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A STOCHASTIC OPTIMAL CONTROL APPROACH TO INTERNATIONAL FINANCE AND FOREIGN DEBT

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Abstract

The recent financial crises, especially the debt crisis in Asia, have led to questions such as: what are their causes, what is an excessive debt and how vulnerable is an economy to external shocks? We develop an economic model of international finance and debt based upon two sources of uncertainty: the productivity of capital and the real interest rate. We use stochastic optimal control-dynamic programming to derive the: optimal consumption, foreign debt, capital, the growth of net worth and the current account. The objective is to maximize the expectation of the discounted value of the utility of consumption over an infinite horizon. Crises – and associated social unrest – occur when the unanticipated shocks produce a significant decline in the utility of consumption. We relate our optimality conditions to the vulnerability of the economy to crises.

The major conclusions are as follows. (1) We derive explicit and implementable closed form equations for the optimum debt/net worth, which maximize the expectation of the discounted value of utility over an infinite horizon. (2) The derived debt/net worth ratio also maximizes the expected growth of net worth, given any fixed consumption/net worth ratio. (3) The vulnerability of an economy to shocks is positively related to the variance of the utility of consumption at any time. We derive a risk-expected return tradeoff. When the debt exceeds the optimum, there is inefficiency. The expected growth of the utility of consumption can be increased, and the vulnerability of the economy – measured by the variance of the utility of consumption – can be decreased by decreasing the debt/net worth ratio.

Keywords: Stochastic optimal control, foreign debt, international finance, current account, vulnerability to external shocks

JEL Classification: C61, D81, D9, F34

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A Stochastic Optimal Control Approach to International Finance and Foreign Debt

The recent debt crises, especially in Asia, have led to the questions: When is the foreign debt excessive? What are early warning signs of "vulnerability"? In the years prior to the crises, the Asian countries were held up as paragons of economic development. They were characterized by outward-oriented growth, which attracted foreign investment, and macroeconomic stability. Inflation was moderate by developing country standards. In the cases of Malaysia and Thailand, the sizeable external current account deficits reflected not public sector budget deficits, but an excess of private investment over private saving. Hence high private saving and capital inflows were financing the growth of capital, which would increase the future productivity of the economy[1].

The literature on debt crises[2] had viewed the vulnerability of countries to debt crises in terms of the concepts "solvency" and "sustainability". "Solvency" was defined as a condition where the ratio of external liabilities/GDP stabilizes. The long run trade surplus that an indebted country must have to keep the ratio of external liabilities/GDP constant was used as a measure of "solvency". "Sustainability" was defined as a condition whereby the resulting trade balance will be consistent with "solvency", if current policies are continued.

Recent thinking has questioned the usefulness of these criteria of vulnerability. The limitations of the existing approach can be seen from equation[3] (1). The time

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[1] International Monetary Fund, World Economic Outlook (WEO), Crisis in Asia, Interim Assessment, December 1997, Washington, DC, ch. II.

[2] We are drawing upon several sources for a discussion of the literature: Kaminsky and Reinhart (1999), Gian Maria Milesi-Ferretti and Assaf Razin (MF-R), Current Account Sustainability, International Finance Section, Princeton Studies in International Finance, #81, October, 1996; International Monetary Fund, World Economic Outlook (WEO), Financial Crises, May, 1998, Washington, DC; the papers presented at World Bank/International Monetary Fund/World Trade Organization Conference on Capital Flows, Financial Crises and Policies, World bank, April 15-16, 1999.

[3] The derivation is as follows: $h(t)=L(t)/Y(t)$, $(1/Y(t))dY(t)/dt = g(t)$ is the growth rate, and the current account deficit $dL/dt = rL - BT$, where BT is the trade balance. $Y(t)=Y(K(t),t)$ is the

subscripts are important. Let $h(t) = L(t)/Y(t)$ be the ratio of external liabilities L (t) to the GDP denoted Y (t). The rate of change in external liabilities/GDP has three components. The first $r(t)f(t)$ is the interest payments at rate $r(t)$ on the debt/GDP. The second term is minus the growth $g(t)$ of GDP times the ratio of the debt/GDP. The third term is minus $B(t) = BT(t)/Y(t)$ the trade balance as a fraction of GDP.

The GDP is $Y(t) = Y[K(t), t]$, where $K(t)$ is capital and $dK(t)/dt$ is the rate of capital formation. The trade balance is $BT(t)$. The growth of GDP is $g(t) = [Y'(K(t),t)]\ [dK(t)/dt\ /\ Y(K(t),t)]$. The ratio $B(t) = BT(t)/Y(t)$ of the trade balance/GDP is 1 less absorption/GDP, where absorption is consumption plus investment: Then $B(t) = [1 - c(t) - i(t)]$ where $c(t)$ is the <u>ratio</u> of consumption to GDP and $i(t) = [dK(t)/dt] / Y(t)$ is the <u>ratio</u> of capital formation/GDP. Equation (1a) uses the simpler notation where $A(t) = [r(t) - g(t)]$ is the interest rate less the growth rate, and $B(t) = [1 - c(t) -i(t)]$ is the trade balance/GDP.

