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Abstract

I characterize the optimal accuracy level r of an unbiased Tullock contest between two players with heterogeneous prize valuations. The designer maximizes the winning probability of the strong player or the winner's expected valuation by choosing a contest with an all-pay auction equilibrium ($r \geq 2$). By contrast, if she aims at maximizing the expected aggregate effort or the winner's expected effort, she will choose a contest with a pure-strategy equilibrium, and the optimal accuracy level $r < 2$ decreases in the players' heterogeneity. Finally, a contest designer who faces a tradeoff between selection quality and minimum (maximum) effort will never (may) chose a contest with a semi-mixed equilibrium.

JEL-Codes: C720, D720.

Keywords: Tullock contest, heterogeneous valuations, accuracy, discrimination, optimal design, all-pay auction.

Marco Sahm
Department of Economics
Otto-Friedrich University of Bamberg
Feldkirchenstraße 21
Germany – 96052 Bamberg
marco.sahm@uni-bamberg.de

1 Introduction

I characterize the optimal accuracy level, sometimes also referred to as decisiveness parameter or discriminatory power, of an *unbiased* Tullock contest between two players with heterogeneous prize valuations under different objectives. As the accuracy level affects efforts, winning probabilities, and payoffs, it is an important tool for designing a contest, particularly when an explicit bias or affirmative action is not feasible. Real world examples are countless and range from defining the type and size of the jury in litigation (number of jurors/judges) to specifying the rules in sports like car racing (technical limitations), table tennis (size of the ball), or soccer (tie breaking regulations).

The analysis thus contributes to the large literature on (optimal) contest design. More specifically, it complements the articles that emphasize the role of the accuracy of the contest success function, e.g., Nti (2004), Alcalde and Dahm (2010), Wang (2010), and, in particular, Ewerhart (2017) who provides a revenue ranking for *optimally biased* contests.

The paper is organized as follows. Section 2 introduces the formal set-up. In Sections 3 and 4, I examine the optimal accuracy level under the assumptions that the designer maximizes the winning probability of the strong player, the winner's expected valuation, the expected aggregate effort, and the winner's expected effort, respectively. Section 5 discusses the optimal solution to different tradeoffs between selection quality and effort. Section 6 concludes.

2 Set-up and Notation

I consider the standard model of a Tullock contest (Tullock, 1980) between two players with linear effort costs and use the same notation as Ewerhart (2017). Player i's probability of winning is

$$p_i = \begin{cases} 1/2 & \text{if } x_1 = x_2 = 0, \\ \frac{(x_i)^r}{(x_1)^r + (x_2)^r} & \text{else,} \end{cases}$$

where x_i denotes the effort of player $i \in \{1, 2\}$ and $r \geq 0$ describes the accuracy level of the contest.[1] Player $i \in \{1, 2\}$ chooses x_i to maximize the payoff $\Pi_i = p_i V_i - x_i$, where the players' valuations for the prize are normalized to $V_1 = 1$ and $V_2 = \omega \in (0, 1)$. I thus refer to player 1 (2) as the strong (weak) player.

Propositions 1 – 4 in Ewerhart (2017) show that, for any given $\omega \in (0, 1)$,

- there is a unique Nash equilibrium, which is in pure strategies, if $0 \leq r \leq \bar{r}$,
- there is a unique Nash equilibrium, which is in semi-mixed strategies, if $\bar{r} < r \leq 2$,
- any Nash equilibrium is an all-pay auction equilibrium[2] in mixed strategies if $2 < r$,

where $\bar{r}$ is an implicit function of ω defined by

$$\bar{r} = 1 + \omega^{\bar{r}} \quad \Leftrightarrow \quad \omega = (\bar{r} - 1)^{1/\bar{r}}. \tag{1}$$

Where appropriate, I mark equilibrium values with an asterisk.

[1]Skaperdas (1996) provides an axiomatic foundation for this type of contest success function.

