for all $j \in \mathbb{N} \setminus N$,

then $\mathbf{x} \succsim \mathbf{y}$ if and only if $\tilde{\mathbf{x}} \succsim \tilde{\mathbf{y}}$.

3.2.5 A population ethics axiom

Our last axiom relates to the effect of adding one person (or infinitely many people) to a population. This kind of question has been addressed in the literature on population ethics stemming from Parfit (1984) and discussed at length in Blackorby, Bossert, and Donaldson (2005). In particular, Blackorby and Donaldson (1984) and Blackorby, Bossert, and Donaldson (1995) argue in favor of the existence of a critical level, such that a person's life contributes positively to the value of a population if and only if the wellbeing of the additional person is above this critical level. In general, the critical level may depend on how wellbeing is distributed.

The following axiom imposes some regularity in the level of the critical level for infinite populations. The axioms asserts that, if it is acceptable to add one person at some level of wellbeing to a population, then it is also acceptable to add infinitely many people at this level.

Axiom (Critical-Level Consistency). For any $\mathbf{x} \in \mathbf{X}$, and any $z \in \mathbb{R}$, $\mathbf{x} \succsim (z, \mathbf{x})$ (resp. $\mathbf{x} \precsim (z, \mathbf{x})$) if and only if $\mathbf{x} \succsim (z\mathbb{1}_{\mathbb{N}}, \mathbf{x})$ (resp. $\mathbf{x} \precsim (z\mathbb{1}_{\mathbb{N}}, \mathbf{x})$).

3.3 The characterization result

It is our main result that the four axioms of subsections 3.2.2–3.2.5, combined with the regularity axioms of subsection 3.2.1, are sufficient to characterize Strongly Anonymous Utilitarianism.

Theorem 1. *The Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$ *is regular. Consider any regular* SWR $\succsim$*. There exists $u \in U$ such that the Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$ *associated with u is a subrelation to $\succsim$ if and only if $\succsim$ satisfies Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-Level Consistency.*

We establish in Appendix A.3 the result that $\succsim^U$ is regular as well as the only-if statement of the theorem. Hence, it remains to be shown that there exists $u \in U$ such that $\succsim^U$ associated with u is a subrelation to any regular SWR satisfying our axioms. To do so, we need three lemmas. We begin with a lemma which shows that any regular SWR satisfying our axioms applies (generalized) utilitarianism to the heads of profiles if the profiles have the same constant tail.

Lemma 2. *If a regular* SWR $\succsim$ *satisfies Limited Inequity, Liminf-Restricted Dominance, and Liminf-Restricted Finite Separability, then there exists* $u \in U$ *such that for all* $\ell \in \mathbb{R}$ *and all* $n \in \mathbb{N}$*, and all* $\mathbf{x}$*,* $\mathbf{y} \in \mathbf{X}_n^\ell$*,*

$$\mathbf{x} \succsim \mathbf{y} \quad \textit{if and only if} \quad \sum_{i=1}^{n} u(x_i) \geq \sum_{i=1}^{n} u(y_i)\,. \tag{1}$$

Proof. See Appendix A.3 for the proof, which is based on the application of the additive representation theorem of Debreu (1960) on $\mathbf{X}_n^\ell$. □

By the next lemma, some of our axioms imply that any profile $\mathbf{x}$ is equally good as the derived reordered profile $\mathbf{x}_{[\,]}$, where components that are rank-ordered, starting with the worst-off.

Lemma 3. *If a regular* SWR $\succsim$ *satisfies Limited Inequity and Critical-Level Consistency, then, for all* $\mathbf{x} \in \mathbf{X}$*,* $\mathbf{x} \sim \mathbf{x}_{[\,]}$*.*

Proof. Assume that $\succsim$ is a regular SWR which satisfies Limited Inequity and Critical-Level Consistency. If $\mathbf{x} \in \mathbf{X}^{\uparrow}$, then the result follows from Strong Anonymity, because $\succsim$ is regular.

Let $\mathbf{x} \notin \mathbf{X}^{\uparrow}$, and consider $\tilde{\mathbf{x}}$ defined as follows: $\tilde{x}_i = x_i$ if $i \in L(\mathbf{x})$ and $\tilde{x}_i = \ell(\mathbf{x})$ if $i \in \mathbb{N} \setminus L(\mathbf{x})$. Since $\succsim$ is regular (and therefore satisfies Weak Continuity, Restricted Continuity, Monotonicity, and Strong Anonymity), and satisfies Limited Inequity and Critical-Level Consistency, Asheim, Kamaga, and Zuber (2022, Proposition 6) implies that, for all $\mathbf{y}$, $\tilde{\mathbf{y}} \in \mathbf{X}$ such that $\mathbf{y} \geq \tilde{\mathbf{y}}$, if $\ell(\mathbf{y}) = \ell(\tilde{\mathbf{y}}) = \ell$ and $\tilde{y}_i \geq \ell$ for all $i \in \mathbb{N}$ such that $y_i > \tilde{y}_i$, then $\mathbf{y} \sim \tilde{\mathbf{y}}$. Thus, we obtain that $\mathbf{x} \sim \tilde{\mathbf{x}}$. By transitivity of $\succsim$, it remains to be shown that $\tilde{\mathbf{x}} \sim \mathbf{x}_{[\,]}$. There are three cases.

Case 1: $|L(\mathbf{x})| = K < +\infty$. By Strong Anonymity, $\tilde{\mathbf{x}} \sim \mathbf{x}_{[\,]}$ since $\mathbf{x}_{[\,]}$ is a permutation of $\tilde{\mathbf{x}}$.

Case 2: $|L(\mathbf{x})| = +\infty$ *and* $|\mathbb{N} \setminus L(\mathbf{x})| = +\infty$. The profile $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{x}_{[\,]})$ is a permutation of $\tilde{\mathbf{x}}$, so that $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{x}_{[\,]}) \sim \tilde{\mathbf{x}}$ by Strong Anonymity. Since $\succsim$ satisfies Weak continuity and Strong anonymity, Asheim, Kamaga, and Zuber (2022, Lemma 5) implies that, for all $\mathbf{y} \in \mathbf{X}$, $\mathbf{y} \sim (\ell, \mathbf{y}) \sim (\ell \mathbb{1}_{\mathbb{N}}, \mathbf{y})$, where $\ell = \ell(\mathbf{y})$. Thus, $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{x}_{[\,]}) \sim \mathbf{x}_{[\,]}$ and, by transitivity of $\succsim$, $\tilde{\mathbf{x}} \sim \mathbf{x}_{[\,]}$.

Case 3: $|L(\mathbf{x})| = +\infty$ *and* $|\mathbb{N} \setminus L(\mathbf{x})| = K < +\infty$. The following profile is a permutation of $\tilde{\mathbf{x}}$: $\mathbf{y}$ such that $y_i = \ell$ for $i = 1, \cdots, K$ and $y_i = x_{[i-K]}$ for $i \geq K+1$. Therefore, by Strong Anonymity, $\mathbf{y} \sim \tilde{\mathbf{x}}$. Repeated application of Asheim, Kamaga, and Zuber (2022, Lemma 5) implies $\mathbf{y} \sim \mathbf{x}_{[\,]}$ and, by transitivity of $\succsim$, $\tilde{\mathbf{x}} \sim \mathbf{x}_{[\,]}$. □

The next lemma shows that the Strongly Anonymous Utilitarian Dominance SWR is a subrelation to any SWR satisfying all our axioms.

Lemma 4. *Consider any regular* SWR $\succsim$*. There exists* $u \in U$ *such that the Strongly Anonymous (Generalized) Utilitarian Dominance* SWR $\succsim^D$ *associated with* u *is a subrelation to* $\succsim$ *if* $\succsim$ *satisfies Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-Level Consistency.*

Proof. Assume that $\succsim$ is a regular SWR which satisfies Limited Inequity, Liminf-Restricted Dominance, and Liminf-Restricted Finite Separability. By Lemma 2, there exists $u \in U$ such that for all $\ell \in \mathbb{R}$ and all $n \in \mathbb{N}$ and for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n^\ell$, $\mathbf{x} \succsim \mathbf{y}$ if and only if $\sum_{i=1}^n u(x_i) \geq \sum_{i=1}^n u(y_i)$. Let $\succsim^D$ be the Strongly Anonymous (Generalized) Utilitarian Dominance SWR associated with u.

Step 1: Assume that $\mathbf{x} \succ^D \mathbf{y}$. Then there exists $n \in \mathbb{N}$ such that $\sum_{r=1}^n u(x_{[r]}) > \sum_{r=1}^n u(y_{[r]})$ and $x_{[r]} \geq y_{[r]}$ for all $r > N$. Define $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\tilde{\mathbf{y}} \in \mathbf{X}$ by

$$\begin{aligned}
\hat{\mathbf{x}} &= (x_{[1]}, \ldots, x_{[n]}, x_{[n+1]}, x_{[n+1]}, \ldots)\,, \\
\hat{\mathbf{y}} &= (y_{[1]}, \ldots, y_{[n]}, x_{[n+1]}, x_{[n+1]}, \ldots)\,, \\
\tilde{\mathbf{y}} &= (y_{[1]}, \ldots, y_{[n]}, x_{[n+1]}, x_{[n+2]}, \ldots)\,.
\end{aligned}$$

We obtain that $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbf{X}_n^\ell$, where $\ell = x_{[n+1]}$ and $\sum_{i=1}^n u(\hat{x}_i) > \sum_{i=1}^n u(\hat{y}_i)$ so that $\hat{\mathbf{x}} \succ \hat{\mathbf{y}}$ by Lemma 2. Since $\succsim$ satisfies Liminf-Restricted Finite Separability, this is equivalent to $\mathbf{x}_{[\,]} \succ \tilde{\mathbf{y}}$. Furthermore, by Monotonicity, $\tilde{\mathbf{y}} \succsim \mathbf{y}_{[\,]}$ and, by transitivity, $\mathbf{x}_{[\,]} \succ \mathbf{y}_{[\,]}$. By Lemma 3 and transitivity of $\succsim$, this implies that $\mathbf{x} \succ \mathbf{y}$.

Step 2: Assume that $\mathbf{x} \sim^D \mathbf{y}$. Then there exists $n \in \mathbb{N}$ such that $\sum_{r=1}^n u(x_{[r]}) = \sum_{r=1}^n u(y_{[r]})$ and $x_{[r]} = y_{[r]}$ for all $r > n$. We can define $\hat{\mathbf{x}}$, and $\hat{\mathbf{y}} \in \mathbf{X}$ like in Step 1, so that $\hat{\mathbf{x}}$, $\hat{\mathbf{y}} \in \mathbf{X}_n^\ell$, where $\ell = x_{[n+1]}$, $\sum_{i=1}^n u(\hat{x}_i) = \sum_{i=1}^n u(\hat{y}_i)$, and $\hat{\mathbf{x}} \sim \hat{\mathbf{y}}$ by Lemma 2. Since $\succsim$ satisfies Liminf-Restricted Finite Separability, this is equivalent to $\mathbf{x}_{[\,]} \sim \mathbf{y}_{[\,]}$. By Lemma 3 and transitivity of $\succsim$, this implies that $\mathbf{x} \sim \mathbf{y}$. □

Lemma 4 shows that all regular SWRs satisfying our axiom must agree with the Strongly Anonymous Utilitarian Dominance SWR $\succsim^D$. As discussed in subsection 3.1, $\succsim^D$ does not satisfies Limited Inequity. However, by Lemmas 2–4 and the results of Appendix A.3, we can characterize the Strongly Anonymous Utilitarian SWR $\succsim^U$ as the minimal regular SWR satisfying our axioms.

Proof of the if-statement of Theorem 1. Assume that $\succsim$ is a regular SWR which satisfies Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-level consistency. By Lemma 4, there exists $u \in U$ such that the Strongly Anonymous (Generalized) Utilitarian Dominance SWR $\succsim^D$ associated with u is a subrelation to $\succsim$. Let $\succsim^U$ be the Strongly Anonymous (Generalized) Utilitarian SWR associated with u. It suffices to show that $\succsim^U$ is a subrelation to $\succsim$.

Part 1: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^U \mathbf{y}$ *implies* $\mathbf{x} \succsim \mathbf{y}$. Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ be such that $\mathbf{x} \succsim^U \mathbf{y}$. By Lemma 3, $\mathbf{y}_{[\,]} \sim \mathbf{y}$. Hence, by transitivity, it suffices to show that $\mathbf{x} \succsim \mathbf{y}_{[\,]}$. Our strategy of proof is to construct a sequence $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ such that $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ converges to $\mathbf{y}_{[\,]}$ in d_1 and such that $\mathbf{x} \succsim \mathbf{y}^n$ for each $n \in \mathbb{N}$. Since $\succsim$ satisfies Weak Continuity, we thus obtain $\mathbf{x} \succsim \mathbf{y}_{[\,]}$.

Construction of the sequence $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$. Since $\mathbf{x} \succsim^U \mathbf{y}$, it follows from Lemma 1 that $(\mathbf{x}_{[\,]}, \mathbf{y}_{[\,]}) \in \mathbf{D}$. Hence, there exists $\delta \in \mathbb{R}_+$ such that $\sum_{r=1}^\infty \left[y_{[r]} - \min\{x_{[r]}, y_{[r]}\}\right] = \sum_{r: y_{[r]} > x_{[r]}} (y_{[r]} - x_{[r]}) = \delta$.

Let $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ be the sequence defined by, for each $n \in \mathbb{N}$,

$$\begin{aligned} y_1^n &= \; u^{-1}\big(u(y_{[1]}) + \min\big\{0, \inf_{k\geq n}\big(v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})\big)\big\}\big) \;, \\ y_r^n &= \; y_{[r]} && \text{for } r \in \{2, 3, \dots, n-1, n\}\,, \\ y_r^n &= \; \min\{x_{[r]}, y_{[r]}\} && \text{for } r \in \{n+1, n+2, \dots\}\,. \end{aligned}$$

Note that $\mathbf{y}^n$ is well-defined since $u \in U$ is finitely non-concave and therefore unbounbed below.

The sequence $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ *converges to* $\mathbf{y}_{[\,]}$ *in* d_1. By the construction of $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$, we have that

$$\begin{aligned} d_1(\mathbf{y}_{[\,]}, \mathbf{y}^n) \;\; &\leq \;\; y_{[1]} - y_1^n + \sum\nolimits_{r=n+1}^{\infty} \big[y_{[r]} - \min\{x_{[r]}, y_{[r]}\}\big] \\ &= \;\; y_{[1]} - y_1^n + \delta - \sum\nolimits_{r=1}^{n} \big[y_{[r]} - \min\{x_{[r]}, y_{[r]}\}\big]. \end{aligned}$$

Moreover, $\inf_{k\geq n}\big(v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})\big) \leq v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]})$ and, since $\mathbf{x} \succsim^U \mathbf{y}$,

$$\lim_{n\to\infty}\left(\inf_{k\geq n}\big(v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})\big)\right) = \lim_{n\to\infty}\big(v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]})\big) \geq 0\,.$$

Hence, $\lim_{n\to\infty} y_1^n = y_{[1]}$. By the definition of δ we obtain that $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ converges to $\mathbf{y}_{[\,]}$ in d_1.

$\mathbf{x} \succsim \mathbf{y}^n$ *for each* $n \in \mathbb{N}$. Consider any $n \in \mathbb{N}$. Note first that

$$\begin{aligned} v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}^n) \;\; &= \;\; v_n^u(\mathbf{x}_{[\,]}) - u(y_1^n) - \sum\nolimits_{r=2}^{n} u(y_{[r]}) \\ &\geq \;\; v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]}) - \inf_{k\geq n}\big(v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})\big) \geq 0. \end{aligned}$$

Also remark that, by definition, $y_r^n = \min\{x_{[r]}, y_{[r]}\} \leq x_{[r]}$ for all $r > n$. Hence, $\mathbf{x} \succsim^D \mathbf{y}^n$. By Lemma 4, this implies $\mathbf{x} \succsim \mathbf{y}^n$.

Part 2: For any $\mathbf{x}, \mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succ^U \mathbf{y}$ *implies* $\mathbf{x} \succ \mathbf{y}$. If $\mathbf{x} \succ^U \mathbf{y}$, then there exists $\Delta \in \mathbb{R}_{++} \cup \{\infty\}$ such that $\lim_{n\to\infty}\big(v_n^u(x_{[\,]}) - v_n^u(y_{[\,]})\big) = \Delta$. Hence, we can find $\tilde{\Delta} \in \mathbb{R}_{++}$ with $\tilde{\Delta} < \Delta$. Define $\tilde{x} \in \mathbb{R}$ by $\tilde{x} = u^{-1}\big(u(x_{[1]}) - (\Delta - \tilde{\Delta})\big)$ if $\Delta \in \mathbb{R}_{++}$ and $\tilde{x} = u^{-1}\big(u(x_{[1]}) - \tilde{\Delta}\big)$ if $\Delta = \infty$ so that, in any case, $\tilde{x} < x_{[1]}$. If $\min_{i\in\mathbb{N}} x_i$ is well-defined and $j \in \mathbb{N}$ is such that $x_j = \min_{i\in\mathbb{N}} x_i$, define $\tilde{\mathbf{x}} \in \mathbf{X}$ by $\tilde{x}_j = \tilde{x}$ and $\tilde{x}_i = x_i$ for all $i \neq j$. Alternatively, if $\min_{i\in\mathbb{N}} x_i$ is not well-defined, define $\tilde{\mathbf{x}} \in \mathbf{X}$ by $\tilde{x}_1 = \tilde{x}$ and $\tilde{x}_i = x_i$ for all $i \geq 2$. In both cases, $\mathbf{x} \succ \tilde{\mathbf{x}}$ since $\succsim$ satisfies Liminf-Restricted Dominance. Hence, by transitivity of $\succsim$ and part 1 of this proof, to establish $\mathbf{x} \succ \mathbf{y}$ it suffices to show that $\tilde{\mathbf{x}} \succsim^U \mathbf{y}$. This follows since, by construction, either, for all $\pi \in \Pi$, $\lim_{n\to\infty}\big(v_n^u(\tilde{\mathbf{x}}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\big) = \tilde{\Delta}$ or, for all $\pi \in \Pi$, $\lim_{n\to\infty}\big(v_n^u(\tilde{\mathbf{x}}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\big) = \infty$. □

4 Extensions of Strongly Anonymous Utilitarianism

The Strongly Anonymous Utilitarian SWR is related to strongly anonymous versions of utilitarian criteria proposed in the literature. In this section, we show that it is a subrelation to the Strongly Anonymous Utilitarian Catching-Up SWR, as given in Definition 1 of subsection 3.1, as well as the Limit of Rank-Discounted Utilitarian SWR and the Strongly Anonymous Cesàro Summation SWR to be defined in subsection 4.2. We also establish the links between the three latter SWR.