(1) $dh(t)/dt = [r(t) - g(t)]\, h(t) - [1 - c(t) -i(t)]$; $g(t) = Y'(K(t),t)\, i(t)$

(1a) $dh(t)/dt = A(t)\, h(t) - B(t)$

The “solvency” criterion is that the steady state debt stabilizes at a finite value $h^* < \infty$. That is, $dh(t)/dt$ converges to zero, so that $A^*h^* = B^*$, where the asterisk indicates a steady state value. This means that the resource transfer B^*- the trade balance as a fraction of GDP - equals the interest payments on the foreign debt adjusted for growth A^*h^*. The “sustainability” condition is that current policies lead to “solvency”. Current policies correspond to the $B(t)$ term, which contains the consumption ratio $c(t)$ and $i(t)$, the ratio of capital formation/GDP. In practice, $h(t)$ is compared with $B(t)/A(t)$ to see if the country’s debt is “too high” for solvency.

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production function. Hence $(1/h(t))dh(t)/dt =[(r(t) - g(t)] - BT(t)/L(t)$; $BT(t) = Y(t) - C(t) - dK(t)/dt$, where $C(t)$ is total consumption.

The inadequacies of this approach diminish its usefulness. First: It is impossible to know if the debt will stabilize unless one knows A(t) and B(t), for all future time T > t the present. Neither A(t) nor B(t) is a constant, and these values are interrelated. In developing countries in particular, trade imbalances $B(t) < 0$ are produced by the rate of capital formation as well as by consumption. The rate of capital formation influences both A(t) and B(t). Policies which affect capital formation affect the growth rate in A(t) as well as the trade balance in B(t).

Second: dynamic efficiency requires that, in the steady state, the interest rate equal to the marginal product of capital be at least as great as the growth rate. However insofar as the interest rate exceeds the growth rate, $A > 0$, and the debt will explode, given current policies B. Third: at what level should the debt be stabilized? Even if the debt stabilized at a finite $h^* = B^*/A^*$, there is no presumption that this ratio is optimal in any sense. The value of f(t) may oscillate considerably even if it remains bounded as $t \Rightarrow \infty$. The resource transfer B(t)dt during any interval of time let alone the steady state value B* may be intolerable insofar as it may require a drastic decline in consumption. It may even violate the non-negativity constraints on c(t) and i(t). Fourth: in a world of uncertainty, the future values of the interest rate r(t) and the productivity of capital Y'[K(t),t)] are unknown. One does not know at what interest rates foreign investors will be willing to continue to lend to the country. The real GDP, which is real value added, is measured in terms of some numeraire, say the GDP deflator. Exports are important parts of the value of GDP and imports are necessary to produce both export and domestic goods. The terms of trade, the ratio of export/import prices, therefore affect real value added or GDP. Hence $Y(t) = Y[K(t),t]$ is a stochastic variable, affected by the terms of trade as well as by the profitability of the investments. Formally, $A(t) = [r(t) - g(t)]$ is a stochastic variable, affected by both the world capital markets and goods markets. The evolution of A(t) is not predictable.

The recent literature has concluded that the “solvency - sustainability” approaches are of limited use. Instead, we want to know: how vulnerable is an economy to external shocks or to the "bad" events in the distribution of shocks? We derive the optimal evolution of the foreign debt in a world of uncertainty in order to maximize the expectation of the discounted value of utility of consumption over an infinite horizon. Consequently, the techniques of stochastic optimal control-dynamic programming should be used. The derived path for the current account and foreign debt when optimal policies are used should serve as a benchmark to determine whether actual current account deficits and foreign debt are "excessive". This metric can be used as a basis for evaluating whether the current policies are sustainable. The state of the art has not been able to quantify and systematize these considerations[4]. Our contribution is to quantify and formalize the "benchmark" - the optimal policies in a stochastic environment. The actual values of the debt and current account are then compared to the benchmark values derived from a stochastic optimal control solution. The deviation from the optimal values "vulnerability" to shocks.

Part 1 is a description of the model in continuous time over an infinite horizon, where the country can be a permanent debtor or creditor. The object is to maximize the expected discounted sum of utility of consumption over an infinite horizon, where both borrowing and lending are possible. The constraint that net worth must be non-negative ensures that there can be no Ponzi schemes. There are two sources of uncertainty. The productivity of capital and the interest rate are both stochastic and may be correlated positively or negatively.

The prototype model in this paper makes several simplifying assumptions, which facilitate the solution. In subsequent papers, (a) We analyze a model in discrete time with a finite horizon. At the end of the final stage, the debt must be repaid. (b) We approach the debt problem in terms of a deterministic differential game instead of the stochastic approach used here. (c) We relax the simplifying

[4] Milesi-Ferretti and Razin (1996;7), whose work represents the state of the art, wrote: "Although we can thus incorporate a broader set of theoretical considerations than can be accommodated in a structural approach using state-of-the-art equilibrium models, we lose our ability to quantify our analysis".

assumption that the levels of capital and debt can be adjusted instantaneously and costlessly.

Part 2 is the solution of the continuous time model for the optimal consumption, foreign debt, capital, the growth rate of net worth and the expected current account/net worth[5].

Part 3 uses the results of part 2 to derive the trade off between the expected utility of consumption and the variance of the utility of consumption. We call this the "expected return-risk trade-off". We prove that if the debt/net worth f exceeds the optimal debt/net worth f*, there is inefficiency. By reducing the debt to f*, the expected utility of consumption can be raised and the variance of the utility of consumption can be reduced. This is the key result of this paper, as shown in part 5 concerning crises.

In part 4 current account deficits under optimal investment and consumption policies are considered. Many economists view current account deficits as signs of vulnerability. We prove that in appropriate ranges for the parameters in the model, permanent expected current account deficits/net worth are optimal. Only when the actual current account deficit exceeds the optimal level is there a warning sign of vulnerability.

