CESifo WORKING PAPERS

9881 2022

August 2022

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CESifo

Impressum:

CESifo Working Papers
ISSN 2364-1428 (electronic version)
Publisher and distributor: Munich Society for the Promotion of Economic Research - CESifo GmbH
The international platform of Ludwigs-Maximilians University's Center for Economic Studies and the ifo Institute
Poschingerstr. 5, 81679 Munich, Germany
Telephone +49 (0)89 2180-2740, Telefax +49 (0)89 2180-17845, email office@cesifo.de
Editor: Clemens Fuest
https://www.cesifo.org/en/wp
An electronic version of the paper may be downloaded
· from the SSRN website: www.SSRN.com
· from the RePEc website: www.RePEc.org
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CESifo *Working Paper No. 9881*

Negative Tax Incidence with Multiproduct Firms

Abstract

A fundamental result in the theory of commodity taxation is that taxes increase consumer prices and reduce supply, aggravating the distortions caused by market power. This result hinges on the assumption that each firm provides a single product. We study the effects of commodity taxes in presence of multiproduct firms that have market power. We consider a monopolist providing two goods and obtain simple conditions such that an ad valorem tax reduces the prices and increases the supply of both goods, thereby increasing total surplus. We show that these conditions can hold in a variety of settings, including add-on pricing, multiproduct retailing with price advertising, intertemporal models with switching costs and two-sided markets.

JEL-Codes: D420, H210, H220.

Keywords: commodity taxation, tax incidence, multi-product firms, monopoly.

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August 4, 2022
We thank Helmuth Cremer, Paul Dobson, Raffaele Fiocco, Luke Garrod, Bruno Jullien, Andrew Rhodes, Jevgenijs Steinbuks, Chris Wilson and participants to presentations at Loughborough University, Bergamo University, IIOC 2022, Public Economic Theory Conference 2022, for useful discussions and comments. All errors are ours.

1 Introduction

Almost every firm in the economy sells more than one product. Transport companies, such as airlines and train operators, sell passages, baggage allowance and onboard meals. Retailers, such as supermarkets and online stores, sell different brands and product categories. Two-sided platforms, that sell different goods to different groups of users, are multiproduct firms as well. For instance, websites, newspapers and TV stations provide content to consumers and ads to firms seeking consumers' attention. A key aspect of multiproduct firms is that, typically, the price and supply of one good affect the profitability of other goods, because in general the demand for each good depends on the price of the others. As a result, multiproduct firms adopt pricing strategies differing from conventional, single-product, ones (Rhodes, 2015; Armstrong and Vickers, 2018).

Like all other firms, multiproduct suppliers are subject to indirect taxes, and their goods are often subject to different taxation regimes. For instance, different goods sold by the same retailer may be subject to different VAT rates (e.g. alcohol and food in a supermarket). Since their pricing strategies differ from single-product firms, it is natural to expect the way multiproduct firms respond to taxation to differ as well. However, the effect of taxation on multiproduct firms is a largely unexplored subject. In this paper, we study indirect taxation in markets where firms provide multiple goods and have market power. We characterize conditions such that taxation reduces prices, increases supply of untaxed and (in the case of ad valorem taxes) taxed goods, and expands total surplus. We also provide several applications where such conditions hold.

We consider a monopolist supplying two goods. We assume for simplicity that these goods have separable cost functions, but their demands are interdependent, in the sense that changes in the price of one good affect the demand for the other. These interdependencies may stem from the goods being substitutes or complements in the traditional sense, but also from search costs, or externalities across the markets in the case of a two-sided platform. Our model can also accommodate the case where the firm sells a single good, but in two successive periods. In this context, the interdependencies between demand in the two periods may arise, for instance, because of switching costs. We focus on a one-sided market in the baseline model, but we show that the analysis also applies to two-sided platforms in an extension.

Our analysis begins by studying how ad valorem taxes affect equilibrium prices and quantities, allowing for different tax rates on each good. We characterize two effects of taxation on prices: a direct effect, which captures how the tax affects the price of a good given the price of the other, and an indirect effect, which captures the change in the price of a good mediated by the tax-induced adjustment in the price of the other. Our main result is that an ad valorem tax on one good can, if the demands for the goods are interdependent, induce a reduction in the price and the supply of both goods. To see how this result can emerge, start by considering the direct effects of this tax. The ad valorem tax targets the revenue from the taxed good, so the supplier has an incentive to reduce such revenue. The revenue decreases with the price of the taxed good if and only if the equilibrium quantity lies on the inelastic part of demand, given the price of the other good. In contrast to a single-product monopolist, a multi-product one can operate on the inelastic part of demand when demands are interdependent. For instance, this occurs when lowering the price of one good stimulates demand for the another, high-margin, one. Consequently, we find that the direct effect of an ad valorem tax is negative (i.e., it tends to reduce the price) when the taxed good is complementary to the other and its marginal cost is small enough.

When demands are interdependent, a tax imposed on a good has a direct effect also on the price of the other good. The sign of this effect depends on how the price of that good affects the demand for the taxed good. The effect is negative only if the quantity of the taxed good increases in the price of the other good, as in the case where the goods are substitutes.

The indirect effect depends on the cross-price derivative of the profit function (which determines whether the two prices move in the same or in opposite directions) and on how the tax affect the other price, i.e. on its direct effect on the other price. We find that the indirect effects of unit and ad valorem taxes on the price of the taxed good are similar. The direct effects of these taxes, however, are very different. The direct effect of a unit tax on the taxed good must be positive (i.e., it tends to increase the price), because the burden of a unit tax is proportional to the quantity of the good. Hence, the tax induces the supplier to reduce such quantity. Given the price of the other good, this can only be achieved by raising the price of the taxed good. Hence, although both unit and ad valorem taxes can result in a lower price of the taxed good, with a unit tax this can only hold if the indirect effect is negative, unlike with an ad valorem tax.

Another important difference between unit and ad valorem taxes regards their effect on output. As mentioned above, the supplier must reduce the output of the good subjet to a unit tax to limit the tax burden. By contrast, an ad valorem tax can stimulate output of both goods when it reduces their price, because the tax burden is proportional to the revenue from the taxed good, rather than its quantity.

The interdependence between demands for the goods is the key ingredient driving the novel effects of taxation that we explore. Indeed, with a single-product firm, or if the demands for the goods are independent, the standard effects of taxation apply. More precisely, only the direct effect of taxes survives, and this effect can only be positive, so that the price of a good increases in the tax rate, whereas its supply decreases. Furthermore, our analysis indicates that the effects of a uniform tax, applied to both goods at the same rate, would also be similar to the conventional ones. For example, we find that a uniform ad valorem tax can only reduce supply.

In the second part of the analysis, we focus on the implications of the above findings for optimal tax policy. When goods are undersupplied in equilibrium (as is typically the case with a supplier that has market power), the government should aim to increase the supply by decreasing their prices. As argued above, prices can decrease with a unit or an ad valorem tax (though under different conditions). However, only differentiated ad valorem taxes may increase the supply of all goods, with unambiguous effects on welfare. When taxation reduces the price of both goods, the (second-best) optimal tax on a single good is strictly positive. This finding is in contrast to the standard prescription - derived in models with single-product suppliers - that the restrictive effects of market power on output can only be addressed with subsidies.

In the final part of the analysis, we show that the conditions such that (ad valorem) taxation results in lower prices and higher supply can hold in several applications characterized by a multiproduct supplier and demand interdependencies. These settings include add-on pricing (Ellison, 2005), multiproduct retailing with advertising (Rhodes, 2015), intertemporal markets with switching costs (Klemperer, 1995), and two-sided markets Armstrong (2006). Overall, the results indicate that imposing an ad valorem tax rate on goods sold at a discount is likely to reduce prices and increase supply of both goods. Our applications suggest that goods fitting this description include loss leaders in supermarkets, "base" goods that firms advertise the price of (e.g., low-cost flight tickets) and new customer deals by providers of subscription services (e.g., mobile or landline internet service providers). In two-sided markets, the above description fits the goods on the "discounted" side of the market, e.g., pay-per-view TV carrying advertising.

The remainder of the paper is organized as follows. Section 2 provides a review of the literature. Section 3 describes the model, and Section 4 derives the equilibrium. Then, Section 5 characterizes the effects of unit and ad valorem taxes, and compares them. In this section we also put more structure on the demand to derive simpler conditions for the price to decrease with the taxes. Section 6 compares the equilibrium and the optimum, and Section 6 derives optimal taxes. Section 7 provides several applications. Section 8 extends the analysis to two-sided markets. Section 9 concludes.

2 Literature review

As one of the oldest subjects in economics, the incidence of indirect taxes on consumer prices has received much attention in the literature (see, e.g., Fullerton and Metcalf, 2002). Many previous studies of commodity taxation have looked at imperfectly competitive markets (Delipalla and Keen, 1992; Anderson et al., 2001; Auerbach and Hines, 2002), focusing on single-product firms. A fundamental result in this literature is that taxes raise prices and reduce supply, aggravating the distortions caused by market power. Weyl and Fabinger (2013) provide general principles for the pass-through of production costs (akin to unit taxes) with single-product suppliers. Their analysis points to the role of market competitiveness and curvature of demand as key determinants of pass-through. We consider instead a multi-product supplier and focus on the role of the interdependency of demands for its products, showing that in this context the pass-through can be negative. Furthermore, our analysis considers the effects of ad valorem as well as unit taxes. Within the literature on taxation in imperfectly competitive markets, only few papers have shown , in specific settings, that taxation can result in lower prices and higher supply. Cremer and Thisse (1994) show this result in a vertically differentiated oligopoly with endogenous quality, while Carbonnier (2014) considered nonlinear, price-dependent tax schedules. D'Annunzio et al. (2020) show that ad valorem taxes can correct underprovision if differentiated tax rates are applied on to the usage and access parts of a multi-part tariff.

The first author to study taxation with multi-product firms was Edgeworth (1925). He provided an example where a monopolist supplying two substitute goods responds to a unit tax on one good by reducing the price of both. This finding is known as Edgeworth's paradox of taxation, and was later re-elaborated by other authors, including Hotelling (1932), Coase (1946) and Salinger (1991), who focused on unit taxes exclusively.In an analysis developed concurrently and independently to ours, Armstrong and Vickers (2022) provide general conditions for the Edgworth's paradox to occur focusing on unit taxes. We consider unit and ad valorem taxes, showing that in many realistic settings ad valorem taxation can not only reduce prices, but also increase supply and total surplus. Moreover, we show that the goods do not need to be substitutes for this result to occur, unlike with unit taxes (Armstrong and Vickers, 2022).

Although the observation that firms provide multiple products is compelling, only a handful of other studies have investigated the effects of taxation in multiproduct settings. Agrawal and Hoyt (2019) consider tax incidence in a setting with multiple products and perfectly competitive firms. The authors show that taxation (on at least two goods) can result in lower prices if the goods are complements. In their model, suppliers do not internalize the interdependencies between demands for different products (indeed, they have no pricing power at all). Rather, the unconventional effect of taxation stems from the feedback effect that taxes on one good have on the demand for its complements or substitutes. Hamilton (2009) considers an oligopoly with endogenous entry and product breadth. He shows that an ad valorem tax on all commodities raises prices and reduces product breadth, but stimulates output per product and entry in the long run. However, the effects of taxation on welfare are negative. We consider a different setup and focus on the short-run effects of taxation (i.e., given the market structure and product breadth).

Our paper also addresses the literature on taxation of two-sided platforms, a particular kind of multiproduct firms. Kind et al. (2008) show that an ad valorem tax can reduce the prices and stimulate supply by a two-sided platform, due to the externalities across markets. We generalize their result and show that the efficiency-enhancing effect of ad valorem taxes can arise whenever a firm provides multiple goods with interdependent demands, even in absence of externalities across markets.[1]

Recently, industrial economists have looked with renewed interest at the behavior of multiproduct firms, focusing primarily on pricing and the effects of mergers (see, e.g., Chen and Rey, 2012; Rhodes, 2015; Armstrong and Vickers, 2018; Johnson and Rhodes, 2021). Unlike single-product firms, multi-product ones care not only for the price of a good, but also for the structure of their prices across markets. Although we concentrate on taxes, we note that they have a similar effect on the behavior of a firm to the fees charged by an upstream provider. Specifically, unit taxes are similar to wholesale prices, whereas ad valorem ones are similar to revenue-sharing arrangements. The empirical literature has provided evidence of negative pass-through of such fees (and costs more generally). Besanko et al. (2005) provide examples of negative pass-through of own- and cross-brand wholesale prices. Also, Froot and Klemperer (1989) find that firms may either increase or decrease their export price in response to an increase in the exchange rate. Luco and Marshall (2020) provide evidence

[1] More recent contributions include Wang and Wright (2017), who show that ad valorem taxes allow efficient price discrimination across goods with different costs and values on a large marketplace platform, Belleflamme and Toulemonde (2018), who show that ad valorem taxes can result in competing two-sided platforms making higher profits, and Tremblay (2018), who considers taxation at the access and the transaction level.

supporting the conjecture that a merger may result in higher prices by a multiproduct supplier (Salinger, 1991), by eliminating double-marginalization. Studying the US carbonated-beverage industry, they conclude that vertical integration increased the price of some products sold by a multiproduct supplier.

3 The model

We consider two goods, 1 and 2, and a numeraire. A monopolist supplier, M, provides goods 1 and 2. We denote by Q_i and p_i the quantity and price of good i, respectively. A representative consumer buys both goods, and the utility function is

$$U(Q_1, Q_2) + y - p_1 Q_1 - p_2 Q_2,$$

where y is the consumer's exogenous income. We assume this function is continuously differentiable and concave.

The demand function $Q_i(p_1, p_2)$ for good $i = 1, 2$ is defined by the equilibrium conditions

$$\frac{\partial U}{\partial Q_i} = p_i, \ i = 1, 2. \tag{3.1}$$

To avoid clutter in the formulas below, we are going to omit the argument of the demand functions from now on. Each demand function is non-increasing in p_i, i.e. $\frac{\partial Q_i}{\partial p_i} \leq 0$. Furthermore, demands depend on the price of the other good: if good i is a substitute (resp. complement) to j, with $i \neq j$, then $\frac{\partial Q_i}{\partial p_j} > 0$ (resp. $\frac{\partial Q_j}{\partial p_i} < 0$). We allow the cross-price derivatives of demand to be asymmetric in sign and/or in magnitude, that is, $\frac{\partial Q_i}{\partial p_j} \neq \frac{\partial Q_j}{\partial p_i}$.