4.1 Characterizing Strongly Anonymous Catching-Up

Note that, for any $u \in U$, the asymmetric and symmetric parts of the Strongly Anonymous (Generalized) Utilitarian SWR $\succsim^C$ associated with u are given as follows: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succ^C \mathbf{y} \text{ if and only if } \limsup_{n\to+\infty} \big(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\big) > 0 \text{ and } \liminf_{n\to+\infty} \big(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\big) \geq 0,$$

$$\mathbf{x} \sim^C \mathbf{y} \text{ if and only if } \lim_{n\to+\infty} \big(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\big) = 0.$$

We characterize $\succsim^C$ by applying an additional axiom.

Axiom (Restricted Tail Continuity). *For any* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}^+$*, if there exists* $m > 0$ *such that, for all* $k \geq m$, $\mathbf{x}_k(z_k\mathbb{1}_\mathbb{N}) \succsim \mathbf{y}_k(z_k\mathbb{1}_\mathbb{N})$*, where* $z_k = \max\{x_k, y_k\}$*, then* $\mathbf{x} \succsim \mathbf{y}$.

This allows us to obtain the following characterization result.

Proposition 1. *The Strongly Anonymous (Generalized) Utilitarian Cathching-Up* SWR $\succsim^C$ *is regular and satisfies Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, Critical-Level Consistency, and Restricted Tail Continuity. If* $\succsim$ *is a regular* SWR *satisfying Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, Critical-Level Consistency, and Restricted Tail Continuity, then there exists* $u \in U$ *such that* $\succsim$ *weakly extends the Strongly Anonymous (Generalized) Utilitarian Cathching-Up* SWR $\succsim^C$ *associated with* u.

Proof. Part 1: $\succsim^C$ *is regular and satisfies the axioms.* By Observations 1 and 2, $\succsim^C$ is a subrelation to $\succsim^U$ and $\succsim^D$. Therefore, the proof of regularity of $\succsim^C$ is very similar to the the proof of regularity of $\succsim^U$ in Proposition 5 of Appendix A.3. Also, by Theorem 1, $\succsim^C$ satisfies Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-Level Consistency.

It remains to be shown that, for any $u \in U$, $\succsim^C$ associated with u satisfies Restricted Tail Continuity. Consider any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}^+$ for which there exists $m > 0$ such that, for all $k \geq m$, $\mathbf{x}_k(z_k\mathbb{1}_\mathbb{N}) \succsim^C \mathbf{y}_k(z_k\mathbb{1}_\mathbb{N})$, where $z_k = \max\{x_k, y_k\}$. By definition, $\mathbf{x}_k(z_k\mathbb{1}_\mathbb{N}) \succsim^C \mathbf{y}_k(z_k\mathbb{1}_\mathbb{N})$ means that $\liminf_{n\to+\infty} \big(w_n^u(\mathbf{x}_k(z_k\mathbb{1}_\mathbb{N})) - w_n^u(\mathbf{y}_k(z_k\mathbb{1}_\mathbb{N}))\big) = v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]}) \geq 0$. So there exists $m > 0$

such that, for all $k \geq m$, $v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]}) = w_k^u(\mathbf{x}) - w_k^u(\mathbf{y}) \geq 0$. Therefore, it is necessarily the case that $\liminf_{n\to+\infty} \big(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\big) \geq 0$ and, thus, $\mathbf{x} \succsim^C \mathbf{y}$.

Part 2: A regular $\succsim$ satisfying the axioms weakly extends $\succsim^C$. Assume that $\succsim$ is a regular SWR satisfying Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, Critical-Level Consistency, and Restricted Tail Continuity. By Lemma 4, there exists $u \in U$ such that the Strongly Anonymous (Generalized) Utilitarian Dominance SWR $\succsim^D$ associated with u is a subrelation to $\succsim$. Let $\succsim^C$ be the Strongly Anonymous (Generalized) Utilitarian Catching-Up SWR associated with u, and let $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ be such that $\mathbf{x} \succsim^C \mathbf{y}$. We want to prove that $\mathbf{x} \succsim \mathbf{y}$. By Lemma 3, $\mathbf{x}_{[\,]} \sim \mathbf{x}$ and $\mathbf{y}_{[\,]} \sim \mathbf{y}$. By transitivity, it suffices to show that $\mathbf{x}_{[\,]} \succsim \mathbf{y}_{[\,]}$. Our strategy of proof is to construct a sequence $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ such that $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ converges to $\mathbf{y}_{[\,]}$ in d_1 and such that $\mathbf{x}_{[\,]} \succsim \mathbf{y}^n$ for each $n \in \mathbb{N}$. Since $\succsim$ satisfies Weak Continuity, we thus obtain $\mathbf{x}_{[\,]} \succsim \mathbf{y}_{[\,]}$.

Construction of the sequence $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$. Let $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ be defined by, for each $n \in \mathbb{N}$,

$$\begin{aligned} y_1^n &= u^{-1}\big(u(y_{[1]}) + \min\big\{0, \inf_{k\geq n}\big(v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})\big)\big\}\big)\ , \\ y_r^n &= y_{[r]} \qquad\qquad \text{for all } r > 1\,. \end{aligned}$$

Note that $\mathbf{y}^n$ is well-defined since $u \in U$ is finitely non-concave and therefore unbounbed below.

The sequence $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ converges to $\mathbf{y}_{[\,]}$ in d_1. By the construction of $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$, $d_1(\mathbf{y}_{[\,]}, \mathbf{y}^n) \leq y_{[1]} - y_1^n$. Moreover, $\inf_{k\geq n}\big(v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})\big) \leq v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})$ for all $k \geq n$ and, since $\mathbf{x} \succsim^C \mathbf{y}$,

$$\lim_{n\to\infty}\left(\inf_{k\geq n}\big(v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})\big)\right) = \liminf_{n\to\infty}\big(v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]})\big) \geq 0\,.$$

Hence, $\lim_{n\to\infty} y_1^n = y_{[1]}$, and we obtain that $\{\mathbf{y}^n\}_{n\in\mathbb{N}}$ converges to $\mathbf{y}_{[\,]}$ in d_1.

$\mathbf{x}_{[\,]} \succsim \mathbf{y}^n$ for each $n \in \mathbb{N}$. Consider any $n \in \mathbb{N}$. Note first that, for each $k \geq n$,

$$\begin{aligned} v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}^n) &= v_k^u(\mathbf{x}_{[\,]}) - u(y_1^n) - \sum\nolimits_{r=2}^{k} u(y_{[r]}) \\ &\geq v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]}) - \inf_{k\geq n}\big(v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}_{[\,]})\big) \geq 0. \end{aligned}$$

For each $k \geq n$, write $z_k = \max\{x_{[k]}, y_k^n\}$, $\tilde{\mathbf{x}}^k = (\mathbf{x}_{[\,]})_k(z_k\mathbb{1}_{\mathbb{N}})$, and $\tilde{\mathbf{y}}^{n,k} = (\mathbf{y}^n)_k(z_k\mathbb{1}_{\mathbb{N}})$. Then

$$v_k^u(\tilde{\mathbf{x}}^k) - v_k^u(\tilde{\mathbf{y}}^{n,k}) = v_k^u(\mathbf{x}_{[\,]}) - v_k^u(\mathbf{y}^n) \geq 0$$

and $\tilde{x}_{[r]}^k = \tilde{y}_{[r]}^{n,k}$ for all $r > n$. Hence, $\tilde{\mathbf{x}}^k \succsim^D \tilde{\mathbf{y}}^{n,k}$. By Lemma 4, this implies $\tilde{\mathbf{x}}^k \succsim \tilde{\mathbf{y}}^{n,k}$. Since $\mathbf{x}_{[\,]}, \mathbf{y}^n \in \mathbf{X}^+$ and, for each $k \geq n$, $(\mathbf{x}_{[\,]})_k(z_k\mathbb{1}_{\mathbb{N}}) \succsim (\mathbf{y}^n)_k(z_k\mathbb{1}_{\mathbb{N}})$, it follows from Restricted Tail Continuity that $\mathbf{x}_{[\,]} \succsim \mathbf{y}^n$. □

4.2 Two other extensions of strongly anonymous utilitarianism

The class of Rank-Discounted Utilitarian SWRs introduced by Zuber and Asheim (2012) is designed to satisfy the axiom of Strong Anonymity while retaining the usual discounted utilitarian form. For any transformation function $u \in U$ and any rank-discount factor $\beta \in (0,1)$, the Rank-Discounted (Generalized) Utilitarian SWR associated with u is characterized by the social welfare function $\rho^u_\beta : \mathbf{X} \to \mathbb{R}$ defined by, for all $\mathbf{x} \in \mathbf{X}$,[11]

$$\rho^u_\beta(\mathbf{x}) = \inf_{\pi \in \Pi} \sum_{i=1}^{\infty} \beta^{i-1} u(x_{\pi(i)}) = \sum_{r=1}^{\infty} \beta^{r-1} u(x_{[r]})\,.$$

By letting the discount factor β approach 1, Rank-Discounted Utilitarianism can be used to define the following class of strongly anonymous (generalized) utilitarian SWRs.

Definition 5. *The Limit of Rank-Discounted (Generalized) Utilitarian* SWR $\succsim^R$. There exists $u \in U$ such that, for any profiles $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^R \mathbf{y}$ if and only if

$$\liminf_{\beta \to 1^-} \left(\rho^u_\beta(\mathbf{x}) - \rho^u_\beta(\mathbf{y})\right) \geq 0\,.$$

This SWR is related to other existing criteria. Asheim and Zuber (2013) consider the limit of rank-discounting when β approaches 0, while Jonsson and Voorneveld (2018) consider the limit of time-discounting when β approaches 1.

For any $u \in U$, the asymmetric and symmetric parts of $\succsim^R$ associated with u are given as follows: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succ^R \mathbf{y} \text{ if and only if } \liminf_{\beta \to 1^-} \left(\rho^u_\beta(\mathbf{x}) - \rho^u_\beta(\mathbf{y})\right) \geq 0 \text{ and } \limsup_{\beta \to 1^-} \left(\rho^u_\beta(\mathbf{x}) - \rho^u_\beta(\mathbf{y})\right) > 0,$$
$$\mathbf{x} \sim^R \mathbf{y} \text{ if and only if } \lim_{\beta \to 1^-} \left(\rho^u_\beta(\mathbf{x}) - \rho^u_\beta(\mathbf{y})\right) = 0.$$

We next introduce a class of strongly anonymous (generalized) utilitarian SWRs based on the notion of Cesàro summation. For any $u \in U$ and any $n \in \mathbb{N}$, define strongly anonymous (generalized) utilitarian partial Cesàro summation $\sigma^u_n : \mathbf{X} \to \mathbb{R}$ by, for all $\mathbf{x} \in \mathbf{X}$,

$$\sigma^u_n(\mathbf{x}) = \inf_{\pi \in \Pi} \frac{1}{n} \sum_{k=1}^{n} \sum_{i=1}^{k} u(x_{\pi(i)}) = \frac{1}{n} \sum_{k=1}^{n} \sum_{r=1}^{k} u(x_{[r]}) = \sum_{r=1}^{n} \frac{n-r+1}{n} u(x_{[r]})\,.$$

Definition 6. *The Strongly Anonymous (Generalized) Utilitarian Cesàro Summation* SWR $\succsim^S$. There exists $u \in U$ such that, for any profiles $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\mathbf{x} \succsim^S \mathbf{y}$ if and only if

$$\liminf_{n \to +\infty} \left(\sigma^u_n(\mathbf{x}) - \sigma^u_n(\mathbf{y})\right) \geq 0\,.$$

[11] Zuber and Asheim (2012) do not require u to be finitely non-concave. However, they show in their Proposition 6 that u has to be finitely non-concave to satisfy the Pigou-Dalton transfer principle.

In the context of discrete dynamic progamming, Veinott (1966, p. 1293) proposes a similar criterion, which is used by Jonsson and Voorneveld (2018) for comparisons with their Limit of Time-Discounted Utilitarianism.

For any $u \in U$, the asymmetric and symmetric parts of $\succsim^S$ associated with u are given as follows: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$,

$$\mathbf{x} \succ^S \mathbf{y} \text{ if and only if } \limsup_{n\to+\infty} \left(\sigma_n^u(\mathbf{x}) - \sigma_n^u(\mathbf{y})\right) > 0 \text{ and } \liminf_{n\to+\infty} \left(\sigma_u^u(\mathbf{x}) - \sigma_n^u(\mathbf{y})\right) \geq 0,$$

$$\mathbf{x} \sim^S \mathbf{y} \text{ if and only if } \lim_{n\to+\infty} \left(\sigma_n^u(\mathbf{x}) - \sigma_n^u(\mathbf{y})\right) = 0.$$

Note that when, for some $n \in \mathbb{N}$, $w_n^u(\mathbf{x}) \geq w_n^u(\mathbf{y})$ ($w_n^u(\mathbf{x}) > w_n^u(\mathbf{y})$ respectively) and $x_{[r]} \geq y_{[r]}$ for all $r > n$, we have $\lim_{\beta\to1^-} \left(\rho_\beta^u(\mathbf{x}) - \rho_\beta^u(\mathbf{y})\right) \geq 0$ and $\lim_{n\to+\infty} \left(\sigma_n^u(\mathbf{x}) - \sigma_n^u(\mathbf{y})\right) \geq 0$ ($\lim_{\beta\to1^-} \left(\rho_\beta^u(\mathbf{x}) - \rho_\beta^u(\mathbf{y})\right) > 0$ and $\lim_{n\to+\infty} \left(\sigma_n^u(\mathbf{x}) - \sigma_n^u(\mathbf{y})\right) > 0$ respectively), implying that $\mathbf{x} \succsim^R \mathbf{y}$ and $\mathbf{x} \succsim^S \mathbf{y}$ ($\mathbf{x} \succ^R \mathbf{y}$ and $\mathbf{x} \succ^S \mathbf{y}$ respectively) for $\succsim^R$ and $\succsim^S$ associated with $u \in U$. We therefore obtain the following result.

Observation 3. *For any $u \in U$, the Strongly Anonymous (Generalized) Utilitarian Dominance* SWR $\succsim^D$ *associated with u is a subrelation to the Limit of Rank-Discounted (Generalized) Utilitarian* SWR $\succsim^R$ *associated with u and the Strongly Anonymous (Generalized) Utilitarian Cesàro Summation* SWR $\succsim^S$ *associated with u.*

4.3 Links between the SWRs

The following proposition relates the three SWRs that we have introduced in this section to each other and to our Strongly Anonymous Utilitarian SWR $\succsim^U$. We do so by showing that they satisfy all our axioms. The proof is relegated to the appendix.

Proposition 2. *(i) Both the Limit of Rank-Discounted (Generalized) Utilitarian* SWR $\succsim^R$ *and the Strongly Anonymous (Generalized) Utilitarian Cesàro Summation* SWR $\succsim^S$ *are regular.*

(ii) The Strongly Anonymous (Generalized) Utilitarian Catching-Up SWR $\succsim^C$ *is weakly extended by $\succsim^S$, which in turn is weakly extended by $\succsim^R$.*

(iii) The Strongly Anonymous (Generalized) Utilitarian SWR $\succsim^U$ *is a subrelation to $\succsim^C$, $\succsim^R$, and $\succsim^S$.*

Proposition 2 shows that there are strong links between the SWRs that we have introduced. It is also important to show that they are distinct. In the following example already discussed in subsection 3.1, we present a pair of well-beging profiles that are indifferent according to $\succsim^C$, $\succsim^R$, and $\succsim^S$ but non-comparable by $\succsim^U$.