[2]I.e., it yields the same expected efforts, winning probabilities and expected payoffs as well as the same expected revenue $\mathcal{R}$ for the contest designer as the unique equilibrium of the corresponding all-pay auction.

3 Maximization of Selection Quality

I first consider different objectives associated with the selection quality of the contest.

3.1 Maximization of the Strong Player's Winning Probability

Ewerhart (2017, Table 1) shows that for all $\omega \in (0,1)$ we have $dp_1^*/dr > 0$ if $0 < r \leq 2$ and $dp_1^*/dr = 0$ if $2 < r$; hence:

Proposition 1. *For any $\omega \in (0,1)$, the designer maximizes the strong player's winning probability by choosing any contest with an all-pay auction equilibrium ($r \geq 2$).*

3.2 Maximization of the Winner's Expected Valuation

Since the winner's expected equilibrium valuation equals

$$EV = p_1^* \cdot 1 + (1 - p_1^*) \cdot \omega = \omega + (1 - \omega)p_1^*$$

and $1 - \omega > 0$ for all $\omega \in (0,1)$, a contest that maximizes the strong player's winning probability also maximizes the winner's expected valuation.

Proposition 2. *For any $\omega \in (0,1)$, the designer maximizes the winner's expected valuation by choosing any contest with an all-pay auction equilibrium ($r \geq 2$).*

4 Effort Maximization

I now consider different objectives associated with effort maximization.

4.1 Maximization of Aggregate Effort

For any $\omega \in (0,1)$, Nti (2004) determines the accuracy level r that maximizes aggregate effort in the range of pure strategy equilibria, i.e., under the constraint $r \leq \bar{r}$. Alcalde and Dahm (2010) show that for any $r \geq 2$, there exists an all-pay auction equilibrium, and for any $r > 2$, any equilibrium is an all-pay auction equilibrium. Epstein et al. (2013) show that, for any $\omega \in (0,1)$, the accuracy level r that maximizes aggregate effort in the range of pure strategy equilibria also leads to a higher aggregate effort than an all-pay auction ($r \geq 2$). Wang (2010) determines a semi-mixed equilibrium for all $\bar{r} < r \leq 2$ and shows that, within this class of equilibria, the aggregate equilibrium effort $\mathcal{R}$ decreases in the accuracy level r for any $\omega \in (0,1)$, i.e., $d\mathcal{R}/dr < 0$ if $\bar{r} < r \leq 2$. Finally, Ewerhart (2017) shows that for any $r \leq 2$ the equilibrium is unique.

Together, these results allow for a unique identification of the optimal accuracy level. More explicitly, Ewerhart (2017, Table 1) shows that for any $\omega \in (0,1)$, aggregate equilibrium effort $\mathcal{R}$ is a continuous function of r. This implies that, for any $\omega \in (0,1)$, the optimal accuracy level must satisfy $r \leq \bar{r}$. It thus coincides with the optimal accuracy level within the region of pure-strategy equilibria as characterized by Nti (2004). I briefly summarize his analysis and add an exact equation for the threshold he approximates (cf. Nti, 2004, Table 1 and Proposition 3).

For any $\omega \in (0,1)$, the optimal accuracy level r maximizes aggregate equilibrium effort $\mathcal{R} = \frac{r\omega^r(1+\omega)}{(1+\omega^r)^2}$ subject to the constraint that $r \leq \bar{r}$. The first order condition

$d\mathcal{R}/dr = (1+w)\frac{w^r}{(1+w^r)^3}F(w,r) = 0$ for an unconstrained maximizer r_A implies

$$F(\omega, r_A) := 1 + \omega^{r_A} + (1-\omega^{r_A})\ln(\omega^{r_A}) = 0. \tag{2}$$

Straightforward calculations show that $F(\omega,0) = 2 > 0$, $\lim_{r_A\to\infty} F(\omega, r_A) = -\infty$, and $\partial F/\partial r_A < 0$ for all $\omega \in (0,1)$. As $d\mathcal{R}/dr$ and F have the same sign, $\mathcal{R}$ is inverted U-shaped and single-peaked. Moreover, $\partial F/\partial\omega > 0$ for all $r_A > 0$. Thus, equation 2 defines an implicit function $r_A(\omega)$ satisfying $dr_A/d\omega = -\frac{\partial F/\partial\omega}{\partial F/\partial r_A} > 0$ for all $\omega \in (0,1)$. Notice from equation (1) that $\bar{r}$ is also an increasing function of $\omega \in (0,1)$.