Part 5 summarizes the results of this paper by explaining how crises can occur when non-optimal policies are followed. This section relates our theoretical-mathematical paper to the empirical literature on the debt and crises. We view a crisis as a situation where shocks arising from interest rates or the productivity of capital decrease the utility of consumption significantly. Social unrest occurs.

Thus the greater the variance of the utility of consumption, the greater is the vulnerability of the economy to external shocks. In part 3, we proved that when the debt exceeds our derived optimal level then: by reducing the debt to the optimal level, the expected utility of consumption can be raised and its variance can be reduced. A Reduction in the variance (risk) corresponds to a reduction in

[5] Mathematical details underlying the derivations in the text are in Appendix A.

the vulnerability of an economy to unfavorable shocks in interest rates or productivity of capital.

1. A Continuous Time Prototype Model

Our prototype model is a simplification of a complex economy that focuses upon the disturbances which have produced crises and is analytically tractable. There is a clear economic interpretation of the derived equations. As more realistic assumptions are introduced, both the solution and economic interpretation become less transparent. The prototype model is proposed as a "benchmark" model. We indicate, either in the text or in footnotes, the specific aspects of the prototype model which are simplifications. We consider two sources of uncertainty. One source concerns the value of GDP and the return on capital. The second concerns the interest rate on loans. It is important and realistic to stress that there is a correlation of these two sources of uncertainty.

The model is in real terms and is formulated in terms of the stochastic calculus. Equation (2) defines net worth (wealth) X as capital K owned by the residents of the nation less international debt L, and (2a) is the change in net worth. A negative L represents foreign assets. Equation (3) states that the change in capital dK is the rate of investment per unit of time I(t) times the length of the period dt.

(2) $X = K - L$ (2a) $dX = dK - dL$.

(3) $dK = I(t)\, dt$

The change in the debt dL is equation (4). Bold letters indicate a stochastic variable. The first term is the interest on the existing debt **r**L at interest rate **r**(t). We assume that the debt is long term but is financed at a variable interest rate[6]. The next two terms are absorption less GDP. Absorption is consumption plus investment: [C(t) + I(t)]dt. The last term is minus the GDP accruing to the residents of the country, denoted **Y**(t) dt.

(4) $dL = \mathbf{r}L\, dt + [C(t) + I(t)]dt - \mathbf{Y}(t)\, dt$.

[6] For example, two Argentine long term bond issues in late 1997 and early 1998 were financed with coupons that reset based upon investor bids in auctions at each rest date. The term L(t) represents net liabilities to foreigners, or net claims if L(t) is negative.

This equation can also be expressed in terms of consumption, equation (4a). The first terms in brackets [$\mathbf{Y}(t)dt$ - $\mathbf{r}L(t)dt$] is GNP, the second term is new debt dL and the third term is capital formation I(t)dt. The GNP, as used in this prototype model, is the real market value of the output of final goods and services produced by the labor and property supplied by the residents of the nation. The GDP plus (minus) income receipts (payments) from (to) abroad is GNP.

The stochastic nature of GNP, the first term in brackets, is the ultimate source for stochastic variations in consumption.

(4a) $C(t)dt = [\mathbf{Y}(t)dt - \mathbf{r}L(t)\,dt] + dL - I(t)dt$.

The stochastic variables affect GDP and the interest rate. Equation (5) is the interest rate on the debt. The first component is the expectation of the interest rate r dt. The second component describes the uncertainty: It is $\sigma_1\, dw_1$ which involves Brownian motion $w_1(t)$. The Brownian motion term is

$dw_1 = \varepsilon\sqrt{dt}$, where $E(\varepsilon) = 0$ and $E(\varepsilon^2) = 1$.

(5) $\mathbf{r}\,dt = r\,dt + \sigma_1\, dw_1$ $\qquad\qquad$ $\mathbf{r}\,dt \sim N(r\,dt, \sigma_1^2\, dt)$

This is a general equation which describes the uncertainty arising from the international financial markets. The interest rate $\mathbf{r= r_1 + (r_2 - r_1) + (r - r_2)}$ has three components. The first component $\mathbf{r}_1$ is the US real long term government bond yield. The second component ($\mathbf{r_2 - r_1}$) is the country risk premium charged to foreign firms or countries who borrow US dollars. The third ($\mathbf{r - r_2}$) is the currency risk premium equal to the expected depreciation of the country's currency. The interest rate r charged on loans in the currency of the country is the sum of the three components. Each component is a stochastic variable. The interest costs on the debt $\mathbf{r}$Ldt are distributed normally with a mean of rL dt and a variance of $(\sigma_1 L\, dw_1)^2 = \sigma^2{}_1 L^2\, dt$.

The real value of GDP accruing to residents of the country is $\mathbf{Y}(t)$ dt described by equations (6a), (6b) which imply (6). It is in the spirit of the "endogenous technical change" models[7]. In the endogenous technical change approach, expected output per unit of capital is b. The real value added or real GDP is a stochastic variable. The real GDP per unit of capital varies for several reasons: (a) variations in the rate of capacity utilization due to the business cycle, (b) the terms of trade - the ratio of export/import prices – vary, (c) good or bad harvests, (d) unexpectedly bad/good returns on investments. The distribution of Y(t)dt has a mean bKdt and a variance of $(\sigma_2 K)^2$ dt.

(6a) $\mathbf{Y}(t) = \mathbf{b}(t)K(t)$

(6b) $\mathbf{b}\ dt \sim N(bdt, \sigma_2^2 dt)$

(6) $\mathbf{Y}(t)\ dt = bK\ dt + \sigma_2 K\ dw_2$. $\quad\quad \mathbf{Y}(t)\ dt \sim N(bK\ dt, (\sigma_2 K)^2\ dt)$,

where $w_2(t)$ is another Brownian motion.