Goods 1 and 2 are provided at a constant unit cost $c_i, i = 1, 2$. Both are subject to indirect taxes that, without loss of generality, we assume fall on the supplier. Therefore, the supplier earns the following profit

$$\pi(p, T) = \sum_{i=1,2} (p_i(1 - t_i) - c_i - \tau_i) Q_i, \tag{3.2}$$

where $t_i \leq 1$ is the ad valorem tax rate and τ_i is the unit tax rate on good i. We denote by p the vector of prices, (p_1, p_2), and by T the vector of tax rates, $(t_1, t_2, \tau_1, \tau_2)$. We assume the profit function π is concave in p.

Consumer surplus is

$$CS(p) \equiv U(Q_1, Q_2) + y - p_1 Q_1 - p_2 Q_2, \tag{3.3}$$

and social welfare, denoted by W, is the sum of CS, π and tax revenue, $\sum_{i=1,2}(p_i t_i + \tau_i) Q_i$. This sum boils down to

$$W(p) = U(Q_1, Q_2) + y - c_1 Q_1 - c_2 Q_2. \tag{3.4}$$

The government faces no revenue requirements and its objective is to maximize W.

In the following, we use superscripts $*$ and e to denote variables in the social optimum and in equilibrium, respectively. Furthermore, we use superscript 0 to denote variables in the "laissez-faire" equilibrium without taxes, i.e. where $t_i = \tau_i = 0, \quad \forall i$.

We end this section with a brief discussion of our setting. The assumption that the demand for one good depends on the price of the other plays a key role in our analysis. Such demand interdependencies can originate from consumer preferences, but can also be due to search costs. Suppose consumers sustain a search cost to learn the prices of the goods supplied by M that are not advertised. Interdependency of demands results from the price of the advertised good driving consumers' decision to search (Lal and Matutes, 1994; Ellison, 2005; Rhodes, 2015). In addition, the model can be interpreted as considering a single good provided in two successive periods, $i = 1, 2$. In this interpretation, demand for the good in one period depends on the price in the previous one (e.g., due to switching costs). Hence, M does not need to be a multiproduct firm in a strict sense. We provide applications of the model in each of these settings in Section 7.

In Section 8, we show that our analysis applies also to the case where M is a two-sided platform bringing together distinct markets connected by externalities (e.g., media content and advertising).

Given the assumption of quasi-linear utility, there is no loss in considering a single representative consumer. With multiple consumers, aggregate demands would depend only on the vector of prices and not on the distribution of income. Note also that, to focus squarely on the implications of demand interdependencies, we assume a linear cost function and ignore interactions in the cost of producing the two goods.[2]

Finally, in Appendix A, we generalize the model by allowing Q_2 to be piecewise continuously

[2] As Hotelling (1932) explains, the cost function does not play a key role in identifying the Edgeworth's paradox under monopoly.

differentiable in p_2. That is, we allow for the possibility that demand for good 2 is kinked, which is consistent with some of the applications in Section 7.

4 Equilibrium

The vector of equilibrium prices, p^e, maximize M's profit and satisfy the following system of first-order conditions:

$$F_i \quad (p, T) \equiv \frac{\partial \pi (p, T)}{\partial p_i} = (1 - t_i) Q_i + (p_i (1 - t_i) - c_i - \tau_i) \frac{\partial Q_i}{\partial p_i} + \\ + (p_j (1 - t_j) - c_j - \tau_j) \frac{\partial Q_j}{\partial p_i} = 0, \quad i, j = 1, 2, \quad j \neq i. \tag{4.1}$$

Notice that the prices p^e are a function of the tax rates T. In the following, however, we omit the argument of the price function to avoid clutter in the formulas. Rearranging (4.1), we obtain

$$p_i^e = \frac{c_i + \tau_i}{1 - t_i} - \frac{Q_i^e}{\frac{\partial Q_i}{\partial p_i}} - \frac{\left(p_j^e (1 - t_j) - c_j - \tau_j\right) \frac{\partial Q_j}{\partial p_i}}{(1 - t_i) \frac{\partial Q_i}{\partial p_i}}, \quad i, j = 1, 2, \quad j \neq i. \tag{4.2}$$

To interpret the above expressions, we focus on the laissez-faire equilibrium:

$$p_i^0 = c_i - \frac{Q_i^0}{\frac{\partial Q_i}{\partial p_i}} - \frac{\left(p_j^0 - c_j\right) \frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}}, \quad i, j = 1, 2, \quad j \neq i. \tag{4.3}$$

The first two terms on the right-hand side of expression (4.3) coincide with the standard single-product monopoly price formula. The last term captures the effect of a change in the price of good i on the profitability from good j, and is therefore distinctive of a multiproduct firm. Clearly, if $\frac{\partial Q_j}{\partial p_i} = 0$ there is no such effect and p_i^0 boils down to the standard monopoly price.

For the sake of discussion, consider $p_j^0 > c_j$ (this condition must hold for at least one of the two goods in equilibrium). If $\frac{\partial Q_j}{\partial p_i} > 0$ (as in, e.g., the case where i is a substitute to j), p_i^0 tends to exceed the standard monopoly price level, because part of the loss in sales when raising p_i is compensated by a higher demand for good j. By contrast, if $\frac{\partial Q_j}{\partial p_i} < 0$ (as in, e.g., the case where i is a complement to j), p_i^0 tends to be below the standard monopoly price, because M is willing to sell good i at a lower price in order to boost demand for the other good. In fact, if $p_j^0 - c_j$ is large enough, good i is a loss leader, i.e. $p_i^0 < c_i$ holds.

Observe that, if $\frac{\partial Q_j}{\partial p_i} < 0$ and $p_j^0 > c_j$ hold, the supplier may set p_i^0 low enough that the equilibrium quantity Q_i^0 lies on the *inelastic* part of the demand curve, i.e., $\frac{p_i^0}{Q_i^0}\frac{\partial Q_i}{\partial p_i} > -1$. More precisely, this condition holds whenever $c_i < \frac{\left(p_j^0 - c_j\right)\frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}}$. This outcome is peculiar to multiproduct pricing with interdependent demands: if demands were independent, the supplier would always operate on the *elastic* part of demand in equilibrium (as would a single-product supplier). We return to this observation when analyzing the effects of taxation below.

5 Effects of taxation

We now analyze the effects of taxation on equilibrium prices. To streamline the exposition, we start from ad valorem taxes and consider unit taxes next.

5.1 Effects of taxation on prices

5.1.1 Ad valorem taxes

To concentrate on the effects of ad valorem taxes, we set $\tau_i = 0$, $\forall i$. Differentiating the expressions in (4.1) with respect to t_i, we find

$$\frac{\partial p_i^e}{\partial t_i} = -\frac{\frac{\partial F_i}{\partial t_i}\frac{\partial F_j}{\partial p_j} - \frac{\partial F_i}{\partial p_j}\frac{\partial F_j}{\partial t_i}}{H}, \quad \frac{\partial p_j^e}{\partial t_i} = -\frac{\frac{\partial F_j}{\partial t_i}\frac{\partial F_i}{\partial p_i} - \frac{\partial F_j}{\partial p_i}\frac{\partial F_i}{\partial t_i}}{H}, \; i,j = 1,2, \quad j \neq i, \tag{5.1}$$

where

$$\frac{\partial F_i}{\partial t_i} = -Q_i^e\left(\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} + 1\right), \quad \frac{\partial F_i}{\partial t_j} = -p_j^e\frac{\partial Q_j}{\partial p_i},$$
$$\frac{\partial F_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2} < 0, \quad \frac{\partial F_i}{\partial p_j} = \frac{\partial F_j}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_1 \partial p_2}, \quad \text{and } H \equiv \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial p_1} > 0.$$

As a benchmark, consider the case of independent demands (i.e. $\frac{\partial Q_2}{\partial p_1} = \frac{\partial Q_1}{\partial p_2} = 0$). The first derivative in (5.1) boils down to $\frac{\partial p_i^e}{\partial t_i} = \frac{\frac{\partial F_i}{\partial t_i}}{\frac{\partial F_i}{\partial p_i}}$. Since the denominator is negative by the second-order conditions of M's maximization problem, $\frac{\partial p_i^e}{\partial t_i}$ is positive if and only if $Q_i^e\left(\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} + 1\right) < 0$. As we argued above, with independent demands the price set by M is the standard, single-product, monopoly price $p_i^e = \frac{c_i}{1-t_i} - \frac{Q_i^e}{\frac{\partial Q_i}{\partial p_i}}$. Hence, the equilibrium quantity Q_i^e lies on the *elastic* part of the demand curve for good i, i.e., $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} < -1$. As one would expect, the price of good i increases in t_i in that case. Furthermore, with independent demands a tax on good i does not affect the price of the other good, $\frac{\partial p_j^e}{\partial t_i} = 0$.

Return now to the case where demands are interdependent. The denominator of the expressions in (5.1) is the determinant of the Hessian matrix, which is positive by the second-order conditions of firm M's problem. Hence, we have

$$\text{sgn}\left(\frac{\partial p_i^e}{\partial t_i}\right) = \text{sgn}\left(-\underbrace{\frac{\partial F_i}{\partial t_i}\frac{\partial F_j}{\partial p_j}}_{\text{direct effect}} + \underbrace{\frac{\partial F_j}{\partial t_i}\frac{\partial F_i}{\partial p_j}}_{\text{indirect effect}}\right),\ i = 1, 2, \quad j \neq i. \tag{5.2}$$

The sign of $\frac{\partial p_i^e}{\partial t_i}$ is determined by two effects. First, there is a *direct* effect of the tax, given the price of the other good, p_j^e. Second, there is an *indirect* effect, due to the fact that, as long as the demand for good i depends on p_j, with $i \neq j$, the tax on i induces a change in p_j^e, which in turn induces an adjustment in p_i^e. Since the derivative $\frac{\partial F_j}{\partial p_j} = \frac{\partial^2 \pi}{\partial p_j^2}$ is negative, the *direct effect* of t_i is negative - and thus tends to *reduce* p_i^e - if and only if the profitability of marginally raising this price gets smaller with the tax, i.e. $\frac{\partial F_i}{\partial t_i} < 0$ holds. This condition is satisfied whenever Q_i^e lies on the *inelastic* part of the demand for good i, i.e. $\frac{p_i^e}{Q_i}\frac{dQ_i}{dp_i} > -1$ holds. To understand this condition, consider that t_i gives the supplier an incentive to change p_i in a way that reduces the revenue from good i, $p_i Q_i$. When Q_i^e is on the inelastic part of demand, this objective can only be achieved by reducing p_i (given p_j^e) . Rearranging (4.2), we find that

$$\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} > -1 \Longleftrightarrow c_i < \frac{\left(p_j^e\left(1-t_j\right) - c_j\right)\frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}}. \tag{5.3}$$

Assuming the profit margin on good j is positive, the above inequality can be satisfied only if $\frac{\partial Q_j}{\partial p_i} < 0$ (e.g., if good i is a complement to j) and the marginal cost c_i is small enough.

The second term in brackets on the right hand side of (5.2) represents the *indirect effect* of t_i on p_i^e, which depends on two factors. First, the change induced by t_i on p_j^e, given p_i^e, that is, on the direct of effect of t_i on p_j^e. This change is captured by the derivative $\frac{\partial F_j}{\partial t_i} = -p_i^e \frac{\partial Q_i}{\partial p_j}$, which is negative if and only if $\frac{\partial Q_i}{\partial p_j} > 0$, as in, e.g., when good j is a substitute to good i. The indirect effect also depends on $\frac{\partial F_i}{\partial p_j} = \frac{\partial^2 \pi}{\partial p_1 \partial p_2}$, i.e., on whether an increase in p_j raises the marginal profitability of increasing p_i. The indirect effect thus tends to reduce p_i^e if and only if either (i) $\frac{\partial Q_i}{\partial p_j} > 0$ and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$, or (ii) $\frac{\partial Q_i}{\partial p_j} < 0$ and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ hold. In words, when taxing good i determines an increase in the price of good j, given the price of the other good, (e.g., because j is a substitute for i) and p_i moves in the same direction as p_j, then the indirect effect pushes p_i upwards. The mechanism works in the opposite direction when good i is a complement to good j.

Summing up, we have found that the supplier can respond to an ad valorem tax on one good by reducing its price, if the sum of the direct and indirect effect is negative. After rearranging (5.2), we obtain the following condition on the marginal cost of good i:

Lemma 1. *The equilibrium price of good i decreases with the ad valorem tax t_i, i.e. $\frac{\partial p_i^e}{\partial t_i} < 0$, if and only if*

$$c_i < \max\left(\frac{\left(p_j^e\left(1-t_j\right)-c_j\right)\frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}} + \frac{\partial^2 \pi}{\partial p_1 \partial p_2}\frac{\partial Q_i}{\partial p_j}\frac{p_i^e\left(1-t_i\right)}{\frac{\partial^2 \pi}{\partial p_i^2}\frac{\partial Q_i}{\partial p_i}}, 0\right), \ i = 1, 2, \quad j \neq i. \tag{5.4}$$

The first term in brackets on the right hand side of this expression is the same as in (5.3). The second term is positive if and only if the indirect effect is negative, which makes the inequality in (5.4) weaker.

Consider now the effect of t_i on the price of the other good, p_j^e. We have

$$\operatorname{sgn}\left(\frac{\partial p_j^e}{\partial t_i}\right) = \operatorname{sgn}\left(-\underbrace{\frac{\partial F_j}{\partial t_i}\frac{\partial F_i}{\partial p_i}}_{\text{direct effect}} + \underbrace{\frac{\partial F_j}{\partial p_i}\frac{\partial F_i}{\partial t_i}}_{\text{indirect effect}}\right), \ i = 1, 2, \quad j \neq i. \tag{5.5}$$

As explained before, the direct effect captures the change induced by the tax on the price of good j given p_i^e. Since the derivative $\frac{\partial F_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2}$ is negative, the sign of this effect only depends on the sign of $\frac{\partial F_j}{\partial t_i} = -p_i^e \frac{\partial Q_i}{\partial p_j}$. Hence, the direct effect of t_i on p_j^e is negative if and only if $\frac{\partial Q_i}{\partial p_j} > 0$. This condition holds if good j is a substitute to good i.