Proposition 3. *For any $\omega \in (0,1)$, aggregate effort is an inverted U-shaped function of the accuracy level. The designer maximizes aggregate effort by choosing a contest with a pure-strategy equilibrium. The optimal accuracy level equals $r = \min\{r_A, \bar{r}\}$ and decreases as the players' heterogeneity increases: $dr/d\omega > 0$.*[3]

Inserting $\omega = (r_A - 1)^{1/r_A}$ into equation (2) implies

$$f(r_A) := r_A + (2 - r_A)\ln(r_A - 1) = 0.$$

It is straightforward to show that f is strictly increasing for all $r_A \in (1,2)$ and has a unique root which I denote by $\bar{r}_A$. Therefore, $r_A < \bar{r}$ if and only if $r_A < \bar{r}_A$ or, equivalently, $\omega < \bar{\omega}_A$, where

$$\bar{\omega}_A := (\bar{r}_A - 1)^{1/\bar{r}_A} \quad \text{and} \quad \bar{r}_A + (2-\bar{r}_A)\ln(\bar{r}_A - 1) = 0. \tag{3}$$

Corollary 1. *The designer maximizes aggregate effort by choosing*

(a) $r = r_A$ *if* $0 < \omega < \bar{\omega}_A$,

(b) $r = \bar{r}$ *if* $\bar{\omega}_A \le \omega < 1$.

Figure 1 illustrates Proposition 3 and Corollary 1. The solid (dotted) curve depicts r_A ($\bar{r}$) as a function of ω. The curves intersect at some point $A \approx (0.2804; 1.2137)$ to the left (right) of which the optimal accuracy level is unconstrained (constrained).

4.2 Maximization of the Winner's Expected Effort

Straightforward calculations show that, for all $\omega \in (0,1)$, the winner's expected equilibrium effort $EX_W = p_1x_1 + p_2x_2$ is also a continuous function of r with $dEX_W/dr < 0$ for $\bar{r} < r \le 2$. Again, these observations imply that, for any $\omega \in (0,1)$, the optimal accuracy level must satisfy $r \le \bar{r}$ and thus coincides with the optimal accuracy level within the region of pure-strategy equilibria.

For any $\omega \in (0,1)$, the optimal accuracy level r maximizes the winner's expected equilibrium effort $EX_W = \frac{r\omega^r(1+\omega^{r+1})}{(1+\omega^r)^3}$ subject to the constraint that $r \le \bar{r}$. The first order condition $dEX_W/dr = \frac{\omega^r}{(1+\omega^r)^4}G(\omega,r) = 0$ for an unconstrained maximizer r_B implies

$$G(\omega, r_B) := 1 + \ln(\omega^{r_B}) + \omega^{r_B}[1 - 2\ln(\omega^{r_B})] + \omega^{r_B+1}[1 + 2\ln(\omega^{r_B})] + \omega^{2r_B+1}[1 - \ln(\omega^{r_B})] = 0. \tag{4}$$

[3] In a recent working paper, Chowdhury et al. (2020, Observation 1.2.1) make a similar observation and provide graphical representations (see also Wang, 2010, Figure 1).