An estimate[8] of the US productivity of capital **b**(t) is graphed in appendix B over the period 1959:1 - 1997:2. It is labeled OUTINV. The Brownian motion assumption in (6b) is an oversimplification because there is a mean reversion tendency, and the output/capital **b**(t) is not normally distributed.

The two stochastic terms dw_1, dw_2, may be interrelated. We consider the general case, equation (7), where the correlation coefficient ρ could be positive, zero or negative, which varies among countries. In our subsequent analysis, we assume that in the advanced countries the correlation ρ is positive, and it is negative in the emerging market countries.

(7) $E(dw_1\ dw_2) = E(\varepsilon_1\ \varepsilon_2)dt = \rho\ dt$.

Equations (2) - (7) describe the underlying model. Substitute equations (2) (3) (5) and (6) into equation (4). On the basis of these equations, we obtain equation

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[7] In our prototype model total GDP is derived from a Leontief production function: Y* = min [bK*(t), a(t)N(t)], where N(t) is labor or materials. We assume that capital is the binding constraint. Y*(t) is total GDP and K*(t) is total capital. The capital owned by residents of the country is K(t) and by foreigners it is $K_f(t) = [K^*(t) – K(t)]$. Hence total GDP, denoted by Y*(t) has two parts: bK(t) accrues to residents of the country and b[K*(t) – K(t)] accrues to foreigners. We define GDP accruing to residents as $Y(t) = Y^*(t) – bK_f(t) = bK(t)$. Coefficient **b** is described in equation (6b), and the increase in a(t) reflects the growth in labor productivity.

[8] Measure b = (dy/dk)/(dk/dt) = g(t)/i(t), where g(t) = (1/y)dy/dt is the growth rate, and i(t) = (dk/dt)/y(t) is the investment ratio. The productivity of capital **b** has the following characteristics.

(8) for our state variable $X(t)$ which is wealth or net worth. It states that the change in net worth $dX(t)$ is GNP = $[Y(t) - r(t)L(t)]dt$ less consumption $C(t)\,dt$ equal to saving $S(t)dt$.

(8) $dX(t) = \mathbf{b}K(t)\,dt - \mathbf{r}L(t)\,dt - C(t)\,dt = \mathbf{S}(t)\,dt$

There are several constraints. First: consumption is positive, or at least non-negative. Second, net worth must be non-negative. The constraint $X(t) > 0$ prevents "Ponzi schemes" where borrowing finances both consumption and interest on the existing debt. If a Ponzi scheme were followed, then debt rises relative to capital and $X = K - L$ will become zero and then negative in finite time. If the net worth is rationally expected to become negative (i.e, the country follows policies that will lead to bankruptcy) the creditors will sell their debt and the interest rates will not be described by (5). Since net worth $X = K - L \geq 0$, this implies that capital $K \geq L$. The capital must be non-negative $K(t) > 0$, but the debt $L(t)$ can be positive, zero or negative. A negative debt implies a creditor position in the international financial markets.

To formulate a stochastic control problem associated with the model, we must specify state and control variables, the dynamics of the state process and the criterion to be optimized.

The controls are consumption $C(t)$ and the debt $L(t) = K(t) - X(t)$. Wealth $X(t)$ is capital $K(t)$ less debt $L(t)$, or capital $K(t) = X(t) + L(t)$ is wealth plus the foreign debt. A simplifying assumption is that the level of the debt can be achieved instantaneously and costlessly[9]. The control on the debt is therefore the same as the control of capital, given the state $X(t)$. If the country has "too much" debt, it sells capital and repays some debt. If the country has "too little" capital, it either sells debt or reduces net foreign assets, and uses the proceeds to acquire capital.

mean 15.84%, standard deviation 15.17%. It is negatively correlated (-0.37) with the unemployment rate and is positively correlated (0.24) with the real long term rate of interest.

[9] It is more realistic to assume that there are "transactions costs" in varying debt (capital). This assumption changes the dimension of the dynamic system. See Fleming (1998, part 2.4), Constantinedes, Bielecki and Pliska for the use of transactions costs in finance models.

Thus one could equivalently take C(t) and K(t) as controls. We impose the state constraint $X(t) > 0$ and the control constraints $C(t) > 0$, $K(t) \geq 0$.

Equation (8) is the change in the state variable net worth $X(t) = K(t) - L(t)$. It is saving equal to GNP less consumption. Equation (i) is the change in the state variable X(t) in terms of control variables[10] consumption C(t) and the debt L(t), and (ii) is in terms of control variables consumption and capital $K(t) = X(t) + L(t)$. Both control variables are adjusted instantaneously and costlessly[11], and the state variable moves differentially.

(i) $dX(t) = \mathbf{b}[X(t) + L(t)]dt - \mathbf{r}L(t)dt - C(t)dt$

$\qquad = \mathbf{r}X(t)dt - (\mathbf{b} - \mathbf{r})[X(t) + L(t)]dt - C(t)dt$

(ii) $dX(t) = \mathbf{b}K(t)dt - \mathbf{r}[K(t) - X(t)]dt - C(t)dt$

Using the stochastic equations for the productivity of capital and interest rate (eqns. 5,6) in (i) we derive stochastic differential equation (8a) for the change in net worth in terms of control variables C(t) and L(t).

stochastic differential equation for net worth

(8a) $dX(t) = [rX(t) + (b-r)(X(t)+L(t)) - C(t)]\, dt - L(t)\sigma_1 dw_1 + (X(t)+L(t))\sigma_2 dw_2$.