Example 1. Assume that $\succsim^U$, $\succsim^C$, $\succsim^S$, and $\succsim^R$ are associated with the identity transformation function $u(z) = z$. For all $n \in \mathbb{N}$, let $S(n) = \sum_{i=1}^{n}(i-1) + 1$. Using $S(n)$, we define the set $G(n)$

of individuals by $G(n) = \{S(n), \ldots, S(n+1)-1\}$ for each $n \in \mathbb{N}$. Therefore, $|G(n)| = n$ for all $n \in \mathbb{N}$. Letting $z = 1 - \ln 2 \approx 0.30685$, consider the following distributions $\mathbf{x}, \mathbf{y} \in \mathbf{X}^+$:

$$\mathbf{x} = (z, \underbrace{\tfrac{2}{3}, \quad \tfrac{2}{3}}_{|G(2)|=2}, \underbrace{\tfrac{2}{3}, \ldots, \tfrac{2}{3}}_{|G(3)|=3}, \underbrace{\tfrac{4}{5}, \ldots, \tfrac{4}{5}}_{|G(4)|=4}, \underbrace{\tfrac{4}{5}, \ldots, \tfrac{4}{5}}_{|G(5)|=5}, \ldots)$$
$$\mathbf{y} = (\tfrac{1}{2}, \overbrace{\tfrac{1}{2}, \quad \tfrac{1}{2}}, \overbrace{\tfrac{3}{4}, \ldots, \tfrac{3}{4}}, \overbrace{\tfrac{3}{4}, \ldots, \tfrac{3}{4}}, \overbrace{\tfrac{5}{6}, \ldots, \tfrac{5}{6}}, \ldots).$$

Formally, $\mathbf{x}$ is defined by $x_1 = z$ and, for each odd $n \in \mathbb{N}$ with $n \geq 3$ and for all $i \in G(n-1) \cup G(n)$,

$$x_i = 1 - \frac{1}{n},$$

and $\mathbf{y}$ is defined by, for each even $n \in \mathbb{N}$ and for all $i \in G(n-1) \cup G(n)$,

$$y_i = 1 - \frac{1}{n}.$$

Note that

$$\mathbf{x} - \mathbf{y} = (z - \tfrac{1}{2}, \underbrace{\tfrac{1}{6}, \quad \tfrac{1}{6}}_{|G(2)|=2} \underbrace{-\tfrac{1}{12}, \ldots, -\tfrac{1}{12}}_{|G(3)|=3}, \underbrace{\tfrac{1}{20}, \ldots, \tfrac{1}{20}}_{|G(4)|=4}, \underbrace{-\tfrac{1}{30}, \ldots, -\tfrac{1}{30}}_{|G(5)|=5}, \ldots).$$

More precisely, for each $n > 1$ and for all $i \in G(n)$,

$$x_i - y_i = (-1)^n \frac{1}{n(n+1)}.$$

Thus, for each $n > 1$,

$$\sum_{i \in G(n)} (x_i - y_i) = (-1)^n \frac{n}{n(n+1)} = (-1)^n \frac{1}{n+1}.$$

The alternating harmonic series converges to $\ln 2$. Hence:

$$\sum_{i=1}^{\infty} (x_i - y_i) = 1 - \ln 2 - \frac{1}{2} + \sum_{n=2}^{\infty} (-1)^n \frac{1}{n+1} = -\ln 2 + \sum_{n=0}^{\infty} (-1)^n \frac{1}{n+1} = 0.$$

Since $\mathbf{x}_{[\,]} = \mathbf{x}$ and $\mathbf{y}_{[\,]} = \mathbf{y}$, we obtain $\mathbf{x} \sim^C \mathbf{y}$ and, from Proposition 2, $\mathbf{x} \sim^S \mathbf{y}$ and $\mathbf{x} \sim^R \mathbf{y}$.

However, for all odd $n \in \mathbb{N}$:

$$\sum_{i < S(n+1): y_i > x_i} \big(y_i - x_i\big) = \sum_{k=1}^{(n+1)/2} \sum_{i \in G(2k-1)} (y_i - x_i) = -1 + \ln 2 + \frac{1}{2} \left(\sum_{k=1}^{(n+1)/2} \frac{1}{k} \right).$$

Therefore, $\lim_{n\to\infty} \sum_{i \leq n: y_i > x_i} \big(y_i - x_i\big) = \infty$ and $(\mathbf{x}, \mathbf{y}) \notin \mathbf{D}$. Likewise, by considering all even $u \in \mathbb{N}$, we obtain that $\lim_{n\to\infty} \sum_{i \leq n: x_i > y_i} \big(x_i - y_i\big) = \infty$ and $(\mathbf{y}, \mathbf{x}) \notin \mathbf{D}$. Since $\mathbf{x}_{[\,]} = \mathbf{x}$ and $\mathbf{y}_{[\,]} = \mathbf{y}$, it follow from Lemma 1 that $\succsim^U$ cannot compare $\mathbf{x}$ and $\mathbf{y}$.

The next example presents a pair of well-begin distributions that are indifferent according to $\succsim^S$ and $\succsim^R$ but non-comparable by $\succsim^C$.

Example 2. Assume that $\succsim^C$, $\succsim^S$, and $\succsim^R$ are associated with the identity transformation function $u(z) = z$. For all $n \in \mathbb{N}$, let $P(n) = \sum_{i=1}^{n} 2^i - 1$. Using $P(n)$, we define the set $G(n)$ of individuals by $G(1) = \{1\}$ and $G(n) = \{P(n-1)+1, \ldots, P(n)\}$ for all $n \geq 2$. Thus, $|G(n)| = 2^n$ for each $n \geq 2$. Consider the following distributions $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}^+$:

$$\begin{aligned}
\mathbf{x} &= (0, \quad \underbrace{0, \ldots, 0}_{|G(2)|=4}, \quad \underbrace{\tfrac{3}{4}, \ldots, \tfrac{3}{4}}_{|G(3)|=8}, \quad \underbrace{\tfrac{3}{4}, \ldots, \tfrac{3}{4}}_{|G(4)|=16}, \quad \underbrace{\tfrac{15}{16}, \ldots, \tfrac{15}{15}}_{|G(5)|=32}, \quad \underbrace{\tfrac{15}{16}, \ldots, \tfrac{15}{16}}_{|G(6)|=64}, \quad \ldots) \\
\mathbf{y} &= (-1, \quad \overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}, \quad \overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}, \quad \overbrace{\tfrac{7}{8}, \ldots, \tfrac{7}{8}}, \quad \overbrace{\tfrac{7}{8}, \ldots, \tfrac{7}{8}}, \quad \overbrace{\tfrac{31}{32}, \ldots, \tfrac{31}{32}}, \quad \ldots).
\end{aligned}$$

Formally, $\mathbf{x}$ is defined by $(x_1, \ldots, x_5) = (0, \ldots, 0)$ and for each even $n \in \mathbb{N}$ and for all $i \in G(n+1) \cup G(n+2)$,

$$x_i = \sum_{k=1}^{n} \frac{1}{2^k} \, ,$$

and $\mathbf{y}$ is defined by $y_1 = -1$ and for each odd $n \in \mathbb{N}$ and for all $i \in G(n+1) \cup G(n+2)$,

$$y_i = \sum_{k=1}^{n} \frac{1}{2^k} \, .$$

Note that

$$\mathbf{x} - \mathbf{y} = (1, \quad \underbrace{-\tfrac{1}{2}, \ldots, -\tfrac{1}{2}}_{|G(2)|=4}, \quad \underbrace{\tfrac{1}{4}, \ldots, \tfrac{1}{4}}_{|G(3)|=8}, \quad \underbrace{-\tfrac{1}{8}, \ldots, -\tfrac{1}{8}}_{|G(4)|=16}, \quad \underbrace{\tfrac{1}{16}, \ldots, \tfrac{1}{16}}_{|G(5)|=32}, \quad \ldots).$$

More precisely, for each $n \in \mathbb{N}$ and for all $i \in G(n+1)$,

$$x_i - y_i = (-1)^n \frac{1}{2^n} \, .$$

Note that $\mathbf{x}_{[\,]} = \mathbf{x}$ and $\mathbf{y}_{[\,]} = \mathbf{y}$. We show that $\succsim^C$ cannot compare $\mathbf{x}$ and $\mathbf{y}$, while $\succsim^R$ and $\succsim^S$ conclude these distributions are indifferent. To this end, we define $\mathbf{s} \in \mathbb{R}^{\mathbb{N}}$ by $s_n = \sum_{i=1}^{n} (x_i - y_i)$ for all $n \in \mathbb{N}$. Thus, $\mathbf{s}$ is given by

$$\mathbf{s} = (\underbrace{1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -1}_{|G(2)|+1}, \quad \underbrace{-\tfrac{3}{4}, -\tfrac{2}{4}, -\tfrac{1}{4}, 0, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}}_{|G(3)|-1}, \quad \underbrace{1, \tfrac{7}{8}, \ldots \tfrac{1}{8}, 0, -\tfrac{1}{8}, \ldots, -\tfrac{7}{8}, -1}_{|G(4)|+1=17}, \quad \ldots).$$

Note that, for each odd $n \in \mathbb{N}$, $s_i = 1$ if $i = P(n)$ and $s_i = -1$ if $i = P(n+1)$. Thus, $\mathbf{x}$ and $\mathbf{y}$ are non-comparable by $\succsim^C$. On the other hand, for each even $n \in \mathbb{N}$ and each odd $m \in \mathbb{N}$ with $m > 1$,

$$\frac{1}{P(n)} \sum_{k=1}^{P(n)} s_k = 0 \quad \text{and} \quad \frac{1}{P(m)-1} \sum_{k=1}^{P(m)-1} s_k = 0 \, .$$

Also, it can be checked that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} s_k = 0 \, .$$

Thus, we obtain $\mathbf{x} \sim^S \mathbf{y}$. By Proposition 2, $\mathbf{x} \sim^R \mathbf{y}$ follows as well.

5 Application to the Ramsey model

We show that the Strongly Anonymous Utilitarianism gives rise to a nonempty set of optimal streams in the Ramsey model of economic growth. Furthermore, if the initial stock of reproducible capital does not exceed the one corresponding to the Golden Rule, then any optimal stream satisfies that wellbeing cannot be increased at some time i without being decreased at some other time i'. Hence, an optimal stream is in this sense efficient, even though the Strongly Anonymous Utilitarian SWR does not satisfy the Pareto principle.

In the Ramsey model, the law of motion governing the stock of reproducible capital, k, is given by a standard increasing, concave gross production function, f. Following Basu and Mitra (2007, Section 5), the function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is assumed to satisfy:

(i) $f(0) = 0$,
(ii) f is continuous, increasing and strictly concave on $\mathbb{R}_+$,
(iii) $\lim_{k\to 0} \frac{f(k)}{k} > 1$ and $\lim_{k\to \infty} \frac{f(k)}{k} < 1$.

Then there exists a unique number $\bar{k} > 0$ such that $f(\bar{k}) = \bar{k}$ and $f(k) > k$ for $k \in (0, \bar{k})$.

A feasible stream from $k \in [0, \bar{k}]$ is a sequence of *capital stocks* $\mathbf{k} = (k_1, \dots, k_i, \dots)$ satisfying:

$$k_1 = k\,, \quad 0 \le k_{i+1} \le f(k_i) \text{ for } i \in \mathbb{N}\,.$$

It follows from the definition of $\bar{k}$ that $\mathbf{k} \in [0, \bar{k}]^{|\mathbb{N}|}$. Hence, $\bar{k}$ is the maximal attainable capital stock if one starts from an initial stock in $[0, \bar{k}]$. Associated with a feasible stream $\mathbf{k}$ from $k \in [0, \bar{k}]$ is a feasible *wellbeing* stream $\mathbf{x} = (x_1, \dots, x_i, \dots)$ from k, defined by

$$x_i = f(k_i) - k_{i+1} \text{ for } i \in \mathbb{N}\,.$$

For any $k \in [0, \bar{k}]$, the set of wellbeing streams associated with feasible capital streams from k is contained in $[0, \bar{k}]^{|\mathbb{N}|}$, keeping in mind that $f(\bar{k}) = \bar{k}$. We say that a feasible wellbeing stream $\hat{\mathbf{x}}$ from k is *optimal* if, for all feasible wellbeing streams $\mathbf{x}$ from k, it is the case that $\hat{\mathbf{x}} \succsim^U \mathbf{x}$.

It follows from the continuity and strict concavity of f that there exists a unique number $k^* \in [0, \bar{k}]$ such $f(k) - k$ is increasing on $[0, k^*]$ and decreasing on $[k^*, \bar{k}]$, implying that $f(k^*) - k^* \ge f(k) - k$ for all $k \in [0, \bar{k}]$. Since, for any $k \in (0, \bar{k})$, $f(k) - k > 0 = f(0) - 0 = f(\bar{k}) - \bar{k}$, we have that $0 < k^* < \bar{k}$. We write $x^* = f(k^*) - k^*$. By keeping the capital stock constant at k^*, a maximum sustainable wellbeing equal to x^* is attained; this corresponds to the *Golden Rule*.

For any $\mathbf{x} \in [0, \bar{k}]^{|\mathbb{N}|}$, let $\ell(\mathbf{x})$ denote the limit infimum of $\mathbf{x}$. Let $\xi : [0, \bar{k}] \to [0, \bar{k}]$ be a stationary wellbeing strategy that for any $i \in \mathbb{N}$ and any $k_i \in [0, \bar{k}]$ determines $x_i = \xi(k_i)$, where $\xi(k) = 0$ if $k < k^*$ and $\xi(k) = x^*$ if $k \ge k^*$. Let $\mathbf{x}^\xi(k)$ be the feasible wellbeing stream from k determined by the strategy ξ. If $k \ge k^*$, then $\mathbf{x}^\xi(k) = x^* \mathbb{1}_\mathbb{N}$. If $0 < k < k^*$, then there exists $n \in \mathbb{N}$ such that $x_i^\xi(k) = 0$ for $i = 1, \dots, n$, and $x_i^\xi(k) = x^*$ for $i > n$. If $k = 0$, then $\mathbf{x}^\xi(k) = 0\mathbb{1}_\mathbb{N}$. Note that, $\ell(\mathbf{x}^\xi(k)) = x^*$ for $k \in (0, \bar{k}]$, while $\ell(\mathbf{x}^\xi(0)) = 0$.

Lemma 5. *For any $k \in (0, \bar{k}]$, if $\mathbf{x}$ is a feasible wellbeing stream from k, then $\ell(\mathbf{x}) \leq x^*$.*

Proof. Fix $k \in (0, \bar{k}]$. It is sufficient to show that there is no feasible $\mathbf{x}$ from k with $\ell(\mathbf{x}) > x^*$. By way of contradiction, suppose that there exist some feasible $\mathbf{x}'$ from k with $\ell(\mathbf{x}') > x^*$. Write $\varepsilon = \big(\ell(\mathbf{x}') - x^*\big)/2$. Then there exist $n \in \mathbb{N}$ such that, for all $i > n$, $x'_i - x^* \geq \varepsilon > 0$. Since $f(k^*) - k^* \geq f(k) - k$ for all $k \in [0, \bar{k}]$, this implies:

$$0 < \varepsilon \leq x'_i - x^* = f(k'_i) - k'_{i+1} - f(k^*) + k^* \leq f(k'_i) - k'_{i+1} - f(k'_i) + k'_i = k'_i - k'_{i+1}\,.$$

Hence, the capital stock will be depleted in a finite time, contradicting the feasibility of $\mathbf{x}'$. □

In his seminal work on optimal growth, Ramsey (1928) introduced an undiscounted utilitarian criterion which computes the total amount by which utility falls short of the maximum possible utility level, referred to as "bliss." We introduce a modified Ramsey criterion where we instead associate bliss with the maximum possible sustainable level of utility $u(x^*)$, given the production function f and the transformation function $u \in U$. Hence, consider the complete preorder represented by $v^{\bar{u}} : \mathbf{X} \to \mathbb{R}_- \cup \{-\infty\}$ where, for all $\mathbf{x} \in \mathbf{X}$,

$$v^{\bar{u}}(\mathbf{x}) = \lim_{n\to\infty} v^{\bar{u}}_n(\mathbf{x}) = \lim_{n\to\infty} \sum_{i=0}^{n} \bar{u}(x_i)\,,$$

with $\bar{u} : \mathbb{R} \to \mathbb{R}_-$ being defined by $\bar{u}(x) = u(x) - u(x^*)$ for all $x < x^*$ and $\bar{u}(x) = 0$ for all $x \geq x^*$. Remark that $v^{\bar{u}}$ is well-defined (it may be $-\infty$) and always takes a non-positive value; indeed it is an infinite sum of non-positive real numbers.

Our strategy of proof is to show that there is a feasible wellbeing stream $\hat{\mathbf{x}}$ which satisfies that $v^{\bar{u}}(\hat{\mathbf{x}})$ is finite and $v^{\bar{u}}(\hat{\mathbf{x}}) \geq v^{\bar{u}}(\mathbf{x})$ for all feasible wellbeing streams $\mathbf{x}$, followed by a demonstration that $\hat{\mathbf{x}}$ is optimal under Strongly Anonymous Utilitarianism. For any $k \in (0, \bar{k}]$, let $\mathbf{X}^*(k)$ denote the set of feasible wellbeing streams $\mathbf{x}$ from k which satisfy that $x_i \in [0, x^*]$ for all $i \in \mathbb{N}$ and $\ell(\mathbf{x}) = x^*$. The following existence result is proven in Appendix A.6.

Lemma 6. *For any $k \in (0, k^*]$, there exists a feasible wellbeing stream $\hat{\mathbf{x}}$ from k such that $v^{\bar{u}}(\hat{\mathbf{x}}) > -\infty$ and $v^{\bar{u}}(\hat{\mathbf{x}}) \geq v^{\bar{u}}(\mathbf{x})$ for all feasible wellbeing streams $\mathbf{x}$ from k. Furthermore, $\hat{\mathbf{x}} \in \mathbf{X}^*(x)$, it is non-decreasing, and there is no feasible wellbeing stream $\mathbf{x}'$ from k such that $\mathbf{x}' \geq \hat{\mathbf{x}}$.*

By Lemma 5, we know that, for any $k \in (0, \bar{k}]$ and any feasible stream $\mathbf{x}$ from k, we have $\ell(\mathbf{x}) \leq x^*$. This implies that $\mathbf{x}_{[\,]} \leq x^* \mathbb{1}_{\mathbb{N}}$. Furthermore, for any two feasible streams from k, $\mathbf{x}$ and $\mathbf{x}'$, we have that $v^{\bar{u}}(\mathbf{x}_{[\,]}) \geq v^{\bar{u}}(\mathbf{x}'_{[\,]}) > -\infty$ implies $\mathbf{x} \succsim^U \mathbf{x}'$.[12] Likewise, $v^{\bar{u}}(\mathbf{x}_{[\,]}) > v^{\bar{u}}(\mathbf{x}'_{[\,]})$ implies $\mathbf{x} \succ^U \mathbf{x}'$.

[12]Recall that $\bar{u}(x) = u(x) - u(x^*)$ for all $x \in (-\infty, x^*]$ and that $\mathbf{x}_{[\,]} \in (-\infty, x^*]^{\mathbb{N}}$. The series $\sum_{r=1}^{n} \big(u(x_{[r]}) - u(x^*)\big)$ is absolutely convergent when it is finite, because $\big(u(x_{[r]}) - u(x^*)\big) = -|u(x_{[r]}) - u(x^*)|$ for any r. Since the difference of two absolutely convergent series is also absolutely convergent (and thus unconditionally convergent), this implies that, for all $\pi \in \Pi$, $\lim_{n\to+\infty} \big(v^u_n(\mathbf{x}_{[\pi]}) - v^u_n(\mathbf{x}'_{[\pi]})\big) = v^{\bar{u}}(\mathbf{x}_{[\,]}) - v^{\bar{u}}(\mathbf{x}'_{[\,]})$, when $v^{\bar{u}}(\mathbf{x}_{[\,]})$ and $v^{\bar{u}}(\mathbf{x}'_{[\,]})$ are finite.