The sign of the indirect effect depends on the cross-derivative $\frac{\partial F_j}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_1 \partial p_2}$ and on the same derivative ($\frac{\partial F_i}{\partial t_i}$) that drives the direct effect of t_i on p_i, discussed above. Therefore, the indirect effect of t_i tends to reduce p_j^e if either (i) $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and Q_i^e lies on the inelastic part of demand for good i or if (ii) $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and Q_i^e is on the elastic part of demand. We have seen above that p_i tends to decrease in t_i (given p_j) if Q_i^e lies on the inelastic part of demand. If the cross profits derivative is positive, the incentive of the monopolist is to move p_j in the same direction as p_i.

Summing up, after rearranging (5.5), we obtain the following

Lemma 2. *The equilibrium price of good j decreases in the ad valorem tax t_i, i.e. $\frac{\partial p_j^e}{\partial t_i} < 0$, if and only if, for $i = 1, 2, \quad j \neq i$*

$$c_i < \max\left(\frac{\left(p_j^e(1-t_j)-c_j\right)\frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}} + \frac{p_i^e(1-t_i)\frac{\partial Q_i}{\partial p_j}}{Q_i\frac{\frac{\partial^2\pi}{\partial p_i^2}}{\frac{\partial Q_i}{\partial p_i}}\frac{\partial^2\pi}{\partial p_1\partial p_2}}, 0\right) \quad \text{if } \frac{\partial^2\pi}{\partial p_1\partial p_2} > 0,$$
$$c_i > \max\left(\frac{\left(p_j^e(1-t_j)-c_j\right)\frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}} + \frac{p_i^e(1-t_i)\frac{\partial Q_i}{\partial p_j}}{Q_i\frac{\frac{\partial^2\pi}{\partial p_i^2}}{\frac{\partial Q_i}{\partial p_i}}\frac{\partial^2\pi}{\partial p_1\partial p_2}}, 0\right) \quad \text{if } \frac{\partial^2\pi}{\partial p_1\partial p_2} < 0. \tag{5.6}$$

We have therefore established that the supplier may respond to an ad valorem tax on either good by decreasing the price of both goods. Furthermore, we have expressed the necessary and sufficient conditions for this price-decreasing effect with respect to the marginal cost of the taxed good.

5.1.2 Unit taxes

We now focus on unit taxes and set ad valorem taxes to zero, i.e. $t_i = 0, \forall i$. Differentiating (4.1) with respect to τ_i, we find

$$\frac{\partial p_i^e}{\partial \tau_i} = -\frac{\frac{\partial F_i}{\partial \tau_i}\frac{\partial F_j}{\partial p_j} - \frac{\partial F_i}{\partial p_j}\frac{\partial F_j}{\partial \tau_i}}{H}, \quad \frac{\partial p_j^e}{\partial \tau_i} = -\frac{\frac{\partial F_j}{\partial \tau_i}\frac{\partial F_i}{\partial p_i} - \frac{\partial F_j}{\partial p_i}\frac{\partial F_i}{\partial \tau_i}}{H}, \; i = 1,2, \quad j \neq i, \tag{5.7}$$

where

$$\frac{\partial F_i}{\partial \tau_i} = -\frac{\partial Q_i}{\partial p_i} > 0, \quad \frac{\partial F_i}{\partial \tau_j} = -\frac{\partial Q_j}{\partial p_i},$$
$$\frac{\partial F_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2} < 0, \quad \frac{\partial F_i}{\partial p_j} = \frac{\partial F_j}{\partial p_i} = \frac{\partial^2\pi}{\partial p_1 \partial p_2}, \quad \text{and } H \equiv \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial p_1} > 0.$$

As with an ad valorem tax, if the demands for the two goods are independent, it is easily shown that $\frac{\partial p_i^e}{\partial \tau_i} > 0$ and $\frac{\partial p_j^e}{\partial \tau_i} = 0$ hold for both goods. When demands are interdependent, the denominator of the expressions in (5.7) is positive by the second-order conditions of firm M's problem. So we have

$$\operatorname{sgn}\left(\frac{\partial p_i^e}{\partial \tau_i}\right) = \operatorname{sgn}\left(-\underbrace{\frac{\partial F_i}{\partial \tau_i}\frac{\partial F_j}{\partial p_j}}_{\text{direct effect}} + \underbrace{\frac{\partial F_i}{\partial p_j}\frac{\partial F_j}{\partial \tau_i}}_{\text{indirect effect}}\right), \; i = 1,2, \quad j \neq i. \tag{5.8}$$

We can again identify a direct and an indirect effect of τ_i on p_i^e. The *direct effect* is unambiguously positive, i.e. it tends to increase the price, because $\frac{\partial F_j}{\partial p_j} = \frac{\partial^2\pi}{\partial p_j^2} < 0$ and $\frac{\partial F_i}{\partial \tau_i} = -\frac{\partial Q_i}{\partial p_i} > 0$ hold. Notice the difference with the direct effect of the ad valorem tax, which can be negative. The reason is that, whereas the burden imposed on M by an ad valorem tax is proportional to the revenue from good i, the burden imposed by τ_i is proportional to the quantity supplied. Thus, given p_j^e, M can reduce the tax burden only by reducing the quantity of good i, hence raising p_i^e. In other words, a unit tax has the same effect as an increase in the cost of production, unlike an ad valorem tax.

The *indirect effect* of τ_i on p_i^e is similar to that of an ad valorem tax: this effect is negative if $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$ and $\frac{\partial Q_i}{\partial p_j} > 0$ (e.g., when good i is a substitute to good j), or if $\frac{\partial^2\pi}{\partial p_1 \partial p_2} < 0$ and $\frac{\partial Q_i}{\partial p_j} < 0$ (e.g., when good i is a complement to good j). Therefore, after rearranging (5.8), we obtain the following

Lemma 3. *The equilibrium price of good i set by the monopolist decreases in the unit tax τ_i, i.e. $\frac{\partial p_i^e}{\partial \tau_i} < 0$, if and only if, for $i = 1, 2, \quad j \neq i$*

$$\begin{aligned} \frac{\partial Q_i}{\partial p_j} &> \frac{\frac{\partial Q_i}{\partial p_i}\frac{\partial^2\pi}{\partial p_j^2}}{\frac{\partial^2\pi}{\partial p_1 \partial p_2}} \qquad \text{if } \frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0, \\ \frac{\partial Q_i}{\partial p_j} &< \frac{\frac{\partial Q_i}{\partial p_i}\frac{\partial^2\pi}{\partial p_j^2}}{\frac{\partial^2\pi}{\partial p_1 \partial p_2}} \qquad \text{if } \frac{\partial^2\pi}{\partial p_1 \partial p_2} < 0. \end{aligned} \tag{5.9}$$

Since the numerator on the right hand side is positive, necessary conditions for p_i^e to decrease with τ_i are that either $\frac{\partial Q_i}{\partial p_j} > 0$ when $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$ holds, and that $\frac{\partial Q_i}{\partial p_j} < 0$ when $\frac{\partial^2\pi}{\partial p_1 \partial p_2} < 0$.

Consider now the effect of τ_i on the price of good j. We have

$$\text{sgn}\left(\frac{\partial p_j^e}{\partial \tau_i}\right) = \text{sgn}\left(-\underbrace{\frac{\partial F_j}{\partial \tau_i}\frac{\partial F_i}{\partial p_i}}_{\text{direct effect}} + \underbrace{\frac{\partial F_j}{\partial p_i}\frac{\partial F_i}{\partial \tau_i}}_{\text{indirect effect}} \right), \ i = 1, 2, \quad j \neq i. \tag{5.10}$$

The *direct effect* is similar to that of an ad valorem tax: since $\frac{\partial F_i}{\partial p_i} = \frac{\partial^2\pi}{\partial p_i^2} < 0$, the direct effect of τ_i on p_j^e is negative if and only if $\frac{\partial Q_i}{\partial p_i} > 0$ holds. Furthermore, the *indirect effect* is negative if and only if $\frac{\partial^2\pi}{\partial p_1 \partial p_2} < 0$ holds, given that $\frac{\partial F_i}{\partial \tau_i} = -\frac{\partial Q_i}{\partial p_i} > 0$. Indeed, as we have seen, the direct of τ_i on p_i is positive. Hence, if the profitability of raising p_j decreases when the price of good i goes up, that is, if the prices move in opposite directions, the indirect effect tends to reduce p_j. By rearranging (5.10), we obtain

Lemma 4. *The equilibrium price of good j decreases in the unit tax τ_i, i.e. $\frac{\partial p_j^e}{\partial \tau_i} < 0$, if and only if, for $i = 1, 2, \quad j \neq i$*

$$\frac{\partial Q_i}{\partial p_j} > \frac{\frac{\partial^2\pi}{\partial p_1 \partial p_2}\frac{\partial Q_i}{\partial p_i}}{\frac{\partial^2\pi}{\partial p_j^2}}, \ i = 1, 2, \quad j \neq i. \tag{5.11}$$

As with $\frac{\partial p_i^e}{\partial \tau_i}$, therefore, a necessary condition for p_j^e to decrease with τ_i is that the good is substitute to i when $\frac{\partial^2\pi}{\partial p_1 \partial p_2} > 0$, or that the good is a complement to i when $\frac{\partial^2\pi}{\partial p_1 \partial p_2} < 0$.

5.1.3 Summary

Table 1 summarizes the effects of taxation on the prices set by a multiproduct monopolist. The first row suggests that the conditions for an ad valorem tax to reduce the price of the taxed good are less stringent than those for a unit tax. Whereas the indirect effect of the two taxes depends on the same factors, there is a fundamental difference regarding the direct effect: this effect is always positive for τ_i, but can be negative for t_i depending on the elasticity of demand in equilibrium. As explained above, the reason is that the burden imposed by the unit tax is proportional to the output of good i, whereas the burden of the ad valorem tax is proportional to the revenue from such good. Thus, p_i^e can decrease with τ_i only if the indirect effect is negative and dominates the direct one. By contrast, even if the indirect effect is positive, p_i^e can decrease with t_i as long as condition (5.4) holds.

The second row of Table 1 summarizes the effects of taxing good i on the price of the other good, j. For either a unit or an ad valorem tax, the direct effect is negative if and only if the demand for good j increases with p_i (e.g., if the goods are substitutes). However, the indirect effects of these taxes are different. The indirect effect of a unit tax is negative if and only if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$. Since, given p_j, the price of good i increases with τ_i, the supplier reduces the price of j only if the marginal profitability of raising this price decreases with the price of good i. Instead, the indirect effect of an ad valorem tax on i may push p_j^e downwards when either $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and Q_i^e is on the elastic part of demand, or when $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and Q_i^e is on the inelastic part of demand.

5.2 Effects of taxation on supply

We now turn to the effect of taxes on the supply of the two goods. The effect on the equilibrium demand Q_j^e an ad valorem tax t_i is

$$\frac{\partial Q_j^e}{\partial t_i} = \frac{\partial Q_j}{\partial p_1}\frac{\partial p_1^e}{\partial t_i} + \frac{\partial Q_j}{\partial p_2}\frac{\partial p_2^e}{\partial t_i},\ i, j = 1, 2. \tag{5.12}$$

When both prices decrease and the goods are not substitutes, the ad valorem tax results in higher supply not only of the untaxed good, but also of the taxed one. As we show in our applications (Section 7), this can be the case in several settings.

Although the derivative $\frac{\partial Q_j^e}{\partial \tau_i}$ would have the same form as (5.12), the effect of a unit tax τ_i on supply is quite different from that of an ad valorem tax. A unit tax can reduce the price of both goods, but the supply of the taxed good always decreases, i.e. $\frac{\partial Q_i^e}{\partial \tau_i} < 0$ (as we show in

	Ad valorem tax, t_i	Unit tax, τ_i
Effect on p_i^e	• $DE < 0$ iff $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} > -1$. • $IE < 0$ if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and $\frac{\partial Q_i}{\partial p_j} > 0$, or if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and $\frac{\partial Q_i}{\partial p_j} < 0$. • Overall: see (5.4).	• $DE > 0$. • $IE < 0$ if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and $\frac{\partial Q_i}{\partial p_j} > 0$, or if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and $\frac{\partial Q_i}{\partial p_j} < 0$. • Overall: see (5.9).
Effect on p_j^e	• $DE < 0$ iff $\frac{\partial Q_i}{\partial p_j} > 0$ • $IE < 0$ if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} > -1$, or if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ and $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} < -1$. • Overall: see (5.6).	• $DE < 0$ iff $\frac{\partial Q_i}{\partial p_j} > 0$. • $IE < 0$ iff $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$. • Overall: see (5.11).

Table 1: Effects of taxation on equilibrium prices. DE stands for "direct effect" and IE for "indirect effect". An effect is negative whenever it tends to reduce the price.

Appendix B.1). The reason is that, unlike with an ad valorem tax, only reducing supply can decrease the burden of a unit tax on the firm's profit. In other words, the unit tax affects the firm's behavior in the same way as an increase in the cost of production.