This is a general description of the dynamics of net worth X(t). The first set of terms in brackets is expected saving, equal to expected GNP less consumption, where expected GNP is $E[Y(t) - rL]\, dt = [bK(t) - rL(t)]\, dt$. The second and third sets of terms concern the stochastic components of GNP: the stochastic component of the productivity of capital $K(t)\sigma_2\, dw_2$ and the stochastic component of interest rates $L(t)\sigma_1 dw_1$. Our model of an economy is a stochastic system, so that there are many paths that the state variable X(t) can take given the controls Γ and the initial data. The stochastic disturbances have been underemphasized in the studies of optimal control because the authors generally use deterministic models, which assume unique (saddle point) paths to the optimal steady state[12].

[10] Saving S(t) is not a control variable since it depends upon the stochastic GNP.
[11] This assumption is modified in our subsequent papers.
[12] Merton's analysis (1990) is based upon stochastic optimal control, and marked a change from the deterministic control approach which used the Maximum Principle of Pontryagin. The article by Infante and Stein (1973) on optimal growth considered a deterministic model where there was

Optimal stochastic control theory attempts to deal with models - such as the one described above - in which random disturbances are important.

only qualitative but not quantitative knowledge of the parameters. Using dynamic programming, they derived feedback control laws - just based upon current observations - which would drive the economy to the optimal trajectory that would exist if there were perfect knowledge. Their feedback control laws are robust to perturbations, whereas the open loopcontrols based upon the Maximum Principle lead to saddle point instability.

2. Derivation of Optimal Consumption, Capital, Debt and the Growth of Net Worth in Continuous Time: over an Infinite Horizon: Prototype Model

Our aim is to establish a "benchmark" for optimal foreign debt, capital and consumption in the prototype stochastic growth model described in part 1. There are many criteria of optimality. In this part we use the expected present value of a HARA utility function as the criterion of optimality. The use of HARA, or the logarithmic function which is a special case of HARA, reduces the dimension of the problem and allows us to solve the model analytically.

We use the <u>dynamic programming</u> method. Equation (9) is our value function. The initial value of wealth $X = X(0)$. The discount rate is δ. Relative risk aversion[13] is $(1 - \gamma)$. The logarithmic utility function is derived when $\gamma = 0$. The optimization is over an infinite horizon. The expectations are taken over the dw_i, $i = 1,2$ where w_i is Brownian motion. The maximum is taken over a set Γ of admissible controls. The controls $C(t)$, $L(t)$ admitted must be such that the state and control constraints are satisfied

<u>Constraints</u>: $X(t) > 0$, $C(t) > 0$, $K(t) = X(t) + L(t) \geq 0$.

Moreover, $C(t)$ and $L(t)$ cannot anticipate future changes[14] in $w_1(s)$, $w_2(s)$ for time $s > t$.

(9) $V(X) = \max_{\Gamma} E \{ \int_0^{\infty} (1/\gamma)\, C(t)^{\gamma} e^{-\delta t}\, dt \} \quad \gamma < 1$ subject to eqn. (8).

The HARA utility function implies that the value function $V(X)$ is homogeneous of degree γ and is equation (10). The proof is as follows. If the state X, and controls C and L are multiplied by a value $\lambda > 0$, then dynamic equation (8) is satisfied. The new value function $V(\lambda X)$ is:

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[13] Relative risk aversion $RRA = - d(\ln U') / d(\ln C) = (- U''/ U')\, C$. When $U(C(t)) = (1/\gamma)C^{\gamma}(t)$, then $RRA = (1-\gamma)$.

[14] For a more precise mathematical description, see Fleming and Rishel (1975, ch. 6) or Fleming and Soner (1992, ch. 3-4).

$$V(\lambda X) = \max_{\Gamma} E \{ \int_0^\infty (1/\gamma) [\lambda C(t)]^\gamma e^{-\delta t} dt \} = \lambda^\gamma V(X).$$

The value function of X is also homogeneous of degree γ. Hence we may write the value function as (9a) where constant $A > 0$ is to be determined. The first two derivatives are (9b) and (9c).

(9a) $V(X) = (A/\gamma) X^\gamma$

(9b) $V_x = A X^{(\gamma-1)}$

(9c) $V_{xx} = A (\gamma-1) X^{(\gamma-2)}$

From equations (8a) and (9), the derived Bellman stochastic dynamic programming (DP) equation is (10a). The expectations have been taken into account in its derivation. In appendix A, we (a) give the conditions on the model parameters such that equation (10) has a solution with $A > 0$, and (b) sketch a formal derivation of equation (10).

$$\text{(10a)}\quad \delta V(X) = \text{Max}_{C,L} \{(1/\gamma)C^\gamma + V_x[(b-r)(X+L) + rX - C] + (V_{xx}/2) [(L^2 \sigma_1^2) + (X+L)^2\sigma_2^2 - 2L(X+L) \rho\sigma_1\sigma_2] \}$$

It is convenient to measure the variables: consumption, capital and debt as fractions of net worth: $C/X = c$, $L/X = f$, $k = K/X$, where lower case letters refer to the ratios. Instead of C and L, we can equivalently take c and f as the control variables. The control constraints are then

$c > 0$, $f = k - 1 \geq -1$, $(k = K/X)$

Use equations (9a) - (9c) in (10a) and derive equation (10) as the dynamic programming equation.