The main result of this section is that there exists an optimal Strongly Anonymous Utilitarian wellbeing stream in the Ramsey model.[13]

Proposition 3. *Fix $u \in U$ and consider the Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$ *associated with u.*

1. *For any $k \in (k^*, \bar{k}]$, any feasible wellbeing stream $\hat{\mathbf{x}}$ from k with $\hat{\mathbf{x}} \geq x^* \mathbb{1}_{\mathbb{N}}$ satisfies that $\hat{\mathbf{x}} \succsim^U \mathbf{x}$ for all feasible wellbeing streams $\mathbf{x}$ from k. The stream $\mathbf{x}^{\xi}(k)$ is one of these.*

2. *For any $k \in (0, k^*]$, there exists a feasible wellbeing stream $\hat{\mathbf{x}}$ from k which satisfies that $\hat{\mathbf{x}} \succsim^U \mathbf{x}$ for all feasible wellbeing streams $\mathbf{x}$ from k. Furthermore, any such optimal stream is non-decreasing, and there is no feasible wellbeing stream $\mathbf{x}'$ from k such that $\mathbf{x}' > \hat{\mathbf{x}}$.*

Proof. Part 1. Assume that $k \in (k^*, \bar{k}]$. Let $\hat{\mathbf{x}}$ be feasible from k with $\hat{\mathbf{x}} \geq x^* \mathbb{1}_{\mathbb{N}}$. Then $\hat{\mathbf{x}}_{[\,]} = x^* \mathbb{1}_{\mathbb{N}}$. By Lemma 5, we know that, for any feasible $\mathbf{x}$ from k, $\ell(\mathbf{x}) \leq x^*$. Hence, for any feasible $\mathbf{x}$ from k, $\mathbf{x}_{[\,]} \leq x^* \mathbb{1}_{\mathbb{N}}$, so that $\mathbf{x}_{[\,]} \leq \hat{\mathbf{x}}_{[\,]}$ and $\hat{\mathbf{x}} \succsim^U \mathbf{x}$. There exist feasible $\hat{\mathbf{x}}$ from k with $\hat{\mathbf{x}} \geq x^* \mathbb{1}_{\mathbb{N}}$ since $\mathbf{x}^{\xi}(k) = x^* \mathbb{1}_{\mathbb{N}}$ is feasible from k. Indeed, $\mathbf{x}^{\xi}(k) \succsim^U \mathbf{x}$ for all feasible $\mathbf{x}$ from k.

Part 2. Assume that $k \in (0, k^*]$. Statement 2 is proven through three steps.

Step 1: If $\mathbf{x} \notin \mathbf{X}^*(k)$ is a feasible wellbeing stream from k, then there exists a wellbeing stream $\mathbf{x}' \in \mathbf{X}^*(k)$ such that $\mathbf{x}' \succ^U \mathbf{x}$. By Lemma 5, we know that, for any feasible $\mathbf{x}$ from k, $\ell(\mathbf{x}) \leq x^*$. If $\ell(\mathbf{x}) < x^*$, then $\mathbf{x}^{\xi}(k) \succ^U \mathbf{x}$ because $\ell(\mathbf{x}^{\xi}(k)) = x^* > \ell(\mathbf{x})$. If $\ell(\mathbf{x}) = x^*$ and $\mathbf{x} \notin \mathbf{X}^*(k)$, then $x_i > x^*$ for some $i \in \mathbb{N}$. Since no stream $\mathbf{x} > x^* \mathbb{1}_{\mathbb{N}}$ is feasible from $k \leq k^*$, there exists i' such that $x_{i'} < x^*$. Furthermore, because f is increasing, there exists ε such that $\mathbf{x}'$ defined as follows is feasible from k: $x'_i = x^*$ for all $i \in \mathbb{N}$ such that $x_i \geq x^*$, $x'_{i'} = x_{i'} + \varepsilon \leq x^*$, and $x'_j = x_j$ for all other j. By construction, $\mathbf{x}' \in \mathbf{X}^*(k)$. But $\mathbf{x}'_{[\,]} > \mathbf{x}_{[\,]}$, implying that $\mathbf{x}'_{[\,]} \succ^U \mathbf{x}_{[\,]}$.

Step 2: There exists an optimal wellbeing stream in $\mathbf{X}^*(k)$. Note that, for any $\mathbf{x} \in \mathbf{X}^*(k)$, we have $v^{\bar{u}}(\mathbf{x}_{[\,]}) = v^{\bar{u}}(\mathbf{x})$. By Lemma 6, we know that there exists $\hat{\mathbf{x}} \in \mathbf{X}^*(k)$ such that $v^{\bar{u}}(\hat{\mathbf{x}}) > -\infty$ and $v^{\bar{u}}(\hat{\mathbf{x}}) \geq v^{\bar{u}}(\mathbf{x})$ for all feasible $\mathbf{x}$ from k. This implies $v^{\bar{u}}(\hat{\mathbf{x}}_{[\,]}) \geq v^{\bar{u}}(\mathbf{x}_{[\,]})$ and thus $\hat{\mathbf{x}} \succsim^U \mathbf{x}$ for all $\mathbf{x} \in \mathbf{X}^*(k)$. Combined with Step 1 and using the transitivity of $\succsim^U$, this implies that $\hat{\mathbf{x}} \succsim^U \mathbf{x}$ for all feasible $\mathbf{x}$ from k.

Step 3: Properties of an optimal wellbeing stream. Assume that $\hat{\mathbf{x}}$ feasible from k satisfies that $\hat{\mathbf{x}} \succsim^U \mathbf{x}$ for all feasible $\mathbf{x}$ from k. Since, for any $\mathbf{x} \in \mathbf{X}^*(k)$, we have $v^{\bar{u}}(\mathbf{x}_{[\,]}) = v^{\bar{u}}(\mathbf{x})$, it follows that $v^{\bar{u}}(\hat{\mathbf{x}}) \geq v^{\bar{u}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbf{X}^*(k)$. If $\mathbf{x} \notin \mathbf{X}^*(k)$ is feasible from k, then it follows from Step 3 of the proof of Lemma 6 that there exists $\mathbf{x}' \in \mathbf{X}^*(k)$ such that $v^{\bar{u}}(\mathbf{x}') > v^{\bar{u}}(\mathbf{x})$. Hence, an optimal wellbeing stream from k satisfies the properties of $\hat{\mathbf{x}}$ of Lemma 6: An optimal $\hat{\mathbf{x}}$ from k is non-decreasing, and there is no feasible $\mathbf{x}'$ from k such that $\mathbf{x}' \geq \hat{\mathbf{x}}$. □

A first implication of Proposition 3 is that, although the Strongly Anonymous Utilitarian SWR $\succsim^U$ cannot compare all wellbeing streams, it has sufficient comparability to find an optimal stream

[13] We consider only streams for an initial stock $k \in (0, \bar{k}]$. If $k = 0$, then the only feasible stream is $\mathbf{x}^{\xi}(0) = 0\mathbb{1}_{\mathbb{N}}$, which is trivially optimal.

that is at least as good as any feasible wellbeing stream in the Ramsey model. Furthermore, since, by Proposition 2, the Strongly Anonymous Utilitarian SWR is a subrelation to $\succsim^C$, $\succsim^R$, and $\succsim^S$, the results of Proposition 3 hold also for these SWRs.

Proposition 3 makes a distinction between two cases. In the first case, with $k \in (k^*, \bar{k}]$, we have that any feasible stream with wellbeing that is at least as high as the highest sustainable wellbeing level, x^*, is optimal. The constant wellbeing stream at the highest sustainable wellbeing level, x^*, is feasible. But although it is optimal, the stream $x^* \mathbb{1}_{\mathbb{N}}$ is not efficient for $k > k^*$. Indeed, the wellbeing stream $\mathbf{x}'$ such that $x'_1 = f(k) - k^*$ and $x'_i = x^*$ for all $i \geq 2$ is feasible. And $x'_1 = f(k) - k^* > f(k^*) - k^* = x^*$, so that $\mathbf{x}' > x^* \mathbb{1}_{\mathbb{N}}$. Note that $\mathbf{x}'$ is also optimal. In contrast, the second case, with $k \in (0, k^*]$, any optimal stream is both efficient and non-decreasing.

6 Discussion

We have proposed and characterized SWRs over infinite wellbeing profiles which are generalized utilitarian and which treat people equally in the strong form of Strong Anonymity. They are *generalized* utilitarian in the sense that they depend on the function u that maps wellbeing into the transformed wellbeing, which is additive and thereby enables us to compare the gain of one to the loss of another. Indeed, the Strongly Anonymous Utilitarian SWR $\succsim^U$ characterized in Theorem 1 is a different criterion for each u in the set U of continuous, strictly increasing, and finitely non-concave transformation functions. If wellbeing corresponds to utility and u is strictly concave, then the Strongly Anonymous Utilitarian SWR $\succsim^U$ is *prioritarian*, having the property of assigning more weight to increases in utility for poor than to increases in utility for rich.[14]

All of these SWRs deem one profile as better than another if the limit inferior of the former strictly exceeds that of the latter. Also, even for profiles with same limit inferior, differences in wellbeing above the common limit inferior do not influence the evaluation. Hence, the distinction between the various SWRs concerns how they compare profiles with the same limit inferior on the basis of the components of wellbeing that do not exceed this common smallest cluster point.

One might argue that these features severely limit the ablicability of the SWRs that we have analyzed. We beg to differ. The common limit inferior might be associated with bliss, as originally suggested by Ramsey (1928), in which case wellbeing above this level is not even feasible, or with the Golden Rule, as we have illustrated in Section 5. Moreover, if the technology has positive net productivity and people are treated equally, then the time profile of wellbeing will be non-decreasing and tend to bliss or the Golden Rule as time goes to infinity (Asheim, Buchholz, and Tungodden, 2001). Thus, the proposed SWRs are applicable and provide evaluation in line already existing utilitarian criteria in the context of such technologies.

[14]Note that much of the literature on infinite population utilitarian criteria combine anonymity with translation scale invariance to obtain utilitarianism in terms of untransformed wellbeing; Basu and Mitra (2007) is an example. Here we obtain generalized utilitarianism by imposing a separability axiom instead of translation scale invariance.

Our SWRs can also be used if the rate of population growth is endogenous. Since, by assumption, total population is always infinite, there cannot be a trade-off between the size of the total population and the level of wellbeing. This implies that there cannot be a *Repugnant Conclusion* (Parfit, 1984), whereby a large population with acceptable but poor lives is preferred to a smaller population with excellent lives, in terms of the total number of people that will ever live. The wellbeing of people might be affected by the spatiotemporal structure of the total population, in the sense that the existence of other people in the proximity might yield benefits or costs. However, having a large population simultaneously at a given point in time does not by itself have value, as evaluated by our SWRs.[15] Hence, if a lower rate of population growth can provide room for higher current wellbeing and faster growth of wellbeing towards bliss, then the lower population growth is preferable. In this sense, our SWRs escape the Repugnant Conclusion when interpreted in terms of the number of people living at a given point in time.

A Appendix: Proofs

A.1 Preliminary results

We first prove some preliminary results that we need to prove our main results.

Proposition 4. *Consider any* SWR $\succsim$*. If there exists $u \in U$ such that the Strongly Anonymous (Generalized) Utilitarian Dominance* SWR $\succsim^D$ *associated with u is a subrelation to $\succsim$, then $\succsim$ satisfies Finite Completeness, Restricted Continuity, Monotonicity, Strong Anonymity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-Level Consistency.*

Proof. Let the SWR $\succsim$ have the property that there exists $u \in U$ such that the Strongly Anonymous (Generalized) Utilitarian Dominance SWR $\succsim^D$ associated with u is a subrelation to $\succsim$.

$\succsim$ *satisfies Finite Completeness.* For any $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbf{X}$ and for any $N \in \mathbb{N}$, let $\tilde{\mathbf{z}} = \mathbf{x}_n\mathbf{z}$ and $\hat{\mathbf{z}} = \mathbf{y}_n\mathbf{z}$. Profiles $\tilde{\mathbf{z}}$ and $\hat{\mathbf{z}}$ differ by a finite number of components, so there exists $k \in \mathbb{N}$ such that $\tilde{z}_{[r]} = \hat{z}_{[r]}$ for all $r \geq k$. And, for any $u \in U$, we necessarily have that either $v^u_k(\tilde{\mathbf{z}}_{[\,]}) - v^u_k(\hat{\mathbf{z}}_{[\,]}) \geq 0$ or $v^u_k(\tilde{\mathbf{z}}_{[\,]}) - v^u_k(\hat{\mathbf{z}}_{[\,]}) \leq 0$ (or both). This implies that either $\mathbf{x}_n\mathbf{z} \succsim^D \mathbf{y}_n\mathbf{z}$ or $\mathbf{y}_n\mathbf{z} \succsim^D \mathbf{x}_n\mathbf{z}$ (or both). Because $\succsim^D$ is a subrelation to $\succsim$, either $\mathbf{x}_n\mathbf{z} \succsim \mathbf{y}_n\mathbf{z}$ or $\mathbf{y}_n\mathbf{z} \succsim \mathbf{x}_n\mathbf{z}$ (or both).

$\succsim$ *satisfies Restricted Continuity.* Consider any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and any $\ell \in \mathbb{R}$. Assume that there exist $\underline{\ell} < \bar{\ell}$ such that $\underline{\ell} \leq \ell \leq \bar{\ell}$ and $(\ell'\mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell'\mathbb{1}_{\mathbb{N}}, \mathbf{y})$ for all $\ell' \in (\underline{\ell}, \bar{\ell}) \setminus \{\ell\}$. It suffices to show the conclusion $(\ell\mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell\mathbb{1}_{\mathbb{N}}, \mathbf{y})$ when $\ell = \underline{\ell}$ and $\ell = \bar{\ell}$. Note that $\succsim^D$ is complete on

$$\left\{(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{X} | \exists \ell \in \mathbb{R} \text{ and } n \in \mathbb{N} \text{ such that } \mathbf{x}_{[\,]}, \mathbf{y}_{[\,]} \in \mathbf{X}^{\ell}_n\right\}.$$

Hence, since $\succsim^D$ is a subrelation to $\succsim$, we obtain that, for each $\ell \in \mathbb{R}$ and $n \in \mathbb{N}$, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ such that $\mathbf{x}_{[\,]}$, $\mathbf{y}_{[\,]} \in \mathbf{X}^{\ell}_n$, $\mathbf{x} \succsim \mathbf{y}$ is equivalent to $\mathbf{x} \succsim^D \mathbf{y}$. There are three cases.

[15] Indeed, adding a person at some time without influencing the wellbeing of other people cannot improve the profile, as it does not influence the evaluation if the added person's wellbeing is as large as the limit inferior and, otherwise, it reduces the wellbeing of finitely ranked people with wellbeing exceeding that of the added person.

Case 1: $\ell > \max\{\ell(\mathbf{x}), \ell(\mathbf{y})\}$. There exists $\ell' \in (\underline{\ell}, \bar{\ell})$ such that $\ell' \geq \max\{\ell(\mathbf{x}), \ell(\mathbf{y})\}$. Write $\mathbf{x}' = (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x})$ and $\mathbf{y}' = (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$. Then $\mathbf{x}'_{[\,]} = \mathbf{x}_{[\,]}$ and $\mathbf{y}'_{[\,]} = \mathbf{y}_{[\,]}$. By definition, $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim^D \mathbf{x}$ and $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y}) \sim^D \mathbf{y}$, so that $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim \mathbf{x}$ and $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y}) \sim \mathbf{y}$ since $\succsim^D$ is a subrelation to $\succsim$. Since, by assumption, $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$, we have $\mathbf{x} \sim \mathbf{y}$ by transitivity. By the same argument, $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim \mathbf{x}$ and $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{y}) \sim \mathbf{y}$, since also $\ell \geq \max\{\ell(\mathbf{x}), \ell(\mathbf{y})\}$. As $\mathbf{x} \sim \mathbf{y}$, we obtain $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell \mathbb{1}_{\mathbb{N}}, \mathbf{y})$ by transitivity.

Case 2: $\ell < \min\{\ell(\mathbf{x}), \ell(\mathbf{y})\}$. If $\ell = \bar{\ell}$, then there does not exist $r \in \mathbb{N}$ such that $\max\{y_{[r]}, \underline{\ell}\} < x_{[r]} \leq \ell$ (or $\max\{x_{[r]}, \underline{\ell}\} < y_{[r]} \leq \ell$). To show this, suppose to the contrary that $\ell = \bar{\ell}$ and there exists $r \in \mathbb{N}$ such that $\max\{y_{[r]}, \underline{\ell}\} < x_{[r]} \leq \ell$. Choose ℓ' and ℓ'' such that $\max\{y_{[r]}, \underline{\ell}\} < \ell' < \ell'' < x_{[r]}$. Since $x_{[r]} \leq \bar{\ell} \leq \ell(\mathbf{y})$, there exists $n > r$ such that $y_{[n]} \geq \ell''$. Write $\mathbf{x}' = (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x})$, $\mathbf{y}' = (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$, $\mathbf{x}'' = (\ell'' \mathbb{1}_{\mathbb{N}}, \mathbf{x})$, and $\mathbf{y}'' = (\ell'' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$. By definition, $\mathbf{x}'_{[\,]}$, $\mathbf{y}'_{[\,]} \in \mathbf{X}_n^{\ell'}$ on which $\succsim^D$ and $\succsim$ coincide, so that $\mathbf{x}' \sim^D \mathbf{y}'$ is equivalent to $\mathbf{x}' \sim \mathbf{y}'$. Likewise, $\mathbf{x}''_{[\,]}$, $\mathbf{y}''_{[\,]} \in \mathbf{X}_n^{\ell''}$ on which $\succsim^D$ and $\succsim$ coincide, so that $\mathbf{x}'' \sim^D \mathbf{y}''$ is equivalent to $\mathbf{x}'' \sim \mathbf{y}''$. By assumption, $\mathbf{x}' \sim \mathbf{y}'$ and $\mathbf{x}'' \sim \mathbf{y}''$, implying that $\mathbf{x}' \sim^D \mathbf{y}'$ and $\mathbf{x}'' \sim^D \mathbf{y}''$. However, by construction

$$w_n^u(\mathbf{x}'') - w_n^u(\mathbf{x}') \geq (n - r)(\ell'' - \ell') > w_n^u(\mathbf{y}'') - w_n^u(\mathbf{y}')\,.$$

Since $w_n^u(\mathbf{x}') = w_n^u(\mathbf{y}')$ and $w_n^u(\mathbf{x}'') = w_n^u(\mathbf{y}'')$ are needed for $\mathbf{x}' \sim^D \mathbf{y}'$ and $\mathbf{x}'' \sim^D \mathbf{y}''$, we obtain a contradiction. Thus, since there does not exist $r \in \mathbb{N}$ such that $\max\{y_{[r]}, \underline{\ell}\} < x_{[r]} \leq \ell$ (or $\max\{x_{[r]}, \underline{\ell}\} < y_{[r]} \leq \ell$), we obtain that $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$ for all $\ell' \in (\underline{\ell}, \ell)$ implies $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim^D (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$ for all $\ell' \in (\underline{\ell}, \ell]$. Therefore $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell \mathbb{1}_{\mathbb{N}}, \mathbf{y})$ since $\succsim^D$ is a subrelation to $\succsim$.