We emphasize that an ad valorem tax can increase the supply of both goods *only* if the tax rates on the two goods are different. To see this, consider a uniform tax rate on both goods. By simple rearrangements, given $t_1 = t_2 = t$ and $\tau_1 = \tau_2 = 0$, we can rewrite the first-order conditions in (4.1) as

$$Q_i + p_i \frac{\partial Q_i}{\partial p_i} + p_j \frac{\partial Q_j}{\partial p_i} - \widetilde{c}_i \frac{\partial Q_i}{\partial p_i} - \widetilde{c}_j \frac{\partial Q_j}{\partial p_i} = 0, \quad i, j = 1, 2, \quad j \neq i,$$

where $\widetilde{c}_1 \equiv \frac{c_1}{1-t}$ and $\widetilde{c}_2 \equiv \frac{c_2}{1-t}$. These equations suggest that a uniform ad valorem tax, t, would have the same effect as an increase in the costs of production, c_1 and c_2, that is equivalent to the introduction of unit taxes. Therefore, as argued above, the effect of this tax can only be such that supply decreases. We summarize the results of this section in the following

Proposition 1. *The equilibrium prices p_1^e and p_2^e decrease with the ad valorem tax t_i or the unit tax τ_i, if and only if the conditions summarized in Table 1 hold. A decrease in both prices is a sufficient condition for the supply of both goods to increase with the ad valorem tax (provided the tax rates on the two goods are different). Instead, the supply of the taxed good i always decreases with a unit tax τ_i.*

The analysis of this section generalizes the results from the literature on the "taxation paradox" with multiproduct firms, initiated by Edgeworth (1925). Most importantly, we extended the analysis to ad valorem taxes, showing that the price-reducing effect is not specific to unit taxes. Furthermore, we find that only ad valorem taxes (when tax rates are different) can induce higher supply of the taxed good, as well as the untaxed one.

5.3 More specific environments

To establish simpler conditions determining the effects of taxes on prices and quantities, we now specify the model in three different directions.

5.3.1 Partially independent demands

We assume the demand for one of the goods (that we take to be good 1 without loss of generality) does not depend on the price of the other. More precisely, we make the following

Assumption 1. $\frac{\partial Q_1}{\partial p_2} = 0$ *and* $\frac{\partial Q_2}{\partial p_1} \neq 0$.

As we show in Section 7, this assumption holds in several settings. For example, suppose consumers decide to purchase good 1 before observing the price of good 2. Suppose also that the supplier can advertise the price of only a subset of its goods, so consumers must search (e.g., visit a store or website) to learn the price of the remaining ones (Lal and Matutes, 1994; Ellison, 2005; Rhodes, 2015). Assuming good 1 is advertised, its demand does not depend on p_2, but on the expectation of this price that consumers form before searching. On the other hand, the demand for good 2 depends on the price of good 1, e.g., because this price drives consumers' decision to visit the store. Similarly, Assumption 1 holds when consumers can make repeated purchases of a good and must decide whether to buy in one period before observing the price in the next. Demand in the first period then depends on the expectation that consumers form about the price in the next. However, demand in the next period can depend on the previous price, e.g., due to switching costs (Klemperer, 1995).

Assumption 1 implies that $\frac{\partial F_2}{\partial t_1} = 0$ in expressions (5.1). Hence, the effect of t_1 on both prices simplifies drastically. The derivatives in (5.1) boil down to

$$\frac{\partial p_1^e}{\partial t_1} = -\frac{\frac{\partial F_1}{\partial t_1}\frac{\partial F_2}{\partial p_2}}{H}, \quad \frac{\partial p_2^e}{\partial t_1} = \frac{\frac{\partial F_2}{\partial p_1}\frac{\partial F_1}{\partial t_1}}{H}. \tag{5.13}$$

Under Assumption 1, there is no direct effect of t_1 on p_2^e and, hence, no indirect effect on p_1^e. Therefore, condition (5.3) is necessary and sufficient for p_1^e to decrease with t_1.[3] Furthermore, if (5.3) and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ hold, p_2^e decreases in t_1. Hence, if both conditions hold, introducing an ad valorem tax on good 1 reduces both prices.

Consider now the effect of the tax on the equilibrium quantities, characterized in (5.12). Given Assumption 1, the demand for good 1 does not depend on p_2, so Q_1^e increases with t_1 if and only if p_1^e decreases with the tax. But since (5.3) can hold only if $\frac{\partial Q_2}{\partial p_1} < 0$, if p_2^e decreases with t_1 as well, Q_2^e must increase too.

Proposition 2. *Given Assumption 1, p_1^e decreases with the ad valorem tax t_1 if and only if (5.3) holds. Furthermore, p_2^e decreases with t_1 if and only if (5.3) and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ hold. Under these conditions, the supply of both goods increases with t_1.*

Proposition 2 establishes simple sufficient conditions such that an ad valorem tax on one of the goods provided by a multiproduct monopolist brings to a *reduction* in the price of all the goods and an *increase* in their supply. We omit the analysis of unit taxes in this section for

[3] Given $\frac{\partial Q_1}{\partial p_2} = 0$, (4.1) implies that $p_2^e > c_2$.

reasons of space. However, as we show in Appendix B.2, unit taxes would have significantly different effects. Most importantly, the price of the taxed good can only increase with a unit tax under Assumption 1.

5.3.2 Linear demands

Suppose now demand functions are linear, i.e.

Assumption 2. $Q_i = \alpha_i - \beta_i p_i - \gamma p_j, \quad i, j = 1, 2, \quad i \neq j,$

where $\alpha_i > 0$ and $\beta_i < 0$. Furthermore, $\gamma > 0$ if the goods are complements (i.e., $\frac{\partial Q_i}{\partial p_j} < 0$), whereas $\gamma < 0$ if the goods are substitutes (i.e., $\frac{\partial Q_i}{\partial p_j} > 0$).

For brevity, we focus on the effects of ad valorem taxes.[4] Given Assumption 2, it is straightforward to show that the cross-price derivative $\frac{\partial^2 \pi}{\partial p_1 \partial p_2}$ is positive if and only if the goods are substitutes, i.e., $\gamma < 0$. This observation helps in streamlining the sign of the derivatives in (5.1). Specifically, the indirect effect of t_i on p_i must be negative, so condition (5.3) is sufficient for the derivative $\frac{\partial p_i}{\partial t_i}$ to be negative. However, since (5.3) can only hold if the goods are complements ($\gamma > 0$), under Assumption 2 this condition also implies that both the direct and the indirect effect of t_i on p_j are positive, so $\frac{\partial p_j}{\partial t_i} > 0$. Therefore, we get the following interesting result:

Proposition 3. *Given Assumption 2, if (5.3) holds, then p_1^e decreases with the ad valorem tax t_1, whereas p_2^e increases.*

If condition (5.3) does not hold, however, the direct and indirect effects of t_i on both prices go in opposite directions. Therefore, the sign of $\frac{\partial p_i}{\partial t_i}$ and $\frac{\partial p_j}{\partial t_i}$ ultimately depends on the overall conditions summarized in Table 1. The same can be said when considering the effects of unit taxes.

5.3.3 Unitary elasticity

Now we consider a case allowing us to simplify (5.1) in a different way as that proposed in Section 5.3.1. Consider the case where, at equilibrium, the elasticity of demand is equal to -1, implying that the following assumption holds

[4] The case of linear demands has been extensively investigated by previous literature on the Edgeworth's paradox, focusing on the effects of unit taxes (see Hotelling, 1932 and Salinger, 1991).

Assumption 3. $\frac{\partial F_i}{\partial t_i} = -Q_i^e \left(\frac{p_i^e}{Q_i^e} \frac{\partial Q_i}{\partial p_i} + 1 \right) = 0.$
We concentrate on the case of ad valorem taxes where the elasticity of demand plays a central role in determining the effects of the introduction of a tax.

When $\frac{p_i^e}{Q_i^e} \frac{\partial Q_i}{\partial p_i} = -1$, the direct effect of the tax t_i on the price of good i cancels out, implying also that there is no indirect effect of the tax on the price of good j. Hence, when computed at equilibrium, equations (5.2) and (5.5) simplify to

$$\operatorname{sgn}\left(\frac{\partial p_i^e}{\partial t_i}\right) = \operatorname{sgn}\left(\underbrace{\frac{\partial F_j}{\partial t_i} \frac{\partial F_i}{\partial p_j}}_{\text{indirect effect}}\right), \qquad \operatorname{sgn}\left(\frac{\partial p_j^e}{\partial t_i}\right) = \operatorname{sgn}\left(-\underbrace{\frac{\partial F_j}{\partial t_i} \frac{\partial F_i}{\partial p_i}}_{\text{direct effect}}\right), \; i = 1, 2, \quad j \neq i.$$

In this setting, we can recover the necessary and sufficient conditions for both prices to decrease in the tax.

Proposition 4. *Given Assumption 3, if and only if good i is a substitute for good j, i.e. $\frac{\partial Q_j}{\partial p_i} > 0$, and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} < 0$ hold, then the introduction of an ad valorem tax on good i entails a reduction in p_i^e and p_j^e, for $i, j = 1, 2$ and $i \neq j$.*

First, it is easy to assess that, because $\frac{\partial F_j}{\partial t_i} = -p_i^e \frac{\partial Q_i}{\partial p_j}$ and the profit function is concave, the direct effect of p_j can be negative if and only if good i is a substitute for good j, i.e. $\frac{\partial Q_j}{\partial p_i} > 0$. Hence, the latter is a necessary and sufficient condition for the price of good j to decrease in t_i. Moreover, if the cross derivative of the profit function is positive, i.e. $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$, the price of the two goods move in the same direction when they are substitutes, implying that a decrease in p_j entails a reduction in p_i as well.

6 Welfare effects of taxation and optimal policy

6.1 Laissez-faire equilibrium vs. social optimum

To set the stage for the analysis of optimal policy, it is useful to characterize the social optimum and compare it to the laissez-faire equilibrium. The socially optimal quantities, Q_1^* and Q_2^*, maximize (3.4) and satisfy the system of equations $\frac{\partial U}{\partial Q_i} = c_i$, $i = 1, 2$. It is straightforward to show that the optimal allocation is decentralized by the optimal prices $p_i^* = c_i$ for $i = 1, 2$.

To compare the laissez-faire to the social optimum, we evaluate the first-order derivatives of the monopolist's problem in (4.1), conditional on zero taxes, at the vector of optimal prices,

p^*. Given concavity of the profit function, we find that

$$p_i^0\left(p_j^*\right) > p_i^* \quad i, j = 1, 2, \quad j \neq i, \tag{6.1}$$

where $p_i^0\left(p_j^*\right)$ denotes the equilibrium price conditional on $p_j = p_j^*$. Because for a given p_j the demand for good i is a decreasing function of p_i, we say that the monopolist underprovides (and overprices) good i in the laissez-faire whenever $p_i^0\left(p_j^*\right) > p_i^*$. This condition holds in this setting due to the supplier's market power. Remark that this finding does not imply that *both* equilibrium prices, $p^0 \equiv (p_1^0, p_2^0)$, exceed the first-best levels, $p^* \equiv (p_1^*, p_2^*)$. [5]

Generally speaking, the allocation and prices in the no-tax equilibrium do not coincide with the welfare-maximizing ones, suggesting that intervention from the government is warranted. Whenever equilibrium prices are too high, the objective should be to reduce them. Quite interestingly, this objective can be achieved by appropriately introducing taxes on one or both goods. For the sake of exposition, we begin by considering a tax on one good, assuming there is no tax on the other one. Next, we study optimal taxation when both goods are taxed.

6.2 Optimal tax on a single good

It is useful to start by looking at the effect of introducing a small tax starting from the laissez-faire. We take the derivative of (3.4) with respect to t_1, conditional on $t_i = \tau_i = 0, \quad \forall i$. Using the first-order conditions of the monopolist's problem (4.1) and the equilibrium conditions of the consumer's problem in (3.1), we can write this derivative as

$$\left.\frac{\partial W}{\partial t_i}\right|_{\left(Q_1^0, Q_2^0\right)} = -Q_1^0 \frac{\partial p_1}{\partial t_i} - Q_2^0 \frac{\partial p_2}{\partial t_i}, \quad i = 1, 2, \tag{6.2}$$

which shows that a sufficient condition for the tax to increase welfare is that its introduction brings to a reduction in the price of both goods.[6] The effect of introducing a small unit tax is equivalent (Salinger, 1991).

We now study the optimal (second-best) tax rate on good i, assuming no tax on the other good. Consider first an ad valorem tax. Given the equilibrium conditions of the consumers'

[5]For example, as we argued above, if $\frac{\partial Q_i}{\partial p_j} < 0$ the firm may use good j as a loss-leader, setting $p_i^0 < c_i$.

[6]Recall that both prices cannot fall if there is a uniform increase in the tax rate on both goods (see Section 5.2).

problem in (3.1), the optimal tax on good i (conditional on $t_j = \tau_1 = \tau_2 = 0$) is such that

$$\frac{\partial W}{\partial t_i} = (p_1 - c_1)\left(\frac{\partial Q_1}{\partial p_1}\frac{\partial p_1}{\partial t_i} + \frac{\partial Q_1}{\partial p_2}\frac{\partial p_2}{\partial t_i}\right) + \\ + (p_2 - c_2)\left(\frac{\partial Q_2}{\partial p_1}\frac{\partial p_1}{\partial t_i} + \frac{\partial Q_2}{\partial p_2}\frac{\partial p_2}{\partial t_i}\right) = 0, \quad i = 1, 2. \tag{6.3}$$

Evaluating the above expression at the equilibrium prices (that satisfy (4.1)) and rearranging, we get the following expression for the (second-best) optimal ad valorem tax on good 1, that we denote by t_i^{SB}:

$$t_i^{SB} = \frac{Q_1 \frac{\partial p_1}{\partial t_i} + Q_2 \frac{\partial p_2}{\partial t_i}}{Q_i\left(1 + \frac{p_i}{Q_i}\frac{\partial Q_i}{\partial p_i}\right)\frac{\partial p_1}{\partial t_i} + p_i \frac{\partial Q_i}{\partial p_j}\frac{\partial p_j}{\partial t_i}}, \quad i, j = 1, 2, \ i \neq j. \tag{6.4}$$

To understand this expression, observe that the denominator captures the change in the tax base ($p_i Q_i$) induced by t_i, through the adjustment in the prices of both goods. Intuitively, the tax induces the supplier to adjust its equilibrium prices so that $p_i Q_i$ shrinks, to reduce the tax expenditure. Hence, the denominator of (6.4) must be negative. Turning to the numerator, we can see that it is negative whenever both prices decrease with the tax rate. Hence, we find that $t_i^{SB} > 0$ (respectively, $t_i^{SB} < 0$) if the tax induces a reduction (respectively, an increase) in the equilibrium prices. Note that, if the demands for the two goods were independent ($\frac{\partial Q_1}{\partial p_2} = \frac{\partial Q_2}{\partial p_1} = 0$), the standard result $t_i^{SB} < 0$ would apply, since then $\frac{\partial p_i}{\partial t_i} > 0$, $\frac{\partial p_j}{\partial t_i} = 0$ and $1 + \frac{p_i}{Q_i}\frac{\partial Q_i}{\partial p_i} < 0$ would hold, as argued in Section 5.1.1.