<u>DP equation for prototype model</u>

$$\text{(10)}\quad \delta/\gamma = b + \max_c \; [(1/\gamma)c^\gamma/A - c] + \max_f [(b-r)f + (\gamma-1)/2 \; (f^2 \sigma_1^2) + (\gamma-1)/2 \; (1+f)^2 \sigma_2^2 - (\gamma-1)(1+f)f \rho\sigma_1\sigma_2]$$

where: $c = C/X$, $f = L/X$.

In equation (9a), we must have $A > 0$. In case $\gamma > 0$, there must be a restriction that the discount factor δ is not "too small"[15]. The restriction is required for $V(X)$ to be finite. In the case where $\gamma = 0$, relative risk aversion is unity, the restriction is satisfied and $A > 0$. In this paper, we often consider a logarithmic utility function, $\gamma = 0$ and then the restriction $A > 0$ is satisfied.

In part 2.1, we derive the optimal consumption, capital and debt ratios. In part 2.2, we derive the growth of wealth.

2.1. Optimal Consumption, Capital and Debt

Our continuous time - infinite horizon model is quite similar to the Merton model, with different emphases. In Merton's model, there is a safe asset with a fixed return r and a risky asset. The only source of income is the interest payments on the portfolio. The returns on the two assets are exogenous[16]. There is no human income. In our model: (a) there is growth in the GDP resulting from capital formation, which is financed by domestic saving and foreign borrowing. The current account deficit is net foreign borrowing. (b) There is no safe asset. Both the productivity of capital and the interest rate are stochastic variables. (c) The correlation between the productivity of capital and the interest rate vary by type of country. (d) Capital and wealth are constrained to be non-negative. The latter ensures that capital $k(t)$ exceeds debt $f(t)$, and prevents Ponzi schemes.

The <u>optimal consumption/net worth</u> ratio c^* in equation (11) is determined by taking the maximum over c in equation (10), where $A > 0$ is a constant determined by equation (A1) in appendix A when $\gamma \neq 0$. Similarly, the <u>optimal debt/net worth</u> ratio f^* is obtained by taking the maximum over f in equation (10), constrained by $f \geq -1$. If $f^* > -1$, it is given by equation (12), simplified as (12a).

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[15] See appendix A.

[16] Fleming and Zariphopoulou (1991) introduced three securities: a risky asset, a safe asset and a debt instrument with a fixed interest rate. The expected return on the risky asset exceeds the fixed interest rate on loans which exceeds the fixed interest rate on the safe asset. There will not be simultaneous investment in the safe asset and borrowing, since the borrowing rate exceeds the interest rate on the safe asset.

Otherwise, f* = -1, endpoint maximum. Optimal capital/net worth k* is (13) simplified as (13a), provided that k* = 1 + f* > 0.

(11) $c^* = C(t)/X(t) = A^{1/(\gamma-1)}$

(12) $f^* = L(t)/X(t) = [(b-r)/(1-\gamma)\, \sigma_2^2(1 + \theta^2 - 2\rho\theta)] - [(1 - \theta\rho)\, \}/(1 + \theta^2 - 2\rho\theta)]$
if $f^* > -1$

(13) $k^* = [(b-r)/(1-\gamma)\, \sigma_2^2(1 + \theta^2 - 2\rho\theta)] + [\theta\,(\theta-\rho)\, \}/\,(1 + \theta^2 - 2\rho\theta)\,]$,
if $k^* = 1 + f^* > 0$.

Note that c*, f* and k* are constants. They do not vary with time t and do not depend on the initial wealth X.

In the case of $\gamma = 0$, the logarithmic utility function[17], the value of A is $(1/\delta)$ the reciprocal of the discount rate. In that case, the optimal consumption/net worth is equation (11a).

(11) $c^* = C(t)/X(t) = \delta$ when $\gamma = 0$

Optimal consumption is a fixed proportion of net worth, where the factor of proportionality is the discount rate or time preference[18]. The economic determinants of the optimal debt ratio f*(t) in (12), and optimal capital/net worth in (13), are understood by writing them as (12a) and (13a) respectively which contain three crucial terms: k_m, λ and $\rho\theta$. See Box 1.

Equations (12a) and (13a) are graphed in figure 1 for the case when the correlation $\rho < 0$. We relate equations (12) and (13) to both the Merton model and to the literature of the economics of futures markets.

[17] See appendix A.

[18] This is the consumption function used in the NATREX dynamic model of the real exchange rate and international debt. This consumption function guarantees that the debt will converge output , denoted by y, and is our intertemporal budget constraint. See Stein in Stein, Allen et al (1995).

BOX 1 Summary of optimal controls and equilibrium

(11a) $c^* = C(t)/X(t) = \delta$ when $\gamma = 0$

(12a) $f^* = \lambda\, k_m + \lambda\, (\rho\theta - 1)$

(13a) $k^* = \lambda k_m + \lambda\theta(\theta - \rho) \geq 0$

Equations (12a), (13a) hold if $k_m + \theta(\theta - \rho) \geq 0$. Otherwise $f^* = -1$, $k^* = 0$.