Likewise, if $\ell = \underline{\ell}$, in which case it suffices to let $\bar{\ell} \in (\ell, \min\{\ell(\mathbf{x}), \ell(\mathbf{y})\})$, there does not exist $r \in \mathbb{N}$ such that $\ell \leq x_{[r]} < \min\{y_{[r]}, \bar{\ell}\}$ (or $\ell \leq y_{[r]} < \min\{x_{[r]}, \bar{\ell}\}$). Similarly, this can be used to show that $(\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$ for all $\ell' \in (\ell, \bar{\ell})$ implies $(\ell \mathbb{1}_{\mathbb{N}}, \mathbf{x}) \sim (\ell \mathbb{1}_{\mathbb{N}}, \mathbf{y})$ also in this subcase.

Case 3: $\min\{\ell(\mathbf{x}), \ell(\mathbf{y})\} \leq \ell \leq \max\{\ell(\mathbf{x}), \ell(\mathbf{y})\}$. Suppose that $\ell(\mathbf{x}) > \ell(\mathbf{y})$ (the case $\ell(\mathbf{x}) < \ell(\mathbf{y})$ is similar). We first show that $(\underline{\ell}, \bar{\ell}) \cap (\ell(\mathbf{y}), \ell(\mathbf{x})) = \emptyset$. Because if there exists $\ell' \in (\underline{\ell}, \bar{\ell})$ such that $\ell(\mathbf{y}) < \ell' < \ell(\mathbf{x})$, then, by assumption, $\mathbf{x}' \sim \mathbf{y}'$, where $\mathbf{x}' = (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{x})$ and $\mathbf{y}' = (\ell' \mathbb{1}_{\mathbb{N}}, \mathbf{y})$. However, $\mathbf{x}' \succ^D \mathbf{y}'$ since $\ell(\mathbf{x}') > \ell(\mathbf{y}')$, and thus $\mathbf{x}' \succ \mathbf{y}'$ since $\succsim^D$ is a subrelation to $\succsim$, leading to a contradiction. Hence, either $\ell = \underline{\ell} = \max\{\ell(\mathbf{x}), \ell(\mathbf{y})\}$, which allows the proof of Case 1 to be adapted, or $\ell = \bar{\ell} = \min\{\ell(\mathbf{x}), \ell(\mathbf{y})\}$, which allows the proof of Case 2 to be adapted.

$\succsim$ *satisfies Monotonicity.* For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, if $\mathbf{x} > \mathbf{y}$, then $\mathbf{x}_{[\,]} > \mathbf{y}_{[\,]}$. Since $u \in U$ is increasing, $u(x_{[1]}) \geq u(y_{[1]})$ and $x_{[r]} \geq y_{[r]}$ for all $r \geq 2$. Hence, $\mathbf{x} \succsim^D \mathbf{y}$ and $\mathbf{x} \succsim \mathbf{y}$ since $\succsim^D$ is a subrelation to $\succsim$.

$\succsim$ *satisfies Strong Anonymity.* For any $\pi \in \Pi$ and $\mathbf{x} \in \mathbf{X}$, if $\mathbf{y} = \mathbf{x}_\pi$, then $\mathbf{x}_{[\,]} = \mathbf{y}_{[\,]}$. Hence, $\mathbf{x} \sim^D \mathbf{y}$ and $\mathbf{x} \sim \mathbf{y}$ since $\succsim^D$ is a subrelation to $\succsim$.

$\succsim$ *satisfies Liminf-Restricted Dominance.* Consider any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ with $x_i > y_i$ for some $i \in \mathbb{N}$, $\liminf_{j \in \mathbb{N}} x_j > y_i$, and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$. Write $n = \big|\{j \in \mathbb{N} | y_j \leq y_i\}\big|$. There are two cases.

Case 1: $x_i \geq \ell(\mathbf{x})$. Write $n = \big|\{j \in \mathbb{N} : y_j \leq y_i\}\big|$. Then $x_{[r]} = y_{[r]}$ for all $r < n$,

$x_{[n]} = y_{[n+1]} > y_{[n]} = y_i$, and $x_{[r]} \geq x_{[r-1]} = y_{[r]}$ for all $r > n$. Hence, $w_n^u(\mathbf{x}) - w_n^u(\mathbf{y}) > 0$ (since $u \in U$ is increasing) and $x_{[r]} \geq y_{[r]}$ for all $r > n$. Therefore, $\mathbf{x} \succ^D \mathbf{y}$ and $\mathbf{x} \succ \mathbf{y}$ since $\succsim^D$ is a subrelation to $\succsim$.

Case 2: $x_i < \ell(\mathbf{x})$. Write $m = \left|\{j \in \mathbb{N} : x_j \leq x_i\}\right|$ and $n = \left|\{j \in \mathbb{N} : y_j \leq y_i\}\right|$. Note that $m \geq n$ since $x_i > y_i$ and $x_j = y_j$ for all $j \in \mathbb{N} \setminus \{i\}$. Then $x_{[r]} = y_{[r]}$ for all $r < n$, $x_{[n]} = \min\{x_i, y_{[n+1]}\} > y_{[n]} = y_i$, $x_{[r]} \geq x_{[r-1]} = y_{[r]}$ for all $r \in \{n+1, n+2, \ldots, m\}$, and $x_{[r]} = y_{[r]}$ for all $r > m$. Hence, $w_m^u(\mathbf{x}) - w_m^u(\mathbf{y}) > 0$ (since $u \in U$ is increasing) and $x_{[r]} = y_{[r]}$ for all $r > m$. Therefore, $\mathbf{x} \succ^D \mathbf{y}$ and $\mathbf{x} \succ \mathbf{y}$ since $\succsim^D$ is a subrelation to $\succsim$.

$\succsim$ *satisfies Liminf-Restricted Finite Separability.* Consider any $\mathbf{x}$, $\mathbf{y}$, $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}} \in \mathbf{X}$ for which there exists a finite subset $N \subset \mathbb{N}$, such that:

(i) $x_i = \tilde{x}_i < \min\{\ell(\mathbf{x}), \ell(\tilde{\mathbf{x}})\}$ for all $i \in N$;

(ii) $y_i = \tilde{y}_i < \min\{\ell(\mathbf{y}), \ell(\tilde{\mathbf{y}})\}$ for all $i \in N$;

(iii) $x_j = y_j$ and $\tilde{x}_j = \tilde{y}_j$ for all $j \in \mathbb{N} \setminus N$.

Write $\bar{x} = \max_{i \in N} x_i$, $m = \left|\{j \in \mathbb{N} : x_j \leq \bar{x}\}\right|$, $\tilde{m} = \left|\{j \in \mathbb{N} : \tilde{x}_j \leq \bar{x}\}\right|$, and $\bar{m} = \max\{m, \tilde{m}\}$. Also, write $\bar{y} = \max_{i \in N} y_i$, $n = \left|\{j \in \mathbb{N} : y_j \leq \bar{y}\}\right|$, $\tilde{n} = \left|\{j \in \mathbb{N} : \tilde{y}_j \leq \bar{y}\}\right|$, and $\bar{n} = \max\{n, \tilde{n}\}$. By definition, we have $x_{[r]} = y_{[r]}$ for $r > \bar{m}$, so that $\mathbf{x} \succsim^D \mathbf{y}$ if and only if $w_{\bar{m}}^u(\mathbf{x}) - w_{\bar{m}}^u(\mathbf{y}) \geq 0$, that is if and only if $\sum_{i \in N} u(x_i) \geq \sum_{i \in N} u(y_i)$. Likewise, we have $\tilde{x}_{[r]} = \tilde{y}_{[r]}$ for $r > \bar{n}$, so that $\tilde{\mathbf{x}} \succsim^D \tilde{\mathbf{y}}$ if and only if $w_{\bar{m}}^u(\tilde{\mathbf{x}}) - w_{\bar{m}}^u(\tilde{\mathbf{y}}) \geq 0$, that is if and only if $\sum_{i \in N} u(\tilde{x}_i) \geq \sum_{i \in N} u(\tilde{y}_i)$. But since $\sum_{i \in N} u(x_i) = \sum_{i \in N} u(\tilde{x}_i)$ and $\sum_{i \in N} u(y_i) = \sum_{i \in N} u(\tilde{y}_i)$, we have that $\mathbf{x} \succsim^D \mathbf{y}$ if and only if $\tilde{\mathbf{x}} \succsim^D \tilde{\mathbf{y}}$. Because $\succsim^D$ is a subrelation to $\succsim$, this implies that $\mathbf{x} \succsim \mathbf{y}$ if and only if $\tilde{\mathbf{x}} \succsim \tilde{\mathbf{y}}$.

$\succsim$ *satisfies Critical-Level Consistency.* Consider any $\mathbf{x} \in \mathbf{X}$ and $z \in \mathbb{R}$. If $z \geq \ell(\mathbf{x})$ then $\mathbf{x}_{[\,]} = \mathbf{y}_{[\,]}$ whenever $\mathbf{y} = (z, \mathbf{x})$; similarly, $\mathbf{x}_{[\,]} = \tilde{\mathbf{y}}_{[\,]}$ whenever $\tilde{\mathbf{y}} = (z\mathbb{1}_{\mathbb{N}}, \mathbf{x})$. Thus, if $z \geq \ell(\mathbf{x})$, $\mathbf{x} \sim^D (z, \mathbf{x}) \sim^D (z\mathbb{1}_{\mathbb{N}}, \mathbf{x})$. Because $\succsim^D$ is a subrelation to $\succsim$, we obtain that $\mathbf{x} \sim^D (z, \mathbf{x}) \sim^D (z\mathbb{1}_{\mathbb{N}}, \mathbf{x})$. If $z < \ell(\mathbf{x})$ and $\mathbf{y} = (z, \mathbf{x})$, then there exists $n = \left|\{j \in \mathbb{N} : y_j \leq z\}\right|$ such that $x_{[r]} = y_{[r]}$ for all $r < n$, $x_{[n]} = y_{[n+1]} > y_{[n]} = z$, and $x_{[r]} \geq x_{[r-1]} = y_{[r]}$ for all $r > n$. Hence, $w_n^u(\mathbf{x}) - w_n^u(\mathbf{y}) > 0$ (since $u \in U$ is increasing) and $x_{[r]} \geq y_{[r]}$ for all $r > n$. Therefore, $\mathbf{x} \succ^D \mathbf{y}$ and $\mathbf{x} \succ \mathbf{y}$ since $\succsim^D$ is a subrelation to $\succsim$. If $z < \ell(\mathbf{x})$ and $\tilde{\mathbf{y}} = (z\mathbb{1}_{\mathbb{N}}, \mathbf{x})$, $\ell(\tilde{\mathbf{y}}) < \ell(\mathbf{x})$ so that $\mathbf{x} \succ^D (z\mathbb{1}_{\mathbb{N}}, \mathbf{x})$ and $\mathbf{x} \succ (z\mathbb{1}_{\mathbb{N}}, \mathbf{x})$ since $\succsim^D$ is a subrelation to $\succsim$. □

Lemma 7. *Let* a, $b \in \mathbb{R}$ *be such that* $a < b$. *If* $u \in U$, *then there exist* $0 < \Gamma < \Lambda$ *satisfying that, for any* c, $d \in \mathbb{R}$ *such that* $a \leq c < d \leq b$,

$$\Gamma|d - c| < |u(d) - u(c)| < \Lambda|d - c|\,.$$

Proof. Assume that $u \in U$; thus, $u : \mathbb{R} \to \mathbb{R}$ is continuous, increasing and finitely non-concave. Let a, $b \in \mathbb{R}$ be such that $a < b$. Consider any c, $d \in \mathbb{R}$ such that $a \leq c < d \leq b$ and let e, $f \in \mathbb{R}$

be such that $e < a$ and $f > b$. Because u is finitely non-concave, there exists $\Theta > 0$ such that

$$\left[\tfrac{u(d)-u(c)}{d-c}\Big/\tfrac{u(a)-u(e)}{a-e}\right] < \Theta \quad \text{and} \quad \left[\tfrac{u(f)-u(b)}{f-b}\Big/\tfrac{u(d)-u(c)}{d-c}\right] < \Theta\,.$$

Write

$$\Gamma = \tfrac{1}{\Theta}\tfrac{u(f)-u(g)}{f-b} \quad \text{and} \quad \Lambda = \Theta\tfrac{u(a)-u(e)}{a-e}\,.$$

Given that u is increasing, $\Gamma > 0$. Furthermore,

$$\left[\tfrac{u(d)-u(c)}{d-c}\Big/\tfrac{u(a)-u(e)}{a-e}\right] < \Theta \quad \text{implies that} \quad u(d)-u(c) < \Lambda(d-c)\,,$$

which is equivalent to $|u(d)-u(c)| < \Lambda|d-c|$. And

$$\left[\tfrac{u(f)-u(b)}{f-b}\Big/\tfrac{u(d)-u(c)}{d-c}\right] < \Theta \quad \text{implies that} \quad u(d)-u(c) > \Gamma(d-c)$$

which is equivalent to $|u(d)-u(c)| > \Gamma|d-c|$. □

Lemma 8. *For any* $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ *and any* $\pi \in \Pi$, *if* $\sum_{i=1}^{\infty} |x_{\pi(i)} - y_i| < +\infty$, *then*

$$\sum_{r=1}^{\infty} \left|x_{[r]} - y_{[r]}\right| \le \sum_{i=1}^{\infty} |x_{\pi(i)} - y_i|\,.$$

Proof. Step 1. We show that for any $n \in \mathbb{N} \setminus \{1\}$, for any $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in \mathbb{R}^n_+$ with $x_1 \le \cdots \le x_n$ and $y_1 \le \cdots \le y_n$, and for any permutations π_1 and π_2 on $\{1, \ldots, n\}$,

$$\sum_{i=1}^{n} |x_i - y_i| \le \sum_{i=1}^{n} |x_{\pi_1(i)} - y_{\pi_2(i)}|\,. \tag{2}$$

We begin by showing that (2) holds if $n = 2$. Let $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2_+$ with $x_1 \le x_2$ and $y_1 \le y_2$. Also, let π_1 and π_2 be permutations on $\{1, 2\}$. Without loss of generality, we assume π_1 is the identity mapping. Thus, we show that

$$|x_1 - y_1| + |x_2 - y_2| \le |x_1 - y_2| + |x_2 - y_1|\,. \tag{3}$$

If $x_1 = x_2$ or $y_1 = y_2$, then (3) holds with equality. Suppose that $x_1 < x_2$ and $y_1 < y_2$. We distinguish two cases. First, suppose that $x_1 \le y_1 < y_2 \le x_2$. Then,

$$|x_1 - y_1| + |x_2 - y_2| < |x_1 - y_2| + |x_2 - y_2| < |x_1 - y_2| + |x_2 - y_1|.$$

Thus, (3) holds. Next, suppose that $x_1 \le y_1$ and $x_2 \le y_2$. Then,

$$\begin{aligned} |x_1 - y_1| + |x_2 - y_2| &= |y_1 - x_1 + y_2 - x_2| \\ &\le |y_1 - x_2| + |y_2 - x_1| = |x_1 - y_2| + |x_2 - y_1|. \end{aligned}$$

Thus, (3) holds.