Consider now the optimal unit tax rate on good 1, τ_i^{SB}, conditional on all other tax rates being zero. Following the same steps as above, we obtain that

$$\tau_i^{SB} = \frac{Q_1 \frac{\partial p_1}{\partial \tau_i} + Q_2 \frac{\partial p_2}{\partial \tau_i}}{\frac{\partial Q_i}{\partial p_1}\frac{\partial p_1}{\partial \tau_i} + \frac{\partial Q_i}{\partial p_2}\frac{\partial p_2}{\partial \tau_i}}. \tag{6.5}$$

This expression is equivalent to (6.4), the main difference being that the denominator is the change in the tax base (Q_i) triggered by the unit tax, via the adjustment in the equilibrium prices. As argued above, this change must be negative. Hence, $\tau_i^{SB} > 0$ if the tax induces a reduction in both prices.

Summing up, we have established that the optimal tax on one of the goods sold by a multiproduct supplier can be positive, even though the goods are underprovided in the laissez-faire, because taxes can reduce prices. This result applies to ad valorem and unit taxes, although the conditions such that these taxes bring to a reduction in prices are different (see

Table 1) .

Proposition 5. *A sufficient condition for the second-best optimal tax on a single good to be positive is that it induces a reduction in the prices of both goods.*

6.3 Taxing both goods

We now let the government set two tax rates, one for each good. We start again from ad valorem taxes and consider unit taxes next.

Ad valorem taxes. Assume that $\tau_1 = \tau_2 = 0$. Intuitively, with two ad valorem tax rates the government can implement the first best allocation, i.e. $Q_i^*, \quad i = 1, 2$. In Section 6.1 we show that the prices that decentralize this allocation are such that $p_i^* = c_i$ for $i = 1, 2$. Plugging these prices in (4.1) and rearranging, we find that the optimal ad valorem taxes satisfy the following system:

$$t_i^* = \frac{Q_i^* - p_j^* t_j^* \frac{\partial Q_j}{\partial p_i}}{Q_i^* \left(1 + \frac{p_i^*}{Q_i^*} \frac{\partial Q_i}{\partial p_i}\right)}, \; i = 1, 2, \quad j \neq i. \tag{6.6}$$

The denominator is positive if and only if Q_i^* lies on the inelastic part of demand for good i. The first term at the numerator is positive, but the second term also depends on the tax rate on the other good. Solving the system in (6.6) above, we obtain

$$t_i^* = \frac{Q_i^* Q_j^* \left(1 + \frac{p_j^*}{Q_j^*} \frac{\partial Q_j}{\partial p_j}\right) - p_j^* \frac{\partial Q_j}{\partial p_i} Q_j^*}{Q_i^* \left(1 + \frac{p_i^*}{Q_i^*} \frac{\partial Q_i}{\partial p_i}\right) Q_j^* \left(1 + \frac{p_j^*}{Q_j^*} \frac{\partial Q_j}{\partial p_j}\right) - p_i^* p_j^* \frac{\partial Q_j}{\partial p_i} \frac{\partial Q_i}{\partial p_j}}, \; i = 1, 2, \quad j \neq i. \tag{6.7}$$

These expressions are quite hard to sign at this level of generality. To simplify, we use Assumption 1, and we obtain

$$t_1^* = \frac{Q_1^* - \frac{Q_2^* c_2 \frac{\partial Q_2}{\partial p_1}}{Q_2^* + c_2 \frac{\partial Q_2}{\partial p_2}}}{Q_1^* \left(1 + \frac{c_1}{Q_1^*} \frac{\partial Q_1}{\partial p_1}\right)}, \quad t_2^* = \frac{1}{Q_2^* \left(1 + \frac{c_2}{Q_2^*} \frac{\partial Q_2}{\partial p_2}\right)}. \tag{6.8}$$

Suppose now the optimal (decentralized) allocation is such that Q_2^* lies on the elastic part of demand for good 2, so $t_2^* < 0$. If the conditions outlined in Proposition 1 hold, then $-1 < \frac{c_1}{Q_1^*} \frac{\partial Q_1}{\partial p_1}$ and $\frac{\partial Q_2}{\partial p_1} < 0$ hold as well. Therefore, we get $t_1^* > 0$ as long as $Q_1^* > \frac{c_2 \frac{\partial Q_2}{\partial p_1}}{Q_2^* + c_2 \frac{\partial Q_2}{\partial p_2}}$. Furthermore, if $-1 < \frac{c_1}{Q_1^*} \frac{\partial Q_1}{\partial p_1}$ and $-1 < \frac{c_2}{Q_2^*} \frac{\partial Q_2}{\partial p_2}$, both t_2^* and t_1^* are positive. In sum, it is possible that the optimal tax rates on one or both goods are strictly positive.

Unit taxes. We now focus on unit taxes and set $t_1 = t_2 = 0$. Proceeding as above, we obtain

$$\tau_i^* = \frac{Q_i^* - \tau_j^* \frac{\partial Q_j}{\partial p_i}}{\frac{\partial Q_i}{\partial p_i}}, \; i = 1, 2, \quad j \neq i. \tag{6.9}$$

The denominator is strictly negative. Hence, aside from the interaction with the other tax rate, the expression indicates that τ_i^* tends to be negative. Solving the system in (6.9) we obtain

$$\tau_i^* = \frac{\frac{\partial Q_j}{\partial p_j} Q_i^* - \frac{\partial Q_j}{\partial p_i} Q_j^*}{\frac{\partial Q_i}{\partial p_i}\frac{\partial Q_j}{\partial p_j} - \frac{\partial Q_j}{\partial p_i}\frac{\partial Q_i}{\partial p_j}}, \; i = 1, 2, \quad j \neq i. \tag{6.10}$$

As in the case of ad valorem taxes, these expressions are fairly difficult to sign. To simplify, suppose Assumption 1 holds. We obtain

$$\tau_1^* = \frac{Q_1^*}{\frac{\partial Q_1}{\partial p_1}} - \frac{Q_2^* \frac{\partial Q_2}{\partial p_1}}{\frac{\partial Q_1}{\partial p_1}\frac{\partial Q_2}{\partial p_2}}, \; \tau_2^* = \frac{Q_2^*}{\frac{\partial Q_2}{\partial p_2}} < 0. \tag{6.11}$$

These expressions indicate that $\tau_1^* < 0$ if $\frac{\partial Q_2}{\partial p_1} < 0$. Instead, the tax rate is positive if $\frac{\partial Q_2}{\partial p_1} > 0$ and large enough in magnitude.

7 Applications

We provide several applications of the model to settings involving a multiproduct firm. For brevity, in these applications we focus only on ad valorem taxes.

7.1 Add-on pricing

We consider a simplified version of the add-on pricing model by Ellison (2005). Good 1 is a "base" good, whereas good 2 is an "add-on". For instance, 1 can be a piece of hardware and 2 after-sale assistance. Furthermore, 1 can be a flight ticket and 2 travel insurance or extra luggage allowance. Each consumer buys at most one unit of each good and has valuation v_i for good $i = 1, 2$. There is a unit mass of consumers. The valuation v_1 is uniformly distributed with support $[0, 1]$, whereas v_2 is identical for all consumers. We assume $v_2 = v > 0$ if and only if the consumer buys good 1 and $v_2 = 0$ otherwise, since good 2 has no value without good 1.

Consumers know their valuations v_i and observe p_1 without visiting M, but observe p_2 only if they visit. However, consumers form rational expectations about this price. Visiting

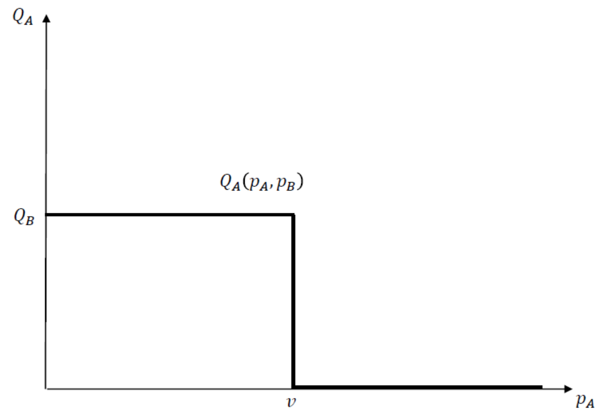

Figure 7.1: Demand for the add-on good.

M entails a search cost s, that is sufficiently small, i.e. $s \to 0$. The timing of the game is as follows: first the government sets t_2 and t_1. Second, M sets p_2 and p_1. Third, consumers observe p_1 and decide whether to visit M. Consumers who visit the store observe p_2 and decide whether to buy 1 and/or 2.

A consumer who visits M buys 2 if and only if she/he buys 1 and $p_2 \leq v$ holds. Thus, conditional on $p_2 \leq v$, the demands for good 1 and 2 are identical, i.e. $Q_2(p) = Q_1(p)$. If $p_2 > v$, no consumer buys 2. Clearly, setting $p_2 > v$ cannot be profitable to M, so we assume henceforth that $p_2 \leq v$. Under this condition, consumers buy both goods if and only if

$$v_1 \geq p_1 - (v - p_2). \tag{7.1}$$

Figure 7.1 illustrates the demand functions for the two goods. Note that the demand for good 2 is either equal to 0 or to 1. In Appendix A we extend the analysis to study corner solutions and we show that there are no significant changes to the analysis presented so far.

To characterize the demand for good 1, consider that, given a small search cost, a consumer visits M if and only if condition (7.1) holds (after replacing p_2 with the expected price, which coincides with the equilibrium price p_2^e in equilibrium). Since $v_1 \sim U[0,1]$, we have

$$Q_1(p) = 1 - p_1 + (v - p_2^e). \tag{7.2}$$

Observe that $\frac{\partial Q_1}{\partial p_2} = 0$ and $\frac{\partial Q_2}{\partial p_1} < 0$ hold, so Assumption 1 applies in this setting.

Given $Q_1 = Q_2$, the profit of M can be written as

$$\pi = (p_2(1 - t_2) - c_2 + p_1(1 - t_1) - c_1) Q_1. \tag{7.3}$$

The solution to the profit maximization problem must be such that $p_2^e = v$ (regardless of the tax rates), because reducing this price below v would not increase demand (see Figure 7.1). Replacing $p_2^e = v$ in (7.2) and maximizing (7.3) with respect to p_1, we obtain

$$p_1^e = \frac{1}{2} + \frac{c_1 + c_2 - v\,(1 - t_2)}{2\,(1 - t_1)}.$$

This expression shows that M curtails the mark-up on the base good in order to boost demand for the add-on, particularly if the latter sells at a high margin. The equilibrium price of good 1 decreases in the ad valorem tax t_1 if and only if p_1^e lies on the inelastic part of the demand curve (see Proposition 2), which holds if and only if costs are small enough

$$\frac{\partial p_1^e}{\partial t_1} < 0 \Leftrightarrow \frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1} > -1 \Leftrightarrow c_1 < v\,(1 - t_2) - c_2.$$

Recall that p_2^e is unaffected by taxation and that at equilibrium $Q_1^e = Q_2^e$. We conclude that, if the above inequality holds, consumption of both goods increases with t_1.

7.2 Multiproduct retailing with price advertising

We consider a simplified version of the model by Rhodes (2015). There is a unit mass of consumers with valuation v_i for good i, distributed according to a distribution $F(v_i)$ with support $[a, b] \subset \mathbb{R}$. This distribution has strictly positive, continuously differentiable, and log-concave density f. The parameter v_i is i.i.d. across products and consumers, who know their individual valuations for each product and buy at most one unit of each. The unit cost of each product is c, with $0 \leq c < b$. We assume M advertises the price of good 1. Hence, consumers observe p_1 at no cost, but must visit M to know p_2, incurring a small search cost s. Consumers form rational expectations about this price. For simplicity, in this application we only consider an ad valorem tax on good 1, t_1, setting all other tax rates at zero. Furthermore, we assume that search and product costs are small enough that a positive mass of consumers searches the firm in equilibrium.

The timing is as follows. At stage one, the government sets t_1. M then chooses p_1 and advertises this price to each consumer, choosing p_2 next. Consumers then learn p_1 and form expectations about p_2. Each consumer decides whether to visit at that point. Finally, consumers who visit learn p_2 and make their purchase decisions.

A consumer that visits M purchases good i if and only if $v_i \geq p_i$. Thus, the consumer visits if and only if her expected surplus is higher than the search cost s, i.e.

$\max(v_2 - p_2^e; 0) + v_1 - p_1 > s$ holds. The demands for good 1 and 2 respectively are

$$Q_1(p) = \int_{p_1}^{b} f(v_1)\, Pr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s \mid v_1\right) dv_1, \tag{7.4}$$

$$Q_2(p) = \int_{p_2}^{b} f(v_2)\, Pr\left(\max(v_2 - p_2^e; 0) + v_1 - p_1 \geq s \mid v_2\right) dv_2. \tag{7.5}$$

Observe that Q_1 depends on p_2^e but not on p_2, so $\frac{\partial Q_1}{\partial p_2} = 0$. Hence, Assumption 1 holds in this setting. Furthermore, $\frac{dQ_2}{dp_1} < 0$ holds.