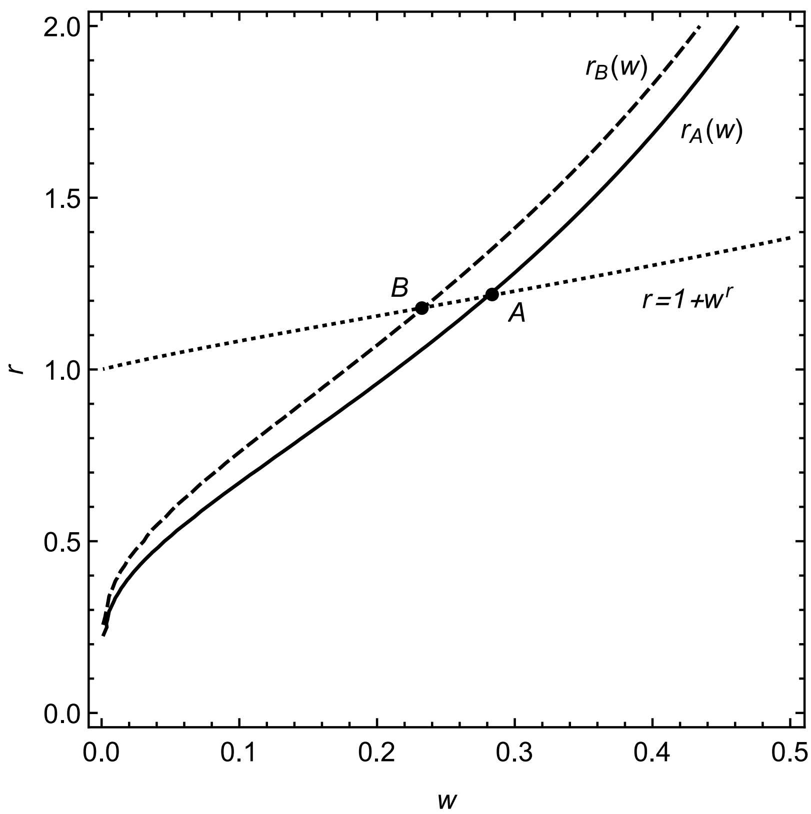

Figure 1: Optimal accuracy level and heterogeneity

Straightforward calculations show that $G(\omega,0) = 2(1+\omega) > 0$, $\lim_{r_B\to\infty} G(\omega,r_B) = -\infty$, and $\partial G/\partial r_B < 0$ for all $\omega \in (0,1)$. As dEX_W/dr and G have the same sign, EX_W is inverted U-shaped and single-peaked.

Proposition 4. *For any $\omega \in (0,1)$, the winner's expected effort is an inverted U-shaped function of the accuracy level. The designer maximizes the winner's expected effort by choosing a contest with a pure-strategy equilibrium. The optimal accuracy level equals $r = \min\{r_B, \bar{r}\}$.*

Moreover, numerical approximations suggest $dr_B/d\omega = -\frac{\partial G/\partial\omega}{\partial G/\partial r_B} > 0$ for all $\omega \in (0,1)$, i.e., the optimal accuracy level decreases as the players' heterogeneity increases.

Inserting $\omega = (r_B-1)^{1/r_B}$ into equation (2) implies

$$g(r_B) := r_B + r_B(r_B-1)^{\frac{1+r_B}{r_B}} + \ln(r_B-1)[3-2r_B+(3-r_B)(r_B-1)^{\frac{1+r_B}{r_B}}] = 0.$$

One can show that g is strictly increasing[4] and has a unique root which I denote by $\bar{r}_B$. Therefore, $r_B < \bar{r}$ if and only if $r_B < \bar{r}_B$ or, equivalently, $\omega < \bar{\omega}_B$, where

$$\bar{\omega}_B := (\bar{r}_B - 1)^{1/\bar{r}_B} \quad \text{and} \quad g(\bar{r}_B) = 0. \tag{5}$$

Corollary 2. *The designer maximizes the winner's expected effort by choosing*

(a) $r = r_B$ *if* $0 < \omega < \bar{\omega}_B$,

(b) $r = \bar{r}$ *if* $\bar{\omega}_B \le \omega < 1$.