Merton point: $k_m = (b-r)/(1-\gamma)\sigma_2^2$ Parameters: $\theta = \sigma_1/\sigma_2$ = standard deviation of interest rate/ standard deviation productivity of capital; ρ = correlation between interest rate and productivity of capital;

$\sigma_2^2 = \text{var } \mathbf{b}$; $\rho = \sigma_{12}/\sigma_1\sigma_2$, $\text{var } (\mathbf{b-r}) = \sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$

$\lambda = (\sigma_2^2/\sigma^2) = 1/(1 + \theta^2 - 2\rho\theta)$; $\mathbf{r} \sim N(r, \sigma_1^2)$, $\sigma_1^2 = \text{var } \mathbf{r}$; $\mathbf{b} \sim N(b, \sigma_2^2)$,

In Merton's model, there is a safe asset whose interest rate is constant, and a risky asset. The uncertainty concerns the return on the risky asset. There is no debt. Term $k_m = (b-r)/(1-\gamma)\sigma_2^2$ is Merton's solution for the ratio of risky assets to net worth, when the interest rate on the safe asset is constant In his model, $k_m \geq 0$. Refer to k_m as the "Merton value" of the ratio of capital/net worth. The value of k_m is the expected value of the difference between the productivity of capital and the interest rate (b-r), divided by the variance (σ_2^2) of the productivity of capital times relative risk aversion (1-γ). If $\gamma < 1$, then the ratio of risky assets/net worth rises with the expected net return (b – r).

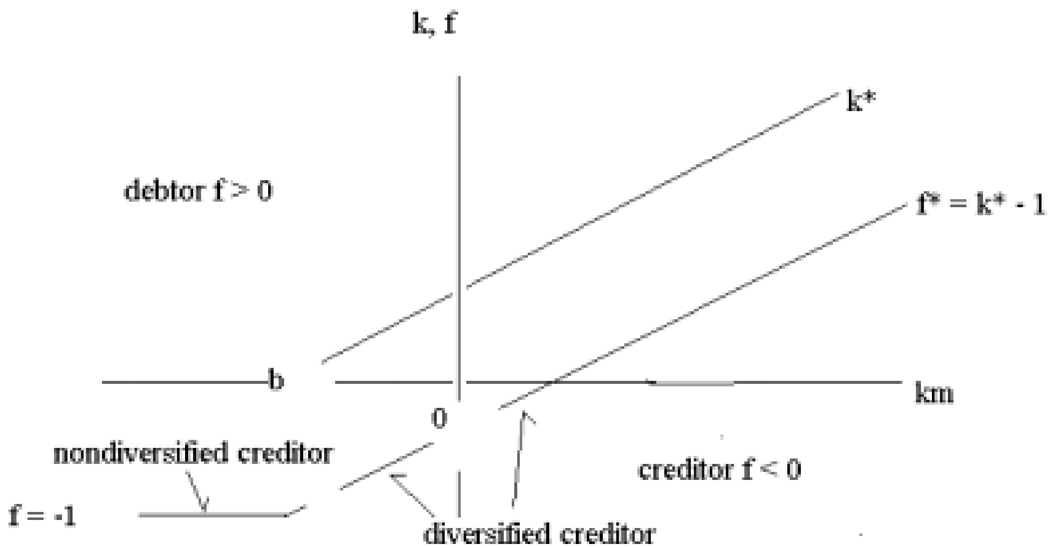

Figure 1. Optimal capital/net worth k*, optimal debt/net worth f*. Merton point km

If the investor is risk neutral $\gamma = 1$, then there is a “bang-bang” solution. For positive expected net return the ratio $k_m = \infty$; otherwise, it is zero.

In our model, there is borrowing as well as lending at an uncertain interest rate. Borrowing can finance either consumption or investment. Here[19] will concentrate upon the financing of capital by debt.

Borrowing to finance capital involves an asset and a liability. The net return is (**b-r**), the difference between the productivity of capital and the real rate of interest. Each component is a stochastic variable. Brownian motion terms dw_1 and dw_2 in the interest rate and the productivity of capital affect the variance of the net return. The two shocks dw_1 and dw_2 may be correlated positively or negatively, or may be independent of each other.

Whereas the k_m term stresses the <u>expectations</u>, the other two terms λ and $(\rho\theta - 1)$ stress the magnitude of, and correlation, between the two shocks which involve <u>deviations</u> from the respective means. The values of these terms are related to the question: will the financing of capital by debt lower or raise the riskiness of the net return? This is similar to the issue in futures markets[20]: Given a position in the spot market, will the sale of a futures contract lower or raise the riskiness of the total position[21]?

We now explain the economic determinants of the slope λ of the optimal capital/net worth and debt/net worth functions. In equation (12) or (12a), a unit change in the Merton point k_m changes both the optimum ratio of debt/net worth and capital/net worth by λ. Term $\lambda = 1/(1 + \theta^2 - 2\rho\theta) = \text{var}(\mathbf{b}) / \text{var}(\mathbf{b- r}) = \sigma_2^2/\sigma^2$ is the ratio of the variance of the productivity of capital ($\sigma_2^2 = \text{var}\ \mathbf{b}$) to the variance of the net return, (denoted by var ($\mathbf{b\text{-}r}$) = σ^2 without any subscript).

If the correlation $\rho > \theta/2$, parameter λ exceeds unity and risky borrowing reduces the riskiness of the net return. In that case, <u>borrowing</u> at an uncertain interest rate is a <u>hedging</u> of the risk of investing in risky capital, just as the sale of a futures contract against some fraction of the spot position reduces the riskiness of the total position. When $\rho > \theta/2$ which implies that $\lambda > 1$, a unit increase in the Merton point k_m induces a greater than unit rise in f* the optimal ratio of debt/net worth, and in k capital/net worth. If $\rho < \theta/2$, then the financing of capital by borrowing at a risky rate increases the riskiness of the total position. Then $\lambda < 1$, debt increases risk and a unit rise in the Merton point induces a smaller rise in both f* and k*.