The vector of equilibrium prices, p^e, maximizes $\pi = (p_1(1-t_1) - c)\, Q_1 + (p_2 - c)\, Q_2$. In this equilibrium, p_2^e satisfies the following

$$p_2^e = -\frac{Q_2^e}{\frac{\partial Q_2}{\partial p_2}} + c,$$

where $\frac{\partial Q_2}{\partial p_2} < 0$ (we provide the expression for this derivative in Appendix B.3). Hence, p_2 is set according to the standard "cost plus mark-up" formula and is strictly above marginal cost.[7] The price p_1^e satisfies the following equation

$$p_1^e = -\frac{Q_1^e}{\frac{dQ_1}{dp_1}} + \frac{c}{1-t_1} + \frac{p_2^e - c}{1-t_1}\frac{dQ_2}{dp_1},$$

where both $\frac{dQ_1}{dp_1}$ and $\frac{dQ_2}{dp_1}$ are negative (we provide the expression for these derivatives in Appendix B.3). Therefore, the necessary and sufficient condition for p_1^e to decrease with t_1, as stated in Proposition 2, is

$$\frac{\partial p_1^e}{\partial t_1} < 0 \Leftrightarrow \frac{p_1^e}{Q_1^e}\frac{dQ_1}{dp_1} > -1 \Leftrightarrow c < \frac{(p_2^e - c)\frac{dQ_2}{dp_1}}{\frac{dQ_1}{dp_1}}. \tag{7.6}$$

Lemma 2 in Rhodes (2015) shows that, when M raises the price of the advertised good, the price of the other good increases as well, i.e. $\frac{\partial p_2^e}{\partial p_1} > 0$. Given this condition, the inequality $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ holds. Hence, as long as (7.6) holds, the conditions stated in Proposition 2 for p_1^e and p_2^e to decrease with the tax rate t_1 are satisfied. These conditions are also sufficient for the output of both goods to increase with t_1.

[7] As shown in Rhodes (2015), this price exceeds the "typical" monopoly price without search costs, because consumers observe p_2 only after searching.

<table>
<tr><td rowspan="2">$i=1$</td><td colspan="2">Buy</td><td colspan="2">Not buy</td></tr>
<tr><td colspan="2">$1-x-p_1$</td><td colspan="2">0</td></tr>
<tr><td rowspan="2">$i=2$</td><td>Buy</td><td>Not buy</td><td>Buy</td><td>Not buy</td></tr>
<tr><td>$1-x-p_2$</td><td>$-s$</td><td>$1-x-p_2-s$</td><td>0</td></tr>
</table>

Table 2: Consumer payoffs in the switching cost model.

7.3 An intertemporal model with switching costs

We consider an intertemporal setting where a firm internalizes the switching costs faced by consumers.[8] Suppose M provides a single product in two time periods, $i = 1, 2$, at a constant unit cost $c < 1$. There is a unit mass of consumers which in each period decides whether to buy either one unit of the good or none. In each period, if a consumer buys, she/he gets utility $1-x$, where x is uniformly distributed on the $[0, 1]$ interval and time-invariant. If a consumer buys (resp. does not buy) M's product in period 1 and she/he does not buy (resp. buys) in period 2, she/he sustains a small switching cost, s.[9] Table 2 summarizes a consumer's payoff in period i.

In period 1, consumers observe p_1 and form rational expectations about p_2. Furthermore, they choose whether to buy the M's product anticipating their payoff at the following stage. M's intertemporal profit is $\pi = \sum_{i=1,2} \left(p_i \left(1-t_i\right) - c\right) Q_i$, where t_i is the ad valorem tax rate in period i. We ignore intertemporal discounting.

We solve the model by backward induction. In period 2, consumers who bought previously sustain the cost s if not buying anymore (all else given), which increases their willingness to pay. Similarly, the switching cost decreases the willingness to pay by consumers who did not buy from M in period 1. Therefore, the demand function $Q_2(p)$ is kinked, as represented in the left panel of Figure 7.2 (see Appendix B.4.1 for the derivation of this function):

$$Q_2(p) = \begin{cases} 1-s-p_2 & \text{if } p_2 < 1-s-Q_1, \\ Q_1 & \text{if } p_2 \in \left[1-s-Q_1; 1+s-Q_1\right], \\ 1+s-p_2 & \text{if } p_2 > 1+s-Q_1. \end{cases} \tag{7.7}$$

Note that $Q_2(p)$ is flat over an interval of values of p_2 such that all old customers buy again, but the price is too high to attract any new customer.

To find p_2^e, we maximize the profit in period 2, $\left(p_2 \left(1-t_2\right) - c\right) Q_2(p)$, with respect to p_2.

[8] See, e.g., Tirole, 1988, and Belleflamme and Peitz, 2015, for a an overview of the literature on this topic.

[9] For example, this cost could capture the effort to learn how to use the product (and, conversely, how to forgo it), e.g., in the case of a particular type of tool or software.

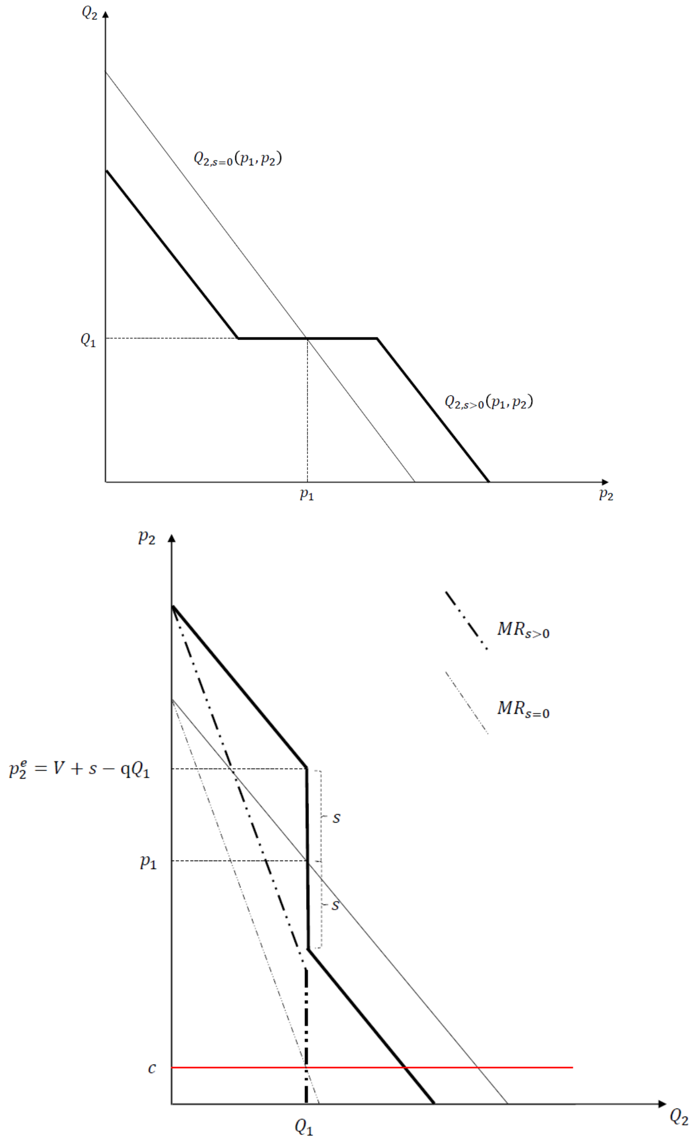

Figure 7.2: Left panel: demand in period 2 with and without the switching cost. Right panel: inverse demand, marginal revenue and equilibrium price in period 2.

As we show in Appendix B.4.1, the solution is such that

$$p_2^e = 1 + s - Q_1. \tag{7.8}$$

Hence we find that $Q_2^e = Q_1$. The price p_2^e coincides with the rightmost kink in the demand function $Q_2(p)$. Exploiting the switching cost, M imposes the largest possible markup conditional on maintaining the previous customer base (see Figure 7.2, right panel).

In period 1 a consumer buys M's good if and only if she/he anticipates she/he will buy again in period 2. That is, consumers correctly anticipate that they will be locked-in. Therefore, the marginal consumer is indifferent between buying in both periods or not buying at all, i.e. $1 - \bar{x}_1 - p_1 + 1 - \bar{x}_1 - p_2^e = 0$ holds. Given $Q_1 = \bar{x}_1$ and (7.8), we get

$$Q_1(p_1) = 1 - p_1 - s.$$

Replacing the latter expression in (7.8), we obtain that $p_2^e = p_1 + 2s$. Consumers expect M to exploit the switching cost in period 2, and already incorporate this cost when determining their willingness to pay in period 1. Note that the demand in period 1 does not depend on p_2, but on the expected equilibrium price, p_2^e. Hence, Assumption 1 holds in this setting.

Given $p_2^e = p_1 + 2s$ and $Q_2^e = Q_1$, we can write M's intertemporal profit as

$$\pi = (p_1(1-t_1) - c + (p_1 + 2s)(1-t_2) - c)\, Q_1.$$

Maximizing the above expression with respect to p_1 we find

$$p_1^e = \frac{1}{2} + \frac{2c - s(4 - t_1 - 3t_2)}{2(2 - t_1 - t_2)}.$$

When all taxes are set to zero, this expression boils down to $p_1^0 = \frac{1}{2} + \frac{c}{2} - s$. The switching cost induces the supplier to reduce its price in period 1, in order to increase the share of locked-in consumers in the next period.

Finally, let us focus on the effects of taxation. Starting from the equilibrium without taxes, the necessary and sufficient condition for the price of good 1 to decrease with t_1 (as stated in Proposition 2) is:

$$\frac{\partial p_1^0}{\partial t_1} < 0 \Leftrightarrow \frac{p_1^0}{Q_1^0}\frac{dQ_1}{dp_1} > -1 \Leftrightarrow c < s. \tag{7.9}$$

Note that, while p_2^e is not directly affected by t_1, it does depend on p_1^e, since we have established above that $p_2^e = p_1^e + 2s$. Thus, (7.9) is necessary and sufficient for prices to decrease with t_1

in *both* periods, starting from the no-tax equilibrium. This condition is also sufficient for the quantities Q_1^e and Q_2^e to increase with the tax.

7.4 An example from Hotelling (1932)

To conclude this section, we test the effects of ad valorem taxes in a setting where the classical Edgeworth's paradox holds. In particular, we consider a simple example with linear demands, borrowed from Hotelling (1932), and show that the introduction of a unit tax on one good produces a reduction of both prices.

Consider the following demand functions for substitute goods:[10]

$$Q_1 = 4 - 10p_1 + 7p_2, \quad Q_2 = 4,2 - 7p_2 + 9,8p_1.$$

Consider a tax t_1 on good 1 and set $t_2 = 0$. Solving the system of first-order conditions in (4.1) we find the following equilibrium prices

$$p_1 = \frac{-452 + 305t_1 + 35c_2\left(2 + 5t_1\right) - 5c_1\left(16 + 35t_1\right)}{8 + 5t_1\left(32 + 35t_1\right)},$$

$$p_2 = \frac{20\left(1 - t_1\right)\left(5t_1 - 27\right) + c_2\left(88 + 255t_1\right) - 50c_1\left(2 + 5t_1\right)}{8 + 5t_1\left(32 + 35t_1\right)}.$$

By deriving both equilibrium prices by t_1 and evaluating the derivatives at $t_1 = 0$, one can show that there exist values of c_1 and c_2 such that both prices decrease when a tax on good 1 is introduced. By equation (5.3), we know that the equilibrium quantity of good 1 never lies on the inelastic part of the demand because goods are substitutes, implying that the direct effect of the tax on p_1 is always positive. Instead, the indirect effect is negative because $\frac{\partial Q_i}{\partial p_j} > 0$ and $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ hold. Also, looking at the effect of the tax on the price of good 2, we know that the direct effect is always negative (because $\frac{\partial Q_i}{\partial p_j} > 0$), while the indirect effect is always positive (because $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ and the equilibrium quantity does not lie on the inelastic part of the demand). Hence, the effect of the tax on p_1 (resp. p_2) can be negative if and only if the indirect (resp. direct) effects is strong enough.

We now look for values of c_1 and c_2 such that both prices decrease when a tax on good 1 is introduced. Equations (5.4) and (5.6) (when $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$ holds) indicate that c_1 has to be low enough for this to occur. Instead, a c_2 high enough favor the negative effects on both prices.

[10] Unlike the linear demands considered in Assumption 2, these functions cannot be derived from a standard utility function because the cross-price parameter is asymmetric.

For instance, if we set $c_1 = 0$ and $c_2 = 9$, it is easy to verify that both prices decrease when a tax on good 1 is introduced. Furthermore, the quantity of good 1 increases when a tax is introduced, while the quantity of good 2 decreases.

8 Two-sided markets

We show that the core of our analysis and our main results also apply to the case where M is a two-sided platform. We modify the setting in the baseline model assuming there are two representative consumers: one buying only good 1 and another buying only 2. The utility function of the consumer in market i is

$$V_i\left(Q_1, Q_2\right) + z_i - p_i Q_i, \quad i = 1, 2,$$

where z_i is the exogenous income of such consumer. The functions $V_i\left(.\right), \quad i = 1, 2$, are continuously differentiable and concave. Importantly, each consumer's utility depends not only on the quantity of the good she consumes, but also on the quantity of the other good. That is, there are cross-market externalities.

The demand $Q_i\left(p_i, Q_j\right)$ for good $i = 1, 2$ is defined by

$$\frac{\partial V_i}{\partial Q_i} = p_i, \quad i, j = 1, 2. \tag{8.1}$$

From now on, we omit the argument of the demand functions. We have $\frac{\partial Q_i}{\partial p_i} < 0$ and $\frac{dQ_i}{dp_j} = \frac{\frac{\partial Q_i}{\partial Q_j}\frac{\partial Q_j}{\partial p_j}}{1-\frac{\partial Q_i}{\partial Q_j}\frac{\partial Q_j}{\partial Q_i}}$. Assuming that $1 - \frac{\partial Q_i}{\partial Q_j}\frac{\partial Q_j}{\partial Q_i} > 0$, the condition $\frac{dQ_i}{dp_j} > 0$ holds if and only if $\frac{\partial Q_i}{\partial Q_j} < 0$, i.e. an increase in Q_j induces a drop in the demand for good i. The key point is that the derivative $\frac{dQ_i}{dp_j}$ is generally not zero. That is, the demands for the two goods are interdependent because, although each consumer only buys one of the two goods, there are externalities across the two markets.

Assuming the same cost function as in the baseline model, the supplier's profit function is isomorphic to (3.2). Therefore, the first-order conditions that define the vector of equilibrium prices, p^e, are as in (4.1). It follows that the effects of taxation are as characterized in Section 5. Therefore, our analysis also applies to two-sided markets. We thus generalize previous findings by Kind et al. (2008), by showing that the effects of taxation that the authors characterized in a two-sided market apply more generally to markets served by a multiproduct firm, even if such markets are "one-sided", provided that the demands for the goods are interdependent. As an illustration, consider the sufficient conditions for t_1 to decrease prices and increase supply that we provide in Proposition 2. These are equivalent to the sufficient conditions that Kind et al. (2008, p. 1535) provide in their main example Specifically, their assumption (b) is tantamount to $\frac{p_i^e}{Q_i^e}\frac{\partial Q_i}{\partial p_i} > -1$. Furthermore, their assumption (a) is the same as Assumption 1 in our setting.