We now consider the case where $n > 2$. Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{R}^n_+$ with $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$. Consider any permutations π_1 and π_2 on $\{1, \ldots, n\}$. Again, without loss of generality, we assume π_1 is the identity mapping. For each $k \in \{0, 1, \ldots, n\}$, we define $(y^k_1, \ldots, y^k_n) \in \mathbb{R}^n_+$ as follows. If $k = 0$, $(y^0_1, \ldots, y^0_n) = (y_{\pi_2(1)}, \ldots, y_{\pi_2(n)})$. For each $k \in \{1, \ldots, n\}$,

$$y^k_i = \begin{cases} y_k & \text{if } i = k \\ y^{k-1}_k & \text{if } i = \min\{j \in \{k, \ldots, n\} : y^{k-1}_j = y_k\} \\ y^{k-1}_i & \text{otherwise.} \end{cases}$$

Note that $(y^n_1, \ldots, y^n_n) = (y_1, \ldots, y_n)$. Further, for each $k \in \{1, \ldots, n\}$, either $(y^k_1, \ldots, y^k_n) = (y^{k-1}_1, \ldots, y^{k-1}_n)$ or $(y^k_1, \ldots, y^k_n)$ is constructed by exchanging two components of $(y^{k-1}_1, \ldots, y^{k-1}_n)$ so as to make them rank-ordered. Thus, from (3), it follows that for each $k \in \{1, \ldots, n\}$,

$$\sum_{i=1}^{n} |x_i - y^k_i| \leq \sum_{i=1}^{n} |x_i - y^{k-1}_i|\,.$$

Thus, we obtain

$$\sum_{i=1}^{n} |x_i - y_i| \leq \sum_{i=1}^{n} |x_i - y_{\pi_2(i)}|\,.$$

Step 2. Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and $\pi \in \Pi$. Assume $\sum_{i=1}^{\infty} |x_{\pi(i)} - y_i| < +\infty$. We show that $\ell(\mathbf{x}) = \ell(\mathbf{y})$. By way of contradiction, suppose $\ell(\mathbf{x}) \neq \ell(\mathbf{y})$. Without loss of generality, we assume $\ell(\mathbf{x}) > \ell(\mathbf{y})$. Let $\varepsilon = (\ell(\mathbf{x}) - \ell(\mathbf{y}))/2$. From the definition of $\ell(\mathbf{x})$, there exists $N \in \mathbb{N}$ such that, for all $n > N$, $x_n > \ell(\mathbf{x}) - \varepsilon$. We define $M \subseteq \mathbb{N}$ by $M = \{n \in \mathbb{N} : y_n < \ell(\mathbf{y}) + \varepsilon$. From the definition of $\ell(\mathbf{y})$, it follows that $|M| = +\infty$. Let $\mathbf{z} = \mathbf{x}_\pi$ and define $\bar{N} = \max\{\pi(n) \in \mathbb{N} : n \in \{1, \ldots, N\}\}$. Then, for all $n > \bar{N}$, $z_n > \ell(\mathbf{x}) - \varepsilon$. Letting $\bar{M} = \{n \in M : n \geq \bar{N}\}$, we obtain $\sum_{i \in \bar{M}} |z_i - y_i| = +\infty$. Thus,

$$\sum_{i=1}^{\infty} |x_{\pi(i)} - y_i| \geq \sum_{i \in \bar{M}} |z_i - y_i| = +\infty\,.$$

This is a contradiction since $\sum_{i=1}^{\infty} |x_{\pi(i)} - y_i| < +\infty$.

Step 3. Let $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ and $\pi \in \Pi$. Suppose that $\sum_{i=1}^{\infty} |x_{\pi(i)} - y_i| < +\infty$. It follows from Step 2 that $\ell(\mathbf{x}) = \ell(\mathbf{y})$. We define $\tilde{\mathbf{x}} \in \mathbf{X}$ by, for each $i \in \mathbb{N}$,

$$\tilde{x}_i = \begin{cases} x_{\pi(i)} & \text{if } x_{\pi(i)} < \ell(\mathbf{x}), \\ \ell(\mathbf{x}) & \text{if } x_{\pi(i)} \geq \ell(\mathbf{x}). \end{cases}$$

Analogously, define $\tilde{\mathbf{y}} \in \mathbf{X}$ by, for each $i \in \mathbb{N}$,

$$\tilde{y}_i = \begin{cases} y_i & \text{if } y_i < \ell(\mathbf{y}), \\ \ell(\mathbf{y}) & \text{if } y_i \geq \ell(\mathbf{y}). \end{cases}$$

Note that for all $i \in \mathbb{N}$, $|\tilde{x}_i - \tilde{y}_i| \leq |x_{\pi(i)} - y_i|$. Thus,

$$\sum_{i=1}^{\infty} |\tilde{x}_i - \tilde{y}_i| \leq \sum_{i=1}^{\infty} |x_{\pi(i)} - y_i| . \tag{4}$$

Note that for any $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ with $N > n$ such that $(\tilde{x}_1, \ldots, \tilde{x}_N)$ and $(\tilde{y}_1, \ldots, \tilde{y}_N)$ contain all the components of $(x_{[1]}, \ldots, x_{[n]})$ and $(y_{[1]}, \ldots, y_{[n]})$, respectively. Hence, by (2),

$$\sum_{i=1}^{n} \left|x_{[i]} - y_{[i]}\right| \leq \sum_{i=1}^{N} |\tilde{x}_i - \tilde{y}_i| \leq \sum_{i=1}^{\infty} |\tilde{x}_i - \tilde{y}_i| .$$

Thus, from (4), the sequence $\{D_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}_+$ defined by $D_n = \sum_{i=1}^{n} \left|x_{[i]} - y_{[i]}\right|$ is bounded above. Since $\{D_n\}_{n\in\mathbb{N}}$ is non-decreasing, it is convergent and we obtain

$$\lim_{n\to+\infty} D_n = \sum_{i=1}^{\infty} \left|x_{[i]} - y_{[i]}\right| \leq \sum_{i=1}^{\infty} |x_{\pi(i)} - y_i| .$$

Thus, the proof is completed. □

A.2 Proof of Lemma 1

Let the Strongly Anonymous (Generalized) Utilitarian SWR $\succsim^U$ be associated with the transformation function $u \in U$.

Step 1: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, *if* $\mathbf{x} \succsim^U \mathbf{y}$, *then* $(\mathbf{x}_{[\,]}, \mathbf{y}_{[\,]}) \in \mathbf{D}$ *and* $\lim_{n\to\infty} \left(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\right) \geq 0$. By Definition 2, there exists $\Delta \in \mathbb{R}_+ \cup \{\infty\}$ such that, for all $\pi \in \Pi$, $\lim_{n\to\infty} \left(v_n^u(\mathbf{x}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\right) = \Delta$. In particular, $\lim_{n\to\infty} \left(v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]})\right) = \lim_{n\to\infty} \left(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\right) = \Delta \geq 0$. Suppose by way of contradiction that $(\mathbf{x}_{[\,]}, \mathbf{y}_{[\,]}) \notin \mathbf{D}$. This implies that

$$\lim_{n\to\infty} \sum_{r\leq n: y_{[r]} > x_{[r]}} \left(y_{[r]} - x_{[r]}\right) = +\infty.$$

Let $a = \min\{\inf(\mathbf{x}_{[\,]}), \inf(\mathbf{y}_{[\,]})\}$ and $b = \max\{\sup(\mathbf{x}_{[\,]}), \sup(\mathbf{y}_{[\,]})\}$. By Lemma 7, there exists $\Gamma > 0$ satisfying that for any c, $d \in \mathbb{R}$ such that $a \leq c < d \leq b$, $\Gamma|d - c| < |u(d) - u(c)|$. Hence, for all $n \in \mathbb{N}$,

$$\Gamma \sum_{r\leq n: y_{[r]} > x_{[r]}} \left(y_{[r]} - x_{[r]}\right) \quad < \sum_{i\leq n: y_{[r]} > x_{[r]}} \left(u(y_{[r]}) - u(x_{[r]})\right),$$

so that

$$\lim_{n\to\infty} \sum_{r\leq n: y_{[r]} > x_{[r]}} \left(u(y_{[r]}) - u(x_{[r]})\right) = +\infty .$$

There exists $\varepsilon > 0$ and $\pi \in \Pi$ which alternates between selecting elements from $\{r \in \mathbb{N} : y_{[r]} > x_{[r]}\}$ and $\{r \in \mathbb{N} : x_{[r]} > y_{[r]}\}$ and which always selects a sufficient number of elements of $\{r \in \mathbb{N} : y_{[r]} > x_{[r]}\}$ before selecting additional elements from $\{r \in \mathbb{N} : x_{[r]} > y_{[r]}\}$ such that, for all $n \in \mathbb{N}$, $v_n^u(\mathbf{x}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]}) \leq -\varepsilon$. This contradicts that $\Delta \in \mathbb{R}_+ \cup \{\infty\}$.

Step 2: For any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, *if* $(\mathbf{x}_{[\,]}, \mathbf{y}_{[\,]}) \in \mathbf{D}$ *and* $\lim_{n\to\infty} (w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})) \geq 0$, *then* $\mathbf{x} \succsim^U \mathbf{y}$. Since $(\mathbf{x}_{[\,]}, \mathbf{y}_{[\,]}) \in \mathbf{D}$, there exists $\delta \in \mathbb{R}_+$ such that $\sum_{i:y_{[r]}>x_{[r]}} \left(y_{[r]} - x_{[r]}\right) = \delta$. By Lemma 7, the sum $\sum_{i\leq n:y_{[r]}>x_{[r]}} \left(u(y_{[r]}) - u(x_{[r]})\right)$ is bounded above by $\Lambda\delta$, and it is by definition increasing. Hence, there exists $\Delta_- \in \mathbb{R}_-$ such that $\sum_{i:y_{[r]}>x_{[r]}} \left(u(x_{[r]}) - u(y_{[r]})\right) = \Delta_-$.

Write $\Delta = \lim_{n\to\infty} \left(w_n^u(\mathbf{x}) - w_n^u(\mathbf{y})\right) = \sum_{r=1}^{\infty} \left(u(x_{[r]}) - u(y_{[r]})\right)$. If $\Delta \in \mathbb{R}_+$, then we have $\Delta_+ = \sum_{i\leq n:x_{[i]}>y_{[i]}} \left(u(x_{[i]}) - u(y_{[i]})\right) = \Delta - \Delta_- < +\infty$. Hence, $\sum_{r=1}^{\infty} \left|u(x_{[r]}) - u(y_{[r]})\right| = \Delta_+ - \Delta_- < +\infty$. We obtain $\mathbf{x} \succsim^U \mathbf{y}$ since absolute convergence implies unconditional convergence. Similarly, if $\Delta = +\infty$, then $\Delta_+ = +\infty$ and $\lim_{n\to\infty} \left(v_n^u(\mathbf{x}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\right) = +\infty$ for all $\pi \in \Pi$, thereby obtaining $\mathbf{x} \succsim^U \mathbf{y}$ also in this case.

A.3 Completing the proof of Theorem 1

The proof of Theorem 1 is completed by means of the following result.

Proposition 5. *The Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$ *is regular. Consider any* SWR $\succsim$. *If there exists* $u \in U$ *such that the Strongly Anonymous (Generalized) Utilitarian* SWR $\succsim^U$ *associated with* u *is a subrelation to* $\succsim$, *then* $\succsim$ *satisfies Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-Level Consistency.*

Proof. Let $u \in U$, and let $\succsim$ be a SWR to which the Strongly Anonymous Utilitarian SWR $\succsim^U$ associated with u is subrelation. By Observation 2, the Strongly Anonymous Utilitarian Dominance SWR $\succsim^D$ associated with u is a subrelation to $\succsim^U$. Hence, since $\succsim^U$ associated with u is a subrelation to the SWR $\succsim$, also $\succsim^D$ associated with u is a subrelation to $\succsim$. So, by Proposition 4, $\succsim$ satisfies Finite Completeness, Restricted Continuity, Monotonicity, Strong Anonymity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-Level Consistency. It remains to be shown that $\succsim^U$ satisfies Weak Continuity and that $\succsim$ satisfies Limited Inequity.

Step 1: $\succsim^U$ *satisfies Weak Continuity.* Consider any $\mathbf{y} \in \mathbf{X}$. Let $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ be a sequence in $\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \succsim^U \mathbf{y}\}$, and assume that $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ converges to $\mathbf{x}^*$ in d_1. We show that $\mathbf{x}^* \succsim^U \mathbf{y}$, which means that $\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \succsim^U \mathbf{y}\}$ is closed in $(\mathbf{X}, d_1)$. Since $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ converges to $\mathbf{x}^*$ in d_1, it follows from Lemma 8 that

$$\lim_{k\to+\infty} \sum_{r=1}^{\infty} \left|x_{[r]}^k - x_{[r]}^*\right| = 0. \tag{5}$$

Because $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ are bounded sequences that converge to $\mathbf{x}^*$ in d_1, then $a = \inf_{k\in\mathbb{N}}\{\inf_t \mathbf{x}_t^k\}$ and $b = \sup_{k\in\mathbb{N}}\{\sup_t \mathbf{x}_t^k\}$ are well-defined finite numbers. By Lemma 7, there exists $\Lambda \in \mathbb{R}_{++}$

such that for all $z, z' \in [a, b]$:

$$|u(z) - u(z')| \leq \Lambda |z - z'| \,. \tag{6}$$

Let $\varepsilon \in (0,\, \Lambda)$. By (5), there exists $K \in \mathbb{N}$ such that

$$\sum_{r=1}^{\infty} \left| x^K_{[r]} - x^*_{[r]} \right| < \frac{\varepsilon}{2\Lambda} \,.$$

From (6), we obtain

$$\sum_{r=1}^{\infty} \left| u(x^K_{[r]}) - u(x^*_{[r]}) \right| \leq \Lambda \sum_{i=1}^{\infty} \left| x^K_{[r]} - x^*_{[r]} \right| < \frac{\varepsilon}{2} \,. \tag{7}$$

On the other hand, since $\mathbf{x}^K \succsim^U \mathbf{y}$, it follows that

$$\lim_{n \to +\infty} \sum_{r=1}^{n} \Big(u(x^K_{[r]}) - u(y_{[r]}) \Big) \geq 0 \,.$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\sum_{r=1}^{n} \Big(u(x^K_{[r]}) - u(y_{[r]}) \Big) \geq -\frac{\varepsilon}{2} \,. \tag{8}$$

Thus, for all $n > N$, we obtain by (7) and (8) that

$$\begin{aligned}
\sum_{r=1}^{n} \Big(u(x^*_{[r]}) - u(y_{[r]}) \Big) &= \sum_{r=1}^{n} \Big(u(x^*_{[r]}) - u(x^K_{[r]}) \Big) + \sum_{r=1}^{n} \Big(u(x^K_{[r]}) - u(y_{[r]}) \Big) \\
&\geq - \sum_{r=1}^{n} \Big| u(x^*_{[r]}) - u(x^K_{[r]}) \Big| + \sum_{r=1}^{n} \Big(u(x^K_{[r]}) - u(y_{[r]}) \Big) \\
&\geq - \sum_{r=1}^{\infty} \Big| u(x^*_{[r]}) - u(x^K_{[r]}) \Big| + \sum_{r=1}^{n} \Big(u(x^K_{[r]}) - u(y_{[r]}) \Big) \\
&> -\varepsilon \,.
\end{aligned}$$

Since $\varepsilon \in (0, \Lambda)$ was chosen arbitrarily, we obtain

$$\liminf_{n \to +\infty} \sum_{r=1}^{n} \Big(u(x^*_{[r]}) - u(y_{[r]}) \Big) = \liminf_{n \to +\infty} \big(v^u_n(\mathbf{x})^*_{[\,]} - v^u_n(\mathbf{y})_{[\,]} \big) = \liminf_{n \to +\infty} \big(w^u_n(\mathbf{x}^*) - w^u_n(\mathbf{y}) \big) \geq 0 \,.$$

Note also that, for each $k \in \mathbb{N}$ and each $r \in \mathbb{N}$, $x^\star_{[i]} + |x^k_{[r]} - x^\star_{[r]}| \geq x^k_{[r]} \geq x^\star_{[r]} - |x^k_{[r]} - x^\star_{[r]}|$ and $y_{[r]} \geq y_{[r]} - |x^k_{[r]} - x^\star_{[r]}|$. Hence, $\max\{x^k_{[r]}, y_{[r]}\} \geq \max\{x^\star_{[r]}, y_{[r]}\} - |x^k_{[r]} - x^\star_{[r]}|$ and $\max\{x^k_{[r]}, y_{[r]}\} - x^k_{[r]} \geq \max\{x^\star_{[r]}, y_{[r]}\} - 2|x^k_{[r]} - x^\star_{[r]}|$ for each $k \in \mathbb{N}$ and each $t \in \mathbb{N}$. We obtain that for each $N \in \mathbb{N}$ and $k \in \mathbb{N}$,

$$\sum_{r=1}^{N} \big[\max\{x^k_{[r]}, y_{[r]}\} - x^k_{[r]} \big] \geq \sum_{r=1}^{N} \big[\max\{x^\star_{[r]}, y_{[r]}\} - x^\star_{[r]} \big] - 2 \sum_{r=1}^{N} |x^k_{[r]} - x^\star_{[r]}| \,.$$

Because $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ are bounded sequences that converge to $\mathbf{x}^*$ in d_1^S, there exists $K \in \mathbb{N}$ such that $\sum_{r=1}^{\infty} |x^K_{[r]} - x^\star_{[r]}|$ is finite. This implies that

$$\sum_{r:y_{[r]}>x_{[r]}} (y_{[r]} - x_{[r]}) = \sum_{r=1}^{\infty} \left[\max\{x^\star_{[i]}, y_{[r]}\} - x^\star_{[r]}\right] = \delta$$

where $\delta \in \mathbb{R}_+$. Hence $(\mathbf{x}_{[\,]}{}^*, \mathbf{y}_{[\,]}) \in \mathbf{D}$.

Finally, the fact that $(\mathbf{x}_{[\,]}{}^*, \mathbf{y}_{[\,]}) \in \mathbf{D}$ and $\liminf_{n\to+\infty} \left(w_n^u(\mathbf{x}^*) - w_n^u(\mathbf{y})\right) \geq 0$ imply that there exists $\Delta \in \mathbb{R}_+ \cup \ \{\infty\}$ such that $\lim_{n\to\infty} \left(w_n^U(\mathbf{x}^*) - w_n^U(\mathbf{y})\right) = \Delta$. Thus, $\mathbf{x}^* \succsim^U \mathbf{y}$ follows. The proof that $\{\mathbf{x} \in \mathbf{X} : \mathbf{y} \succsim^U \mathbf{x}\}$ is closed in $(\mathbf{X},\, d_1)$ is analogous.