[4] I used the software *Mathematica* to verify that $dg/dr_B > 0$ for all $r_B \in (1,2)$.

Figure 1 illustrates Proposition 4 and its Corollary. The dashed (dotted) curve depicts r_B ($\bar{r}$) as a function of ω. The curves intersect at some point $B \approx (0.2337; 1.1799)$ to the left (right) of which the optimal accuracy level is unconstrained (constrained). While maximizing the aggregate effort is equivalent to maximizing the players' average effort (with equal weights), maximizing the winner's expected effort is equivalent to maximizing the players' weighted average effort with a higher equilibrium weight $p_1 > p_2$ on the stronger player. Intuitively, the solution to this problem is thus a compromise between the maximization of aggregate effort and the maximization of the strong player's winning probability. As a result, $r_B \geq r_A$ for all $\omega \in (0,1)$. Hence, the range of heterogeneities ω for which r_B is constrained by $\bar{r}$ must be larger than that for which r_A is constrained by $\bar{r}$, i.e., $0 < \bar{\omega}_B < \bar{\omega}_A$.

In the next section, I characterize the optimal compromise between conflicting objectives more generally.

5 Conflicting Objectives

Contest designers often have multiple objectives which may conflict. During a pre-election, for example, a political party tries to select the best candidate but, at the same time, limit pre-election efforts in order to save resources for the main election campaign (Bruckner and Sahm, 2022). By contrast, the organizer of a qualifying competition tries to select the best athlete and provoke as much effort as possible because a highly intense competition attracts more attention from spectators and sponsors.

5.1 Tradeoff between Selection Quality and Minimum Effort

Obviously, the contest that minimizes aggregate effort is purely random: an accuracy level of $r = 0$ leads to zero efforts. The previous analysis thus suggests that a designer who optimally solves a tradeoff between selection quality and minimum aggregate effort (rent dissipation) will never choose a contest with a semi-mixed equilibrium because, in this range, an increasing accuracy implies both, better selection and lower efforts. More precisely, for any $\omega \in (0,1)$, he will choose an all-pay auction ($r^* \geq 2$) if and only if he puts sufficiently much weight on selection quality. Otherwise, he will choose an accuracy level $r^* < \min\{r_A, \bar{r}\}$ that leads to a pure-strategy equilibrium. A smaller upper-bound for the optimal r^* is then given by the (smallest) accuracy level r that equates the aggregate effort in the pure-strategy equilibrium and the expected aggregate effort of the all-pay auction equilibrium:

$$\frac{r\omega^r(1+\omega)}{(1+\omega^r)^2} = \frac{(1+\omega)\omega}{2} \quad \Leftrightarrow \quad H(\omega, r) := (1+\omega^r)^2 - 2r\omega^{r-1} = 0.$$

5.2 Tradeoff between Selection Quality and Maximum Effort

By contrast, a designer who optimally solves a tradeoff between selection quality and maximum aggregate effort will always choose an accuracy level r that is larger than the one that maximizes aggregate effort. In particular, he may choose a contest with a semi-mixed equilibrium (if he puts sufficiently much weight on selection quality), and will definitely chose an accuracy level $r \geq \bar{r}$ if $\bar{\omega}_A \leq \omega < 1$ (see Corollary 1).

6 Conclusion

I have examined the optimal accuracy level r of an unbiased Tullock contest between two players with heterogeneous prize valuations under different objectives. The designer maximizes the winning probability of the strong player or the winner's expected valuation by choosing a contest with an all-pay auction equilibrium ($r \geq 2$). By contrast, if she aims at maximizing the expected aggregate effort or the winner's expected effort, she will choose a contest with a pure-strategy equilibrium, and the optimal accuracy level $r < 2$ decreases in the players' heterogeneity. Finally, a contest designer who faces a tradeoff between selection quality and minimum (maximum) effort will never (may) chose a contest with a semi-mixed equilibrium.

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