The main difference with respect to the case of a "one-sided" market concerns the social optimum and the optimal policy. With two-sided markets, the social welfare function is such that

$$W = \sum_{i=1,2} \left(V_i\left(Q_1, Q_2\right) + z_i\right) - c_1 Q_1 - c_2 Q_2. \tag{8.2}$$

The quantities maximizing this function, Q_1^* and Q_2^*, satisfy the system of equation $\frac{\partial V_j}{\partial Q_i} - c_i = 0,\ i = 1, 2$. Given (8.1), the optimal allocation is decentralized by the following prices

$$p_i^* = c_i - e_i^*, \quad \text{where } e_i^* \equiv \left.\frac{\partial V_j}{\partial Q_i}\right|_{\left(Q_1^*, Q_2^*\right)},\ i, j = 1, 2,\ j \neq i. \tag{8.3}$$

e_i^* captures the marginal external effect of changes in Q_i on the utility of consumers in the other market. As shown by Kind et al. (2008), the presence of these externalities implies that a monopolist two-sided platform does not necessarily underprovide a good, given the price of the other good, unlike in a "one-sided" market (see Equation (6.1)). Indeed, proceeding as in Section 6.1, we find that

$$p_i^0\left(p_j^*\right) > p_i^* \Leftrightarrow Q_i^* > e_i^* \frac{\partial Q_i}{\partial p_i} + e_j^* \frac{\partial Q_j}{\partial p_i}, \quad i, j = 1, 2, \quad j \neq i, \tag{8.4}$$

where $p_i^0\left(p_j^*\right)$ denotes the equilibrium price conditional on $p_j = p_j^*$. Suppose, to illustrate, that cross-market externalities are negative for one of the goods and absent for the other (i.e., $e_1^* < 0$ and $e_2^* = 0$). Under these conditions, inequality (8.4) may be violated for $i = 1$, because the right hand side of the second inequality is positive. Hence, the monopolist may *underprice* (and overprovide) good 1, because it does not fully internalize the negative effect of a change in Q_1 on the utility of consumers of good 2.

With overprovision, the government's objective should be to increase the price of the good and reduce its supply. The optimal set of tax rates may therefore involve a subsidy, if the price price decreases with the tax rate. However, if there is underprovision, positive taxes may address the problem if prices decrease. However, For reasons of space, and because that analysis has already been provided by Kind et al. (2008), we do not analyze the optimal tax policy in this setting.

8.1 Application

We provide an application based on Armstrong (2006). Consider a platform serving two groups of users, indexed by $i = 1, 2$. The utility of a user in group i is

$$u_i = \alpha_i Q_j - p_i, \quad i, j = 1, 2, \quad j \neq i, \tag{8.5}$$

where p_i is the price set by the platform for users in group i and Q_j is the number of users in group $j \neq i$. If $\alpha_i > 0$, the utility of users in group i increases with size of the other group j. An example is a gaming console, where group 1 are players while group 2 are game developers. We also consider the case where users in one group, say 2, benefit from participation by users in the other group, but not the other way round, i.e. $\alpha_1 \leq 0$ and $\alpha_2 > 0$. An example is a media platform (e.g., an online website or a TV station), where group 2 are advertisers and group 1 are viewers.

We assume the number of users that join the platform in each group is

$$Q_i = \phi_i (u_i), \quad i = 1, 2, \tag{8.6}$$

with $\phi_i' > 0$. Combining (8.5) and (8.6), one obtains the following own- and cross-price derivatives of demand:

$$\frac{\partial Q_i}{\partial p_i} = -\phi_i' < 0, \; \frac{\partial Q_j}{\partial p_i} = -\frac{\phi_1' \phi_2' \alpha_j}{1 - \phi_1' \phi_2' \alpha_1 \alpha_2}, \quad i, j = 1, 2; j \neq i. \tag{8.7}$$

Assuming $1 > \phi_i' \alpha_i \phi_j' \alpha_j$, we have that $\frac{\partial Q_j}{\partial p_i} > 0$ if and only if $\alpha_j < 0$. That is, the effect of increasing p_i on demand of the other side of the market depends on the externality that group i generates on group j: if greater participation on side i reduces the utility of users on side j ($\alpha_j < 0$), then a higher price on side i will increase demand on side j, and viceversa.

The platform sustains a unit cost c_i per each user in group i. Hence, its profit is $\pi = Q_1 (p_1 (1 - t_1) - c_1) + Q_2 (p_2 (1 - t_2) - c_2)$. As we show in Appendix B.5.1, we have (ignoring unit taxes)

$$p_i^e = \frac{c_i - \phi_j \alpha_j (1 - t_j)}{1 - t_i} + \frac{\phi_i}{\phi_i'} \; i, j = 1, 2, \; j \neq i. \tag{8.8}$$

Setting $t_1 = t_2 = 0$, expression (8.8) boils down to

$$p_i^0 = c_i + \frac{\phi_i}{\phi_i'} - \phi_j \alpha_j \; i, j = 1, 2, \; j \neq i. \tag{8.9}$$

This is a standard monopoly price formula (marginal cost plus mark up), except for the third term that accounts for the marginal external effect that users in group i produce on users in the other group. If $\alpha_j > 0$, the platform has an incentive to reduce the price of good i to raise the willingness to pay on the other side.

To illustrate the effects of taxation, let us focus on t_1. As we show in Appendix B.5.2, p_1 and p_2 decrease with this tax rate if the following conditions hold

$$\frac{p_1^e}{Q_1^e} \frac{\partial Q_1}{\partial p_1} > -1 \Leftrightarrow c_1 < Q_2^e \alpha_2 (1 - t_2), \; \alpha_1 \leq 0 \text{ and } \frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0. \tag{8.10}$$

The first condition states that Q_1^e lies on the inelastic part of the demand curve for good 1. This condition can hold only if participation by group 1 (e.g., viewers) produces a positive externality on group 2 (e.g. advertisers), i.e. $\alpha_2 > 0$. The second condition applies whenever users in group 2 produce either a negative or no externality on users in group 1. The third condition applies whenever a higher price of one good makes raising the other price more profitable to the platform. Given $\alpha_1 = 0$ and $\alpha_2 \neq 0$ (i.e., Assumption 1 holds), the conditions in (8.10) correspond to the sufficient conditions provided in Proposition 2. However, note that t_1 reduces both prices even if $\alpha_1 < 0$, as long as the other conditions in (8.10) hold. Hence, Assumption 1 is not necessary for both prices to decrease with the ad valorem tax.

9 Concluding remarks

A fundamental result in the theory of commodity taxation is that taxes increase consumer prices and reduce supply, aggravating the distortions caused by market power. This result hinges on the assumption that each firm provides a single product. We have studied the effects of commodity taxation in presence of a multiproduct monopolist. We consider a firm providing two goods and obtain simple conditions such that an ad valorem tax reduces the prices and increases the supply of both goods. By contrast, even if a unit tax can reduce both prices, the supply of the taxed good always decreases. Whenever both goods are underprovided and imposing a tax on one good increases both quantities, the tax has a positive effect on welfare.

This paper broadens previous findings on the Edgeworth's paradox by considering general demand functions and studying unit as well as ad valorem taxes. We show that taxes can induce a price decrease in a variety of settings, including add-on pricing, multiproduct retailing with price advertising, and intertemporal models with switching costs. Moreover, we generalize previous findings on the effects of taxation in two-sided markets, showing that these effects apply more generally to markets served by a multiproduct firm, even if "one-sided", provided that the demands for the goods are (at least partially) interdependent.

As a final remark, we note that the effects of taxation that we characterized should apply more generally to other settings, in particular regarding vertical relations. Specifically, unit taxes are similar to wholesale prices, whereas ad valorem ones are similar to revenue-sharing arrangements. We plan to explore the implications of the mechanisms we identified for vertical relations among multiproduct firms in future research.

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Appendix

A Kinked demands and corner solutions

In this section we briefly show how our analysis can account for kinks in demand. We assume that, while $Q_1(p)$ is everywhere continuously differentiable in p_1 and p_2, $Q_2(p)$ is continuously differentiable in p_1, but piecewise continuously differentiable in p_2. To avoid repetition, we concentrate on the case where the equilibrium price p_2^e coincides with a kink in $Q_2(p)$. One can expect this to be the case, for example, if $Q_2(p)$ is highly inelastic with respect to p_2 when approaching the kink from the left, but drops sharply beyond it. To streamline the exposition, and consistently with the applications we present in Section 7, we assume p_2^e exceeds c_2 and is unaffected by the tax rates. Furthermore, we ignore cross-market externalities. Under the above assumptions, we can treat p_2^e as a parameter and thus focus on the effect of taxation on p_1[11]

Given p_2^e, the equilibrium price p_1^e satisfies the first-order conditions in (4.1), so that p_1^e has the same form as characterized in (4.2). Differentiating (4.1), we obtain the following effects

[11] Although the government cannot affect p_2^e with taxes by assumption, it can in principle use other instruments (e.g. introduce a price ceiling). We ignore this possibility because in all the applications in Section 7 where demand is kinked, given p_1^e, the equilibrium price p_2^e does not generate any distortion.

of ad valorem taxes

$$\frac{\partial p_1^e}{\partial t_1} = \frac{Q_1^e\left(\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1}+1\right)}{\frac{\partial \pi}{\partial p_1}}, \quad \frac{\partial p_1^e}{\partial t_2} = \frac{p_2^e\frac{\partial Q_2}{\partial p_1}}{\frac{\partial \pi}{\partial p_1}}. \tag{A.1}$$

Since $\frac{\partial \pi}{\partial p_1} < 0$ by the second-order conditions of the supplier's maximization problem, p_1^e decreases with t_1 if and only if Q_1^e lies on the inelastic part of the demand for good 1 (given p_2^e), i.e. $\frac{p_1^e}{Q_1}\frac{\partial Q_1}{\partial p_1} > -1$. This condition holds if and only if the unit cost c_1 is small enough (see (5.3)). Furthermore, p_1^e decreases with t_2 if and only if $\frac{\partial Q_2}{\partial p_1} < 0$. Given p_2^e, a reduction in p_1^e is sufficient to determine an increase in the equilibrium quantity of good 1 and, if $\frac{\partial Q_2}{\partial p_1} > 0$, of good 2 as well.

Consider now unit taxes. Differentiating (4.1), we obtain

$$\frac{\partial p_1^e}{\partial \tau_1} = \frac{\frac{\partial Q_1}{\partial p_1}}{\frac{\partial \pi}{\partial p_1}} > 0, \qquad \frac{\partial p_1^e}{\partial \tau_2} = \frac{\frac{\partial Q_2}{\partial p_1}}{\frac{\partial \pi}{\partial p_1}}.$$

The effect of τ_1 on p_1^e is standard, while the effect of τ_2 depends on the sign of $\frac{\partial Q_2}{\partial p_1}$.

Focus now on the optimal government policy. Without externalities across markets (i.e. $e_1 = e_2 = 0$) and given p_2^e, the price of good 1 in the no-tax equilibrium exceeds the optimal level (thus, good 1 is underprovided). This can be established following the same procedure as for the inequalities in (**??**). Hence, the government should reduce p_1^e, starting from the no tax equilibrium, in order to increase welfare. As we show below, given p_2^e, the government can implement the optimum by adopting tax rates that satisfy the following conditions:

$$t_1 = \frac{Q_1^e - p_2^e t_2 \frac{\partial Q_2}{\partial p_1}}{Q_1^e\left(\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1}+1\right)}, \quad \text{or} \quad \tau_1 = \frac{Q_1^e - \tau_2\frac{\partial Q_2}{\partial p_1}}{\frac{\partial Q_1}{\partial p_1}}. \tag{A.2}$$

Note that, to ease exposition, we presented the expressions for the optimal ad valorem and unit taxes separately, i.e. setting the other tax rates to zero. The above tax rates indicate that the government can correct the distortion either in a conventional way, i.e. with a unit subsidy ($\tau_1 < 0, \tau_2 = 0$) or an ad-valorem subsidy (provided that $\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1} < -1$). More interestingly, the government can also correct the distortion by imposing an ad valorem tax on good 1 ($t_1 > 0, t_2 = 0$) if and only if $\frac{p_1^e}{Q_1^e}\frac{\partial Q_1}{\partial p_1} > -1$ holds (which is necessary and sufficient for p_1^e to decrease with t_1).[12]

[12]In principle the government could also correct the distortion by taxing good 2 only, if $\frac{\partial Q_2}{\partial p_1} > 0$. However, this finding is specific to the case where p_2^e is a corner solution. As we show in the main text, if this price responds to taxation as well, the effects of a tax on good 2 on welfare are more complex.

Proposition A1. *Suppose p_2^e coincides with a kink in demand and there are no externalities across markets. The price p_1^e increases with τ_1 and the optimal unit tax rates are such that $\tau_1 < 0, \tau_2 = 0$. Furthermore, if and only if (5.3) holds for $i = 1$, p_1^e decreases with t_1 and the optimal ad valorem tax rates are such that $t_1 > 0, t_2 = 0$.*

Derivation of (A.2)

Given p_2^e and $e_1 = e_2 = 0$, the first-order conditions for welfare maximization with respect to p_1 is

$$\frac{\partial W}{\partial p_1} = (p_1 - c_1)\frac{\partial Q_1}{\partial p_1} + (p_2^e - c_2)\frac{\partial Q_2}{\partial p_1} = 0,$$

whereas the first-order conditions for profit maximization with respect to p_1 is

$$\frac{\partial \pi}{\partial p_1} = (1 - t_1) Q_1 + (p_1 (1 - t_1) - c_1 - \tau_1)\frac{\partial Q_1}{\partial p_1} + (p_2^e (1 - t_2) - c_2 - \tau_2)\frac{\partial Q_2}{\partial p_1} = 0.$$

Setting all unit taxes to zero, replacing the prices and quantities that satisfy $\frac{\partial W}{\partial p_1} = 0$ in the above equation, we obtain the leftmost expression in (A.2). Setting all ad valorem taxes to zero, replacing the prices and quantities that satisfy $\frac{\partial W}{\partial p_1} = 0$ in the above equation, we obtain the rightmost expression in (A.2).