Step 2: Any SWR $\succsim$ *to which* $\succsim^U$ *is a subrelation satisfies Limited Inequity.* We first show that $\succsim^U$ associated with u satisfies Limited Inequity. Since $u \in U$, u is a finitely non-concave function. Hence, there exists $\Gamma \in \mathbb{R}_{++}$ such that:

$$\Gamma > \left[\tfrac{u(z_4)-u(z_3)}{z_4-z_3} \Big/ \tfrac{u(z_2)-u(z_1)}{z_2-z_1}\right]$$

for all $z_1 < z_2 \leq z_3 < z_4$. Consider any $\mathbf{x},\ \mathbf{y} \in \mathbf{X}$ with $y_i < x_i \leq x_j < y_j$ for some $i,\ j \in \mathbb{N}$, $x_k = y_k$ for all $k \in \mathbb{N} \setminus \{i, j\}$, and $y_j - x_j = (1/\Gamma)\,(x_i - y_i)$. There are three cases:

1. If $y_j < \ell(\mathbf{y})$, then writing $n = \left|\{k \in \mathbb{N} : y_k \leq y_j\}\right|$, $x_{[k]} = y_{[k]}$ for all $k > n$ and

$$\begin{aligned} v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]}) &= u(x_i) + u(x_j) - u(y_i) - u(y_j) \\ &= \tfrac{u(x_i)-u(y_i)}{\Gamma}\left[\Gamma - \tfrac{u(y_j)-u(x_j)}{y_j-x_j} \Big/ \tfrac{u(x_i)-u(y_i)}{x_i-y_i}\right] > 0 \end{aligned}$$

   because u is increasing. Hence, $\mathbf{x} \succsim^D \mathbf{y}$ and $\mathbf{x} \succsim^U \mathbf{y}$ since $\succsim^D$ is a subrelation to $\succsim^U$.

2. If $x_j < \ell(\mathbf{y}) \leq y_j$, then, for any $n > \left|\{k \in \mathbb{N} : x_k \leq x_j\}\right|$, $y_{[n]} \geq x_{[n]} > x_j$ and $v_n^u(\mathbf{x}_{[\,]}) - v_n^u(\mathbf{y}_{[\,]}) = u(x_i) + u(x_j) - u(y_i) - u(y_{[n]})$. Hence, since $y_{[n]}$ is non-decreasing and converges to $\ell(\mathbf{y})$, we have, for all $\pi \in \Pi$,

$$\begin{aligned} \lim_{n\to\infty} \left(v_n^u(\mathbf{x}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\right) &= u(x_i) + u(x_j) - u(y_i) - u(\ell(\mathbf{y})) \\ &\geq u(x_i) + u(x_j) - u(y_i) - u(y_j) \\ &= \tfrac{u(x_i)-u(y_i)}{\Gamma}\left[\Gamma - \tfrac{u(y_j)-u(x_j)}{y_j-x_j} \Big/ \tfrac{u(x_i)-u(y_i)}{x_i-y_i}\right] > 0 \end{aligned}$$

   because u is increasing, implying that $\mathbf{x} \succsim^U \mathbf{y}$.

3. If $\ell(\mathbf{y}) \leq x_j$, then, for all $\pi \in \Pi$,

$$\lim_{n\to\infty} \left(v_n^u(\mathbf{x}_{[\pi]}) - v_n^u(\mathbf{y}_{[\pi]})\right) = u\big(\min\{x_i, \ell(\mathbf{y})\}\big) - u\big(\min\{y_i, \ell(\mathbf{y})\}\big) \geq 0$$

   because $x_i > y_i$ and u is increasing, implying that $\mathbf{x} \succsim^U \mathbf{y}$.

Hence, $1/\Gamma$ is in the set $\mathcal{E}$ determined for $\succsim^U$, implying that $\succsim^U$ satisfies Limited Inequity.

However, any SWR $\succsim$ to which $\succsim^U$ is a subrelation has also the property that $\mathbf{x} \succsim \mathbf{y}$ for any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$ as considered above. Hence, $1/\Gamma$ is also in the set $\mathcal{E}$ determined for $\succsim$, implying that also $\succsim$ satisfied Limited Inequity. □

A.4 Proof of Lemma 2

Step 1: Representing a complete SWR $\succsim_n^\ell$ *on* $(-\infty, \ell]^n$. Let $\ell \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geq 3$. We define the binary relation $\succsim_n^\ell$ on $(-\infty, \ell]^n$ as follows. For all $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in (-\infty, \ell]^n$,

$$(x_1, \ldots, x_n) \succsim_n^\ell (y_1, \ldots, y_n) \quad \text{if and only if} \quad \mathbf{x}_n\mathbf{z} \succsim \mathbf{y}_n\mathbf{z}\,, \tag{9}$$

where $\mathbf{z} = \ell \mathbb{1}_{\mathbb{N}}$. Since $\succsim$ satisfies Finite Completeness and Weak Continuity, $\succsim_n^\ell$ is a continuous complete preorder on $\mathbb{R}_+^N$. Since $\succsim$ satisfies Strong Anonymity and Liminf-Restricted Finite Separability, we have that $\succsim_n^\ell$ satisfies the independence condition of Debreu (1960). Furthermore, each coordinate of $\mathbb{R}_+^n$ is essential in the sense of Debreu (1960) because $\succsim$ satisfies Liminf-Restricted Dominance. Thus, it follows from Theorem 3 in Debreu (1960) that there exist n real-valued functions $u_n^{1\ell}, \ldots, u_n^{n\ell}$ on $\mathbb{R}_+$ such that, for all $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in (-\infty, \ell]^n$,

$$(x_1, \cdots, x_n) \succsim_n^\ell (y_1, \cdots, y_n) \quad \text{if and only if} \quad \sum_{i=1}^n u_n^{i\ell}(x_i) \geq \sum_{i=1}^n u_n^{i\ell}(y_i)\,.$$

Also, the functions $u_n^{1\ell}, \ldots, u_n^{n\ell}$ are determined up to a common positive affine transformation.

Since $\succsim_n^\ell$ is a continuous complete pre-order on $\mathbb{R}_+^n$ and that $\succsim$ satisfies Strong Anonymity and Liminf-Restricted Dominance, the functions $u_n^{1\ell}, \ldots, u_n^{n\ell}$ must be the same continuous and increasing function. By defining $u_n^\ell = u_n^{i\ell}$ for all $i \in \{1, \ldots, n\}$, we obtain the following representation of $\succsim_n^\ell$ on $(-\infty, \ell]^n$, for $\ell \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n \geq 3$: There exists a continuous and increasing real-valued function u_n^ℓ on $(-\infty, \ell]$ such that, for all $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_n) \in (-\infty, \ell]^n$,

$$(x_1, \cdots, x_n) \succsim_n^\ell (y_1, \cdots, y_n) \quad \text{if and only if} \quad \sum_{i=1}^n u_n^\ell(x_i) \geq \sum_{i=1}^n u_n^\ell(y_i)\,. \tag{10}$$

Note that u_n^ℓ is determined up to a positive affine transformation.

Step 2: Link between the complete SWR*s* $\succsim_n^\ell$ *for different* $n \in \mathbb{N}$. Fix $\ell \in \mathbb{R}$ and write $u^\ell = u_3^\ell$. We show that u^ℓ has the property, for all $n \geq 3$ it is the case that, for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n^\ell$,

$$\mathbf{x} \succsim \mathbf{y} \quad \text{if and only if} \quad \sum_{i=1}^n u^\ell(x_i) \geq \sum_{i=1}^n u^\ell(y_i)\,. \tag{11}$$

It is straightforward to prove this if $n = 3$. Thus, let $n > 3$. Then, it follows from (9) and

(10) that for any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_3^\ell$,

$$\begin{aligned}(x_1,x_2,x_3) \succsim_3^\ell (y_1,y_2,y_3) \;&\Leftrightarrow\; \mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; (x_1,x_2,x_2,\ell,\ldots,\ell) \succsim_n^\ell (y_1,y_2,y_3,\ell,\ldots,\ell)\\ &\Leftrightarrow\; \sum_{i=1}^{3} u_n^\ell(x_i) \geq \sum_{i=1}^{3} u_n^\ell(y_i).\end{aligned}$$

Recall that u^ℓ is determined up to a positive affine transformation. Thus, there exist $a_n^\ell \in \mathbb{R}_{++}$ and $b_n^\ell \in \mathbb{R}$ such that $u_n^\ell = a_n^\ell u^\ell + b_n^\ell$. Thus, it follows from (9) and (10) that for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_\mathbf{n}^\ell$,

$$\begin{aligned}\mathbf{x} \succsim \mathbf{y} \;&\Leftrightarrow\; (x_1,\ldots,x_n) \succsim_n^\ell (y_1,\ldots,y_n) \;\Leftrightarrow\; \sum_{i=1}^{n} u_n^\ell(x_i) \geq \sum_{i=1}^{n} u_n^\ell(y_i)\\ &\Leftrightarrow\; \sum_{i=1}^{n} a_n^\ell u^\ell(x_i) + nb_n^\ell \geq \sum_{i=1}^{n} a_n^\ell u^\ell(y_i) + nb_n^\ell \;\Leftrightarrow\; \sum_{i=1}^{n} u^\ell(x_i) \geq \sum_{i=1}^{n} u^\ell(y_i)\,.\end{aligned}$$

Let $n = 2$ and $\ell \in \mathbb{R}$ and assume that $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_2^\ell$. Since $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_3^\ell$, (9) and (10) imply that

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \sum_{i=1}^{3} u^\ell(x_i) \geq \sum_{i=1}^{3} u^\ell(y_i) \;\Leftrightarrow\; \sum_{i=1}^{2} u^\ell(x_i) \geq \sum_{i=1}^{2} u^\ell(y_i)\,.$$

Likewise, for $n = 1$, we trivially have that $\mathbf{x} \succsim \mathbf{y}$ is equivalent to $u^\ell(x_1) \geq u^\ell(y_1)$ for $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_1^\ell$ because u^ℓ is increasing and $\succsim$ satisfies Liminf-Restricted Dominance.

Step 3: Extension to all $\ell \in \mathbb{R}$. We show that there exists a continuous and increasing function $u : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $\ell \in \mathbb{R}$, all $n \in \mathbb{N}$, and all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n^\ell$,

$$\mathbf{x} \succsim \mathbf{y} \quad \text{if and only if} \quad \sum_{i=1}^{n} u(x_i) \geq \sum_{i=1}^{n} u(y_i)\,. \tag{12}$$

From Step 2, we know that, for any $\ell \in \mathbb{R}$, there exists a continuous and increasing function $u^\ell \colon (-\infty,\ell] \to \mathbb{R}$ such that (11) holds for all $n \in \mathbb{N}$ and for all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n^\ell$. Since, for each $\ell \in \mathbb{R}$, u^ℓ is unique up to a positive affine transformation, we can without loss of generality normalize these functions for all $\ell \geq 1$ so that $u^\ell(0) = 0$ and $u^\ell(1) = 1$.

We show first that $u^\ell = u^{\ell'}$ on $(-\infty,\ell']$ for all ℓ, $\ell' \in \mathbb{R}$ satisfying $\ell > \ell' \geq 1$. Let $\ell > \ell' \geq 1$ and consider any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_3^\ell$ and $\mathbf{x}'$, $\mathbf{y}' \in \mathbf{X}_3^{\ell'}$ such that $(x_1,x_2,x_3) = (x_1',x_2',x_3')$ and $(y_1,y_2,y_3) = (y_1',y_2',y_3')$, implying that none of the three first components exceeds ℓ'. Since $\succsim$ satisfies Liminf-Restricted Finite Separability:

$$\mathbf{x} \succsim \mathbf{y} \;\Leftrightarrow\; \mathbf{x}' \succsim \mathbf{y}' \;\Leftrightarrow\; \sum_{i=1}^{3} u^{\ell'}(x_i) \geq \sum_{i=1}^{3} u^{\ell'}(y_i)\,.$$

Because $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_3^\ell$ and u^ℓ in representation (11) is unique up to an affine transformation, there exists $a^\ell \in \mathbb{R}_{++}$ and b^ℓ such that $u^\ell = a^\ell u^{\ell'} + b^\ell$ on $(-\infty,1]$. On the other hand, since by assumption u^ℓ and $u^{\ell'}$ are normalized with $u^\ell(0) = u^1(0) = 0$ and $u^\ell(1) = u^{\ell'}(1) = 1$, we obtain

$b^\ell = 0$ and $a^\ell = 1$ so that $u^\ell(x) = u^{\ell'}(x)$ for all $x \in (-\infty, \ell']$.

Likewise, we obtain that, for all $\ell \in \mathbb{R}$ with $\ell < 1$, all $n \in \mathbb{N}$, and all $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_n^\ell$,

$$\mathbf{x} \succsim \mathbf{y} \quad \Leftrightarrow \quad \sum_{i=1}^{n} u^1(x_i) \geq \sum_{i=1}^{n} u^1(y_i)\,. \tag{13}$$

Define the function $u\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{aligned}&\text{for all } z \leq 1, \quad u(z) = u^1(z).\\ &\text{for all } z > 1, \quad u(z) = u^\ell(z) \text{ for some } \ell \geq z.\end{aligned}$$

Since $u^\ell = u^{\ell'}$ on $(-\infty, \ell']$ for all ℓ, $\ell' \in \mathbb{R}$ satisfying $\ell > \ell' \geq 1$, u is a well-defined real-valued function on $\mathbb{R}$. Also, it follows that, for all $\ell \in \mathbb{R}$ and all $z \in (-\infty, \ell]$, $u(z) = u^\ell(z)$. Thus, u is continuous and increasing on $\mathbb{R}$ since, for all $\ell \in \mathbb{R}$, u^ℓ in continuous and increasing on $(-\infty, \ell]$.

Step 4: u is finitely non-concave. Since $\succsim$ satisfies Limited Inequity, there exists $\varepsilon > 0$ such that, for all $z_1 < z_2 \leq z_3 < z_4$ with $z_4 - z_3 = \varepsilon(z_2 - z_1)$, we have that $\mathbf{x} \succsim \mathbf{y}$ if $\mathbf{x}$ and $\mathbf{y}$ are defined by $x_1 = z_2$, $x_2 = z_3$, $y_1 = z_1$, $y_2 = z_4$ and $x_i = y_i = z_4$ for all $i \geq 3$. Because $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}_{z_4}^2$, Step 3 implies that $u(z_2) + u(z_3) \geq u(z_1) + u(z_4)$. This can be written:

$$\varepsilon \tfrac{u(z_4)-u(z_3)}{z_4-z_3} = \tfrac{u(z_4)-u(z_3)}{z_2-z_1} \leq \tfrac{u(z_2)-u(z_1)}{z_2-z_1}.$$

So, for all $z_1 < z_2 \leq z_3 < z_4$ such that $z_4 - z_3 = \varepsilon(z_2 - z_1)$:

$$\tfrac{1}{\varepsilon} \geq \tfrac{u(z_4)-u(z_3)}{z_4-z_3} \Big/ \tfrac{u(z_2)-u(z_1)}{z_2-z_1}.$$

Writing $\mathbf{X}_\varepsilon = \{(z_1, z_2, z_3, z_4) \in \mathbb{R}^4 | z_1 < z_2 \leq z_3 < z_4 \text{ and } z_4 - z_3 = \varepsilon(z_2 - z_1)\}$ and

$$G_u(\varepsilon) = \sup_{(z_1,z_2,z_3,z_4)\in\mathbf{X}_\varepsilon} \tfrac{u(x_4)-u(x_3)}{x_4-x_3} \big/ \tfrac{u(x_2)-u(x_1)}{x_2-x_1}\,,$$

Chateauneuf, Cohen, and Meilijson (2005, Lemma 1) show that $G_u = G_u(\varepsilon)$ for any $\varepsilon \in \mathbb{R}_{++}$. Our results above imply that $G_u(\varepsilon) < 1/\varepsilon$. Hence, $G_u < +\infty$ and u is finitely non-concave.

A.5 Proof Proposition 2

Let the Limit of Rank-Discounted Utilitarian SWR $\succsim^R$, the Strongly Anonymous Utilitarian Cesàro Summation SWR $\succsim^S$, the Strongly Anonymous Utilitarian Catching-Up SWR $\succsim^C$, and the Strongly Anonymous Utilitarian SWR $\succsim^U$ all be associated with u.

Proof of part (i): $\succsim^R$ and $\succsim^S$ are regular SWR*s.* By Observation 3 and Proposition 2, $\succsim^R$ and $\succsim^S$ satisfy Finite Completeness, Restricted Continuity, Monotonicity, and Strong Anonymity. From Definition 4, it remains to be shown that $\succsim^R$ and $\succsim^S$ satisfy Weak Continuity.

$\succsim^S$ satisfies Weak Continuity. Consider any $\mathbf{y} \in \mathbf{X}$. Let $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ be a sequence in $\{\mathbf{x} \in \mathbf{X} :$

$\mathbf{x} \succsim^S \mathbf{y}\}$, and assume that $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ converges to $\mathbf{x}^*$ in d_1. We show that $\mathbf{x}^* \succsim^S \mathbf{y}$, which means that $\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \succsim^S \mathbf{y}\}$ is closed in $(\mathbf{X},\, d_1)$. For any $\mathbf{x}^k_{[\,]},\, \mathbf{x}^*_{[\,]} \in \mathbf{X}^+$, we obtain

$$\lim_{n\to+\infty} \sum_{r=1}^{n} \frac{n-r+1}{n} \left| u(x^k_{[r]}) - u(x^*_{[r]}) \right| \leq \sum_{r=1}^{\infty} \left| u(x^k_{[r]}) - u(x^*_{[r]}) \right| .$$

Thus, applying the same argument as that used in Step 1 of the proof of Proposition 5, it follows from the local Lipschitz continuity of u (Lemma 7) and the convergence of $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ that there exists $K \in \mathbb{N}$ such that

$$\lim_{n\to+\infty} \sum_{r=1}^{n} \frac{n-r+1}{n} \left| u(x^K_{[r]}) - u(x^*_{[r]}) \right| < \frac{\varepsilon}{2} , \tag{14}$$

where $\varepsilon \in (0,\, \Lambda)$ and $\Lambda \in \mathbb{R}_{++}$ are obtained in Lemma 7. Furthermore, since

$$\liminf_{n\to+\infty} \sum_{r=1}^{n} \frac{n-r+1}{n} \left(u(x^K_{[r]}) - u(y_{[r]}) \right) \geq 0 ,$$

follows from the fact that $\mathbf{x}^K \succsim^S \mathbf{y}$, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\sum_{r=1}^{n} \frac{n-r+1}{n} \left(u(x^K_{[r]}) - u(y_{[r]}) \right) \geq -\frac{\varepsilon}{2} . \tag{15}$$

Thus, employing an argument analogous to that used in Step 1 of the proof of Proposition 5, and using (14) and (15) instead of (7) and (8), it follows that, for all $n > N$,

$$\sum_{r=1}^{n} \frac{n-r+1}{n} \left(u(x^*_{[r]}) - u(y_{[r]}) \right) > -\varepsilon .$$

Hence, since $\varepsilon \in (0,\, \Lambda)$ was chosen arbitrarily, we obtain

$$\liminf_{n\to+\infty} \sum_{r=1}^{n} \frac{n-r+1}{n} \left(u(x^*_{[r]}) - u(y_{[r]}) \right) = \liminf_{n\to+\infty} \left(\sigma^u_n(\mathbf{x}^*) - \sigma^u_n(\mathbf{y}) \right) \geq 0 .$$

Thus, $\mathbf{x}^* \succsim^S \mathbf{y}$ follows. The proof that $\{\mathbf{x} \in \mathbf{X} : \mathbf{y} \succsim^S \mathbf{x}\}$ is closed in $(\mathbf{X},\, d_1)$ is analogous.