B Proofs of results not given in the text

B.1 Effect of unit tax on quantity of taxed good

To prove the claim in the most direct way, we provide the solution of M's profit maximization problem under the alternative assumption that quantities are the decision variables, rather than prices. Under the assumptions of our baseline model, the demand system defined by (3.1) and (8.1) is invertible. Let $p_i(Q_i, Q_j)$ be the inverse demands for goods $i = 1, 2$. The first-order conditions of the monopolist's problem (assuming $t_1 = t_2 = \tau_j = 0$ for simplicity) write as

$$G_i \quad : \quad \frac{\partial \pi}{\partial Q_i} = Q_i\frac{\partial p_i}{\partial Q_i} + (p_i - c_i - \tau_i) + Q_j\frac{\partial p_j}{\partial Q_i} = 0, \quad i, j = 1, 2, \quad j \neq i. \tag{B.1}$$

To determine the effect of a change in τ_i on the equilibrium quantity Q_i^e, we totally differentiate (B.1) to obtain

$$\frac{\partial Q_i^e}{\partial \tau_i} = -\frac{\frac{\partial G_i}{\partial \tau_i}\frac{\partial G_j}{\partial Q_j} - \frac{\partial G_i}{\partial Q_j}\frac{\partial G_j}{\partial \tau_i}}{H}, \; i, j = 1, 2, \quad j \neq i, \tag{B.2}$$

where

$$\frac{\partial G_i}{\partial \tau_i} = -1, \quad \frac{\partial G_i}{\partial \tau_j} = 0,$$

$$\frac{\partial G_j}{\partial Q_j} = \frac{\partial^2 \pi}{\partial Q_j^2} < 0, \quad \frac{\partial G_i}{\partial Q_j} = \frac{\partial^2 \pi}{\partial Q_1 \partial Q_2}, \quad \text{and} \quad H \equiv \frac{\partial G_1}{\partial Q_1}\frac{\partial G_2}{\partial Q_2} - \frac{\partial G_1}{\partial Q_2}\frac{\partial G_2}{\partial Q_1} > 0.$$

The denominator of (B.2) is positive by second-order conditions of the maximization problem. The numerator, $\frac{\partial G_i}{\partial \tau_i}\frac{\partial G_j}{\partial Q_j}$, is equal to $-\frac{\partial^2 \pi}{\partial Q_i^2}$, which is positive. Hence, we obtain that $\frac{\partial Q_i^e}{\partial \tau_i} < 0$.

B.2 Effect of unit taxes under Assumption

Given Assumption 1, we have $\frac{\partial F_2}{\partial \tau_1} = -\frac{\partial Q_1}{\partial p_2} = 0$. Thus, the derivatives in (5.1) for $i = 1$ simplify to

$$\frac{\partial p_1^e}{\partial \tau_1} = -\frac{\frac{\partial F_1}{\partial \tau_1}\frac{\partial F_2}{\partial p_2}}{\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial p_1}} > 0, \quad \frac{\partial p_2^e}{\partial \tau_1} = \frac{\frac{\partial F_2}{\partial p_1}\frac{\partial F_1}{\partial \tau_1}}{\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial p_1}}. \tag{B.3}$$

Only the direct effect of τ_1 on p_1^e survives, implying that the price necessarily increases with the tax. Furthermore, only the indirect effect of τ_1 on p_2^e matters under Assumption 1, so p_2^e increases with τ_1 if and only if $\frac{\partial^2 \pi}{\partial p_1 \partial p_2} > 0$. Therefore, under Assumption 1 the effect of a unit tax is significantly different from that of ad valorem tax.

B.3 Demands in Section 7.2

Starting from (7.4) and (7.5), we obtain the following derivatives:

$$\frac{\partial Q_2}{\partial p_2} = -f(p_2)\, Pr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s\right) < 0$$

and

$$\begin{aligned} \frac{dQ_1}{dp_1} &= -f(p_1)\, Pr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s\right) + \\ &\quad + \int_{p_1}^{b} f(v_1) \frac{dPr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s \mid v_1\right)}{dp_1} dv_1 < 0, \end{aligned}$$

where

$$\begin{aligned} \frac{dPr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s \mid v_1\right)}{dp_1} &= \frac{\partial Pr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s \mid v_1\right)}{\partial p_1} + \\ &+ \frac{\partial Pr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s \mid v_1\right)}{\partial p_2^e}\frac{\partial p_2^e}{\partial p_1} \end{aligned}$$

Given that $\frac{\partial p_2^e}{\partial p_1} > 0$ (Rhodes, 2015, Lemma 2), both derivatives on the right hand side of the above expression must be nonpositive. However, note that each may take a different value depending on whether$v_2 \lesseqgtr p_2^e$.

Finally, we have

$$\begin{aligned}\frac{dQ_2}{dp_1} &= -f(p_2)\,Pr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s\right)\frac{\partial p_2^e}{\partial p_1} + \\ &\quad + \int_{p_2}^{b} f(v_2)\,\frac{dPr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s \mid v_2\right)}{dp_1} dv_2 < 0.\end{aligned}$$

where

$$\begin{aligned}\frac{dPr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s \mid v_2\right)}{dp_1} &= \frac{\partial Pr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s \mid v_2\right)}{\partial p_1} + \\ &\quad + \frac{\partial Pr\left(\max\left(v_2 - p_2^e; 0\right) + v_1 - p_1 \geq s \mid v_1\right)}{\partial p_2^e}\frac{\partial p_2^e}{\partial p_1}\end{aligned}$$

Given that $\frac{\partial p_2^e}{\partial p_1} > 0$ (Rhodes, 2015, Lemma 2), both derivatives on the right hand side of the above expression must be nonpositive. However, note that each may take a different value depending on whether $v_2 \lesseqgtr p_2^e$.

B.4 Proofs of the results in Section 7.3

B.4.1 Characterizing expression (7.7)

Let $\bar{x}_1$ denote the marginal consumer in period 1. All consumers such that $x \in [0, \bar{x}_1]$ bought M's product in period 1 and thus incur s if they do not buy again. Within this set of consumers, the marginal consumer in period 2, denoted $\bar{x}_2$, is such that $1 - \bar{x}_2 r - p_2 = -s \Rightarrow \bar{x}_2 = \frac{1+s-p_2}{r}$ holds. Clearly, if and only if $\bar{x}_2 \geq \bar{x}_1$, all consumers who bought in period 1 buy in the next period as well. The consumers who did not buy in period 1 are such that $x \in [\bar{x}_1, 1]$. These consumers thus incur s if they buy in period 2. Hence, the marginal consumer within this group, denoted $\tilde{x}_2$, is such that $1 - \tilde{x}_2 r - s - p_2 = 0 \Rightarrow \tilde{x}_2 = \frac{1-s-p_2}{r}$. Clearly, if and only if $\tilde{x}_2 \leq \bar{x}_1$, no consumer that did not buy previously buys in period 2. Note that $\bar{x}_2 > \tilde{x}_2$ for $s > 0$. We can therefore write the demand for M's product in period 2 as

$$Q_2 = \min\left(\bar{x}_1, \bar{x}_2\right) + \max\left(\tilde{x}_2 - \bar{x}_1, 0\right).$$

Recalling that $Q_1 = \bar{x}_1$, we can rewrite the above expression as in (7.7).

B.4.2 Establishing the equilibrium price of good 2

We first show that the subgame perfect equilibrium cannot be such that $p_2^e > 1 + s - Q_1^e r$, i.e. $Q_2^e < Q_1^e$. If $p_2 > 1 + s - Q_1^e r$ holds, we have $Q_2 = \frac{1+s-p_2}{r}$ given (7.7). The profit in period $i = 2$ is thus $\pi_2 = \left(\frac{1+s-p_2}{r}\right)(p_2(1-t_2) - c)$. The maximizer of this function is $p_2 = \frac{1}{2}\left(1 + s + \frac{c}{(1-t_2)}\right)$, and, given this price, we get $Q_2 = \frac{1}{2r}\left(1 + s - \frac{c}{(1-t_2)}\right)$. For consistency, the condition $Q_2 < Q_1^e$, i.e. $\frac{1}{2r}\left(1 + s - \frac{c}{(1-t_2)}\right) < Q_1^e$ must hold. We now check that this condition cannot hold on the equilibrium path. If $Q_2 < Q_1^e$ holds, $\pi_2 = \left(\frac{1+s-p_2}{r}\right)(p_2(1-t_2) - c)$ is independent of p_1. Hence, when choosing p_1, M maximizes the profit function $\pi_1 = (p_1(1-t_1) - c) Q_1$ with $Q_1 = \bar{x}_1 = \frac{1-p_1}{r}$. The maximizer of this function is $p_1 = \frac{1}{2}\left(1 + \frac{c}{1-t_1}\right)$, which would imply that $Q_1^e = \frac{1}{2r}\left(1 - \frac{c}{1-t_1}\right)$. However, The condition $Q_2 = \frac{1}{2r}\left(1 + s - \frac{c}{(1-t_2)}\right) < \frac{1}{2r}\left(1 - \frac{c}{1-t_1}\right)$ can hold only if $t_2 > t_1$, which we ruled out by assumption.

In addition, the subgame perfect equilibrium cannot be such that $p_2^e < 1 + s - Q_1^e r$, i.e. $Q_2 > Q_1^e$. To see this, suppose there is no switching cost, i.e. $s = 0$. Then M's profit in period 2 is independent of p_1. Furthermore, consumer demands are identical in the two periods, which implies that $p_1 = p_2$ and $Q_1 = Q_2$ in equilibrium and that p_2 must be such that the supplier's marginal revenue in period 2 equals c. Suppose now that $s > 0$. As Figure 7.2 suggests, at $Q_2 = Q_1^e$ the marginal revenue drops sharply, because the marginal consumer did not buy from M in period 1. Hence, to attract this consumer the supplier must reduce p_2 sharply. Therefore, the marginal revenue at $Q_2 > Q_1^e$ must be smaller than c, which implies that M would be better off increasing p_2 and thus reducing Q_2.

Based on the above arguments, we can restrict attention to the case where $p_2^e \in [1 - s - Q_1^e; 1 + s - Q_1^e]$. Any value of p_2 within this interval results in the same quantity Q_2, and this quantity equals Q_1^e. Therefore, it must be that the equilibrium price is at the upper bound of the interval, i.e. $p_2^e = 1 + s - Q_1^e$.

B.5 Proof of the results in Section ??

B.5.1 Equilibrium prices set by the platform

Given (8.5) and (8.6), we can express the prices set by the platform as a function of the utility levels provided to each group:

$$p_i(u_i, u_j) = \alpha_i Q_j - u_i = \alpha_i \phi_j(u_j) - u_i, \; i, j = 1, 2; j \neq i, \tag{B.4}$$

We can write the expression for the profit made by the platform as

$$\pi(u_i, u_j) = \phi_1(u_1)(p_1(u_1, u_2)(1 - t_1) - c_1) + \phi_2(u_2)(p_2(u_1, u_2)(1 - t_2) - c_2), \tag{B.5}$$

where the price is as in (B.4). Since the platform's objective only depends on the utility levels (u_1, u_2), there is no loss in proceeding as if these utility levels were the platform's decision variables. The first-order conditions of the problem are such that

$$\frac{\partial \pi}{\partial u_i} = \phi_i^{'}((\alpha_i \phi_j - u_i)(1 - t_i) - c_i) - \phi_i + \phi_i^{'} \phi_j \alpha_j = 0 \quad i, j = 1, 2 \quad i \neq j.$$

Denote the profit-maximizing utility levels as u_i^e, that satisfy the above system of equations. We find:

$$u_i^e = -\frac{c_i}{1 - t_i} - \frac{\phi_i}{\phi_i^{'}} + \left(\alpha_i + \alpha_j \frac{1 - t_j}{1 - t_i}\right) \phi_j.$$

Replacing them in (B.4), we get the equilibrium prices provided in (8.8).

B.5.2 Effects of taxation

Assume now the monopolist's problem is solved maximizing with respect to prices. Let F_i be the first-order derivative $\frac{\partial \pi}{\partial p_i}, i = 1, 2$. The equilibrium prices, p_i^e, must satisfy the system of equations $F_i = 0, i = 1, 2$. Hence, (5.2) and (5.5) hold. In this setting, we have

$$\frac{\partial F_i}{\partial t_i} = -Q_i^e \left(\frac{p_i^e}{Q_i^e} \frac{\partial Q_i}{\partial p_i} + 1\right) = \frac{-\frac{\partial Q_i}{\partial p_i}\left(c_i - Q_j^e \alpha_j (1 - t_j)\right)}{1 - t_i}, \; i, j = 1, 2; j \neq i.$$
$$\frac{\partial F_i}{\partial t_j} = -p_j^e \frac{dQ_j}{dp_i} = \frac{\phi_1' \phi_2' \alpha_j}{1 - \phi_1' \phi_2' \alpha_1 \alpha_2} p_j^e, \; i, j = 1, 2; j \neq i.$$
$$\frac{\partial F_i}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_i^2} < 0, \text{and} \quad \frac{\partial F_i}{\partial p_j} = \frac{\partial F_j}{\partial p_i} = \frac{\partial^2 \pi}{\partial p_1 \partial p_2}.$$

Consider the effect of t_1 on p_1. Given the above expressions, the direct effect characterized in expression (5.2) is negative if and only if $c_1 < Q_2^e \alpha_2 (1 - t_2)$, whereas the indirect effect is nonpositive if and only if $\alpha_1 \frac{\partial^2 \pi}{\partial p_1 \partial p_2} \leq 0$. As for the effect of t_1 on p_2, the direct effect characterized in equation (5.5) is nonpositive if and only if $\alpha_1 \leq 0$, whereas the indirect effect is nonpositive if and only if $(c_1 - Q_2^e \alpha_2 (1 - t_2)) \frac{\partial^2 \pi}{\partial p_1 \partial p_2} \leq 0$.