$\succsim^R$ *satisfies Weak Continuity.* Consider any $\mathbf{y} \in \mathbf{X}$. Let $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ be a sequence in $\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \succsim^R \mathbf{y}\}$, and assume that $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ converges to $\mathbf{x}^*$ in d_1. We show that $\mathbf{x}^* \succsim^R \mathbf{y}$, which means that $\{\mathbf{x} \in \mathbf{X} : \mathbf{x} \succsim^R \mathbf{y}\}$ is closed in $(\mathbf{X},\, d_1)$. For any $\mathbf{x}^k_{[\,]},\, \mathbf{x}^*_{[\,]} \in \mathbf{X}^+$, we obtain, for all $\beta \in (0,\, 1)$,

$$\sum_{r=1}^{\infty} \beta^{r-1} \left| u(x^k_{[r]}) - u(x^*_{[r]}) \right| \leq \sum_{r=1}^{\infty} \left| u(x^k_{[r]}) - u(x^*_{[r]}) \right| .$$

Thus, applying the same argument as that used in Step 1 of the proof of Proposition 5, it follows from the local Lipschitz continuity of u (Lemma 7) and the convergence of $\{\mathbf{x}^k\}_{k\in\mathbb{N}}$ that there exists $K \in \mathbb{N}$ such that for all $\beta \in (0,\, 1)$,

$$\sum_{r=1}^{\infty} \beta^{r-1} \left| u(x^K_{[r]}) - u(x^*_{[r]}) \right| < \frac{\varepsilon}{2} , \tag{16}$$

where $\varepsilon \in (0, \Lambda)$ and $\Lambda \in \mathbb{R}_{++}$ are obtained in Lemma 7. Furthermore, since

$$\liminf_{\beta \to 1^-} \left(\rho^u_\beta(\mathbf{x}^K) - \rho^u_\beta(\mathbf{y}) \right) \geq 0$$

follows from the fact that $\mathbf{x}^K \succsim^R \mathbf{y}$, there exists $\beta^* \in (0, 1)$ such that for all $\beta \in (\beta^*, 1)$,

$$\rho^u_\beta(\mathbf{x}^K) - \rho^u_\beta(\mathbf{y}) \geq -\frac{\varepsilon}{2}\,. \tag{17}$$

Thus, employing an argument analogous to that used in Step 1 of the proof of Proposition 5, and using (16) and (17) instead of (7) and (8), it follows that, for all $\beta \in (\beta^*, 1)$,

$$\begin{aligned}
\rho^u_\beta(\mathbf{x}^*) - \rho^u_\beta(\mathbf{y}) &= \rho^u_\beta(\mathbf{x}^*) - \rho^u_\beta(\mathbf{x}^K) + \rho^u_\beta(\mathbf{x}^K) - \rho^u_\beta(\mathbf{y}) \\
&= \sum_{r=1}^{\infty} \beta^{r-1} \left(u(x^*_{[r]}) - u(x^K_{[r]}) \right) + \rho^u_\beta(\mathbf{x}^K) - \rho^u_\beta(\mathbf{y}) \\
&\geq - \sum_{r=1}^{\infty} \beta^{r-1} \left| u(x^*_{[r]}) - u(x^K_{[r]}) \right| + \rho^u_\beta(\mathbf{x}^K) - \rho^u_\beta(\mathbf{y}) \\
&> -\varepsilon.
\end{aligned}$$

Hence, since $\varepsilon \in (0, \Lambda)$ was chosen arbitrarily, we obtain

$$\liminf_{\beta \to 1^-} \left(\rho^u_\beta(\mathbf{x}^*) - \rho^u_\beta(\mathbf{y}) \right) \geq 0\,.$$

Thus, $\mathbf{x}^* \succsim^R \mathbf{y}$ follows. The proof that $\{\mathbf{x} \in \mathbf{X} : \mathbf{y} \succsim^R \mathbf{x}\}$ is closed in $(\mathbf{X}, d_1)$ is analogous.

Proof of part (iii): $\succsim^U$ is a subrelation to $\succsim^C$, $\succsim^R$, and $\succsim^S$. In view of Theorem 1 it suffices to show that $\succsim^C$, $\succsim^R$, and $\succsim^S$ are regular and satisfy Limited Inequity, Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-Level Consistency. In the case of $\succsim^C$, this follows directly from Proposition 1. It follows from part (i) that $\succsim^R$ and $\succsim^S$ are regular and, by Observation 3 and Proposition 2, satisfy Liminf-Restricted Dominance, Liminf-Restricted Finite Separability, and Critical-Level Consistency. Hence, it remains to be shown that $\succsim^R$ and $\succsim^S$ satisfy Limited Inequity. This proof is omitted because it is similar to the proof that $\succsim^U$ satisfies this axiom in Step 2 of the proof Proposition 5.

Proof of part (ii): $\succsim^S$ weakly extends $\succsim^C$. In view of Proposition 1 and the proof of part (iii) it suffices to show that $\succsim^S$ satisfies Restricted Tail Continuity. This proof is omitted because it is similar to the proof that $\succsim^C$ satisfies this axiom in part 2 of the proof Proposition 1.

Proof of part (ii): $\succsim^R$ weakly extends $\succsim^S$. It follows from Jonsson and Voorneveld (2018, Lemma 1) that, for any $\mathbf{x}$, $\mathbf{y} \in \mathbf{X}$, $\liminf_{n \to +\infty} \left(\sigma^u_n(\mathbf{x}) - \sigma^u_n(\mathbf{y}) \right) \leq \liminf_{\beta \to 1^-} \left(\rho^u_\beta(\mathbf{x}) - \rho^u_\beta(\mathbf{y}) \right)$. Thus, $\mathbf{x} \succsim^S \mathbf{y}$ implies $\mathbf{x} \succsim^R \mathbf{y}$, showing that $\succsim^R$ weakly extends $\succsim^S$.

A.6 Proof of Lemma 6

Step 1: Existence of a maximum for $v^{\bar{u}}$. The Ramsey model, as described in Section 5, satisfies Assumptions 1 and 2 of Brock (1970). It therefore follows from Theorem 1 of Brock (1970) that, for any $k \in (0, k^*]$, there exists a feasible $\hat{\mathbf{x}}$ from k such that $\limsup_{n\to\infty} \left(v_n^{\bar{u}}(\hat{\mathbf{x}}) - v_n^{\bar{u}}(\mathbf{x})\right) \geq 0$ for all feasible $\mathbf{x}$ from k, where $\hat{\mathbf{x}} = x^*\mathbb{1}_{\mathbb{N}}$ if $k = k^*$. Recall that $v^{\bar{u}} : \mathbf{X} \to \mathbb{R}_- \cup \{-\infty\}$ where, for all $\mathbf{x} \in \mathbf{X}$, $v^{\bar{u}}(\mathbf{x}) = \lim_{n\to\infty} v_n^{\bar{u}}(\mathbf{x})$ is well-defined (it may be $-\infty$) and always takes a non-positive value; indeed it is an infinite sum of non-positive real numbers. Furthermore, since $\mathbf{x}^{\xi}(k)$ is feasible and satisfies $v^{\bar{u}}(\mathbf{x}^{\xi}(k)) > -\infty$, it follows that $v^{\bar{u}}(\hat{\mathbf{x}}) \geq v^{\bar{u}}(\mathbf{x}^{\xi}(k)) > -\infty$. Therefore, Theorem 1 of Brock (1970) implies that there exists a feasible $\hat{\mathbf{x}}$ from k such that $v^{\bar{u}}(\hat{\mathbf{x}}) \geq v^{\bar{u}}(\mathbf{x})$ for all feasible $\mathbf{x}$ from k, where $\hat{\mathbf{x}} = x^*\mathbb{1}_{\mathbb{N}}$ if $k = k^*$.

Step 2: If a wellbeing stream $\hat{\mathbf{x}}$ is maximum for $v^{\bar{u}}$, then $\hat{k}_i \leq k^*$ for all $i \in \mathbb{N}$. By definition, $\hat{k}_1 = k \leq k^*$. For $n \geq 2$, by way of contradiction, suppose $\hat{k}_n > k^*$ and $\hat{k}_i \leq k^*$ for $i \in \{1, \dots, n-1\}$. Then $\mathbf{x}'$ defined by $x'_i = \hat{x}_i$ for $i \in \{1, \dots, n-2\}$, $x'_{n-1} = f(\hat{k}_{n-1}) - k^* > f(\hat{k}_{n-1}) - \hat{k}_n = \hat{x}_{n-1}$, and $x'_i = x^*$ for $i \geq n$ is feasible from k since $k'_n = f(\hat{k}_{n-1}) - x'_{n-1} = k^*$. Since $\hat{x}_{n-1} < x'_{n-1} \leq x^*$ and $x'_i = \hat{x}_i$ for $i \in \{1, \dots, n-2\}$, it follows that $\sum_{i=1}^{n-1} \left(\bar{u}(x'_i) - \bar{u}(\hat{x}_i)\right) > 0$. Furthermore, $\sum_{i=n}^{\infty} \left(\bar{u}(x'_i) - \bar{u}(\hat{x}_i)\right) = -\sum_{i=n}^{\infty} \bar{u}(\hat{x}_i) \geq 0$. Hence $v^{\bar{u}}(\mathbf{x}') > v^{\bar{u}}(\hat{\mathbf{x}})$, which contradicts that $v^{\bar{u}}(\hat{\mathbf{x}}) \geq v^{\bar{u}}(\mathbf{x})$ for all feasible $\mathbf{x}$ from k.

Step 3: If a wellbeing stream $\hat{\mathbf{x}}$ is maximum for $v^{\bar{u}}$, then $\hat{\mathbf{x}} \in \mathbf{X}^*(k)$. By way of contradiction, suppose that $\hat{\mathbf{x}} \notin \mathbf{X}^*(k)$. Note that $\ell(\hat{\mathbf{x}}) = x^*$ since $\ell(\hat{\mathbf{x}}) < x^*$ would contradict $v^{\bar{u}}(\hat{\mathbf{x}}) > -\infty$ and $\ell(\hat{\mathbf{x}}) > x^*$ would contradict Lemma 5. Hence, $\hat{x}_i > x^*$ for some $i \in \mathbb{N}$. Since no stream $\mathbf{x} > x^*\mathbb{1}_{\mathbb{N}}$ is feasible from $k \leq k^*$, there exists i' such that $\hat{x}_{i'} < x^*$. Furthermore, because f is increasing, there exists ε such that $\mathbf{x}'$ defined as follows is feasible from k: $x'_i = x^*$ for all $i \in \mathbb{N}$ such that $\hat{x}_i \geq x^*$, $x'_{i'} = \hat{x}_{i'} + \varepsilon \leq x^*$, and $x'_j = \hat{x}_j$ for all other j. By construction, $\mathbf{x}' \in \mathbf{X}^*(k)$ and $v^{\bar{u}}(\mathbf{x}') > v^{\bar{u}}(\hat{\mathbf{x}})$. This contradicts that $\hat{\mathbf{x}}$ is maximum for $v^{\bar{u}}$.

Step 4: If a wellbeing stream $\hat{\mathbf{x}}$ is maximum for $v^{\bar{u}}$, then $\hat{\mathbf{x}}$ is non-decreasing. By way of contradiction, suppose that $\hat{x}_n > \hat{x}_{n+1}$ for some $n \in \mathbb{N}$. By Step 2, $\hat{k}_n \leq k^*$, while, by Step 1, $\hat{k}_n = k^*$ implies that $\hat{x}_i = x^*$ for all $i \geq n$, contradicting $\hat{x}_n > \hat{x}_{n+1}$. Hence, $\hat{k}_n < k^*$. By Step 3, $\hat{x}_{n+1} < \hat{x}_n \leq x^*$. There are two subcases:

(i) If $f(\hat{k}_n) - \hat{x}_{n+1} > k^*$, then $\mathbf{x}'$ with $x'_i = \hat{x}_i$ for $i \in \{1, \dots, n-1\}$, $x'_n = f(\hat{k}_n) - k^* > \hat{x}_{n+1}$, and $x'_i = x^*$ for $i \geq n+1$ is feasible from k. Since $\min\{x^*, x'_n\} > \hat{x}_{n+1}$, it follows that $\bar{u}(x'_n) > \bar{u}(\hat{x}_{n+1})$, which implies $\sum_{i=1}^{n-1} \left(\bar{u}(x'_i) - \bar{u}(\hat{x}_i)\right) = 0$ and $\sum_{i=n}^{\infty} \left(\bar{u}(x'_i) - \bar{u}(\hat{x}_i)\right) > 0$. Hence, $v^{\bar{u}}(\mathbf{x}') > v^{\bar{u}}(\hat{\mathbf{x}})$, which contradicts that $\hat{\mathbf{x}}$ is maximum for $v^{\bar{u}}$.

(ii) If $f(\hat{k}_n) - \hat{x}_{n+1} \leq k^*$, then $\mathbf{x}'$ with $x'_i = \hat{x}_i$ for $i \notin \{n, n+1\}$, $x'_{n+1} = \hat{x}_n$, and $x'_n = \sup X'_n$, where $X'_n = \{x \in [0, f(\hat{k}_n)] : f(f(\hat{k}_n) - x) - x'_{n+1} \geq \hat{k}_{n+2}\}$, is feasible from k since (a) $\hat{x}_{n+1} \in X'_n$ implying that X'_n is nonempty, (b) f is continuous implying that X'_n is compact and contains its supremum, and (c) by construction, $k'_{t+2} = f(f(\hat{k}_n) - x'_n) - x'_{n+1} = \hat{k}_{t+2}$. To verify that $\hat{x}_{n+1} \in X'_n$, write $k''_{n+1} = f(\hat{k}_n) - \hat{x}_{n+1}$ and $k''_{n+2} = f(\hat{k}_{n+1}) - \hat{x}_n$, implying that $\hat{x}_{n+1} \in X'_n$ is equivalent to $k''_{n+2} \geq \hat{k}_{n+2}$ since $x'_{n+1} = \hat{x}_n$. To show this, note that

$$f(\hat{k}_n) - \hat{k}_{n+1} + f(\hat{k}_{n+1}) - \hat{k}_{n+2} = \hat{x}_n + \hat{x}_{n+1} = \hat{x}_{n+1} + \hat{x}_n = f(\hat{k}_n) - k''_{n+1} + f(k''_{n+1}) - k''_{n+2}\,.$$

Hence, $k''_{n+2} - \hat{k}_{n+2} = f(k''_{n+1}) - k''_{n+1} - \big(f(\hat{k}_{n+1}) - \hat{k}_{n+1}\big) > 0$ since

$$\hat{k}_{n+1} = f(\hat{k}_n) - \hat{x}_n < f(\hat{k}_n) - \hat{x}_{n+1} = k''_{n+1} \leq k^*$$

and $f(k) - k$ is increasing on $[0, k^*]$. Since k''_{n+2} is strictly greater than $\hat{k}_{n+2}$, it follows from the continuity of f that $\min\{x^*, x'_n\} > \hat{x}_{n+1}$ and $\bar{u}(x'_n) > \bar{u}(\hat{x}_{n+1})$, which implies $\sum_{i=1}^{n-1} \big(\bar{u}(x'_i) - \bar{u}(\hat{x}_i)\big) = 0$ and $\sum_{i=n}^{\infty} \big(\bar{u}(x'_i) - \bar{u}(\hat{x}_i)\big) > 0$. Hence, $v^{\bar{u}}(\mathbf{x}') > v^{\bar{u}}(\hat{\mathbf{x}})$, which contradicts that $\hat{\mathbf{x}}$ is maximum for $v^{\bar{u}}$.

Step 5: If a wellbeing stream $\hat{\mathbf{x}}$ is maximum for $v^{\bar{u}}$, then there is no feasible wellbeing stream $\mathbf{x}'$ from k such that $\mathbf{x} > \hat{\mathbf{x}}$. If $k = k^*$, then $\hat{\mathbf{x}} = x^* \mathbb{1}_{\mathbb{N}}$ is efficient. If $k < k^*$, then $x^* \mathbb{1}_{\mathbb{N}}$ is not feasible. Furthermore, $\hat{\mathbf{x}}$ is a nondecreasing stream in $\mathbf{X}^*(k)$, implying that $\hat{x}_1 < x^*$. By way of contradiction, suppose there is $\mathbf{x}'$ feasible from k such that $\mathbf{x}' > \hat{\mathbf{x}}$. If $x'_1 > \hat{x}_1$ then $v^{\hat{u}}(\mathbf{x}') > v^{\hat{u}}(\hat{\mathbf{x}})$ contradicting that $\mathbf{x}$ is maximum for $v^{\hat{u}}$. If $x'_n > \hat{x}_n$ for $n \geq 2$, then we can construct a feasible stream $\mathbf{x}''$ such that $x''_1 > x'_1$, $x''_n = \hat{x}_n$ and $x''_i = x'_i$ for all $i \notin \{1, n\}$; indeed f is increasing and we just need to accumulate less capital in period 1. Then $v^{\hat{u}}(\mathbf{x}'') > v^{\hat{u}}(\hat{\mathbf{x}})$, which contradicts that $\hat{\mathbf{x}}$ is maximum for $v^{\hat{u}}$.

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