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[19] In section 4, we discuss borrowing to finance consumption. The NATREX model cited above shows the great difference in the trajectory of the real exchange rate when borrowing finances consumption rather than capital formation.

[20] See Stein (1986, ch.2) for an analysis of hedging of risk in futures markets.

[21] In the economics of futures market, the firm decides to produce quantity s and must decide on the quantity of futures to be sold. The price of output p is uncertain, with a variance called var p. The price of the commodity specified in the futures contract is also uncertain. (The production and hedging decisions are simultaneously determined). If there were no sales of futures, the variance of profits is: s^2 var p. The ratio y of the variance of profits on the total spot and futures position/variance of the price of the as a function of the short or long position in futures, denoted x, is a parabola with a minimum at x = s, where s is total output. The correlation between the price of output p relevant to the firm and the price of the commodity specified in the futures contract is r. At the minimum risk point, the variance of the profits is $(1\text{-}r^2)\ s^2$var p. Hence the sale of debt to finance capital corresponds to the sale of a futures contract. The hedging substitutes a basis risk,

When the Merton point $k_m = 0$, such that the expected net return b-r = 0, is it optimal to have some risky assets and be a debtor? This is equivalent to asking if the intercept term in the debt equation (12a) is positive. In part 3, we explain the economic meaning of the intercept term and relate it to the efficient frontier between the expected growth of consumption (expected "return") and the variance of the growth of consumption (risk). We prove that when the ratio of debt/net worth is equal to the intercept term $\lambda(\rho\theta - 1)$, the variance of logarithm of net worth and of consumption are minimized.

The maximization with respect to debt f does not contain the discount rate. That is, f* is independent of the discount rate[22]. The discount rate determines solely the optimal consumption ratio.

We summarize the results so far as follows. When there is uncertainty-risk aversion, and a correlation between the return on capital and the interest rate, the situation is described in figure 1. There are several noteworthy regions. These conditions refer to the values of capital and debt when they are at their optimal values described by (12a) and (13a).

(a) The country will be a net debtor for Merton points $k_m > (1 - \rho\theta) = 0a$.

(b) The country will be a diversified creditor, holding both international debt and risky capital, for Merton points $(1-\rho\theta) > k_m > (1-\rho\theta) - 1/\lambda = \theta(\rho-\theta) = 0b$.

(c) The country will not hold risky capital when the Merton point $k_m < \theta(\rho-\theta)$.

We have taken the rate of interest[23] and the productivity of capital to be stochastic variables with a correlation ρ which is positive zero or negative.

2.2 The optimal growth of net worth

the spread between the futures and spot price, for the total risk of variations in the spot price. See Stein (1976: 35-36).

[22] The independence of the optimal debt and the discount rate does not occur in all models. In particular, if the debt must be repaid at a terminal date then the discount rate does affect the optimal debt. The independence issue is examined in our subsequent papers.

[23] All countries cannot be creditors or debtors. The world rate of interest equilibrates the supply of and demand for debt such that there will be debtor and creditor countries. We have omitted the analysis of the determination of the world rate of interest, just to save space.

We have just seen that the controls c^* and f^* which optimize the expected discounted HARA utility criterion (9) are constants. Since $k^* = 1 + f^*$, equivalently c^* and k^* are optimal controls. Let us now show that f^* also optimizes the growth of net worth, among all controls such that capital/net worth and debt/net worth are constant: $c(t) = C(t)/X(t) = c$, $f(t) = L(t)/X(t) = f$ for all t. This is proposition 1. We also verify in proposition 2 that bankruptcy cannot occur. We continue to work with the case where $\gamma = 0$, the logarithmic utility function.

In the literature, a sustained current account deficit is often viewed as a cause for alarm. The current account deficit is the rate of change of the debt $dL(t)/dt$. The optimal debt $L(t)$ is a constant multiple f of net worth $X(t)$. Therefore the current account deficit as a proportion of net worth will be $[dL(t)/dt]/X(t) = f[dX(t)/dt]/X(t)$. We therefore derive the growth of net worth, when constant policies are followed. The result is used in section 4 below to derive the expected optimal current account deficit.

PROPOSITION 1: For any ratio c of consumption/net worth, $c > 0$, the ratio of capital/net worth (debt/net worth) which maximizes the expected growth rate of net worth is the same as the optimal ratio of capital/net worth (debt/net worth) in equation (12a) or (13a) which maximizes the expected discounted value of utility. Both are independent of the discount rate.

proof: Let the consumption/net worth ratio $c = C(t)/X(t)$, $c > 0$ and capital/net worth ratio $k = K(t)/X(t) = 1 + f$ be constant. The growth of net worth equation (8b) is written as (14) abbreviated as (14a), where no optimality conditions are imposed.

(14) $dX = [(r - c) + (b-r)k]X\,dt + (1-k)X\sigma_1 dw_1 + k\sigma_2 X dw_2.$

(14a) $dX = AX\,dt + B_1 X dw_1 + B_2 X\,dw_2.$

$A = [\,(r - c) + (b-r)k\,] = [\,(r - c) + (b-r)(1+f)\,];$

$B_1 = (1-k)\sigma_1 = -f\sigma_1 \quad B_2 = k\sigma_2 = (1+f)\sigma_2$

Using the stochastic calculus, equation (14) or (14a) implies equation (15). The expected growth of net worth over an interval dt is E(d ln X) = g dt defined in (16b)-(16c).

(15) $d \ln X = g\, dt + (B_1\, dw_1 + B_2\, dw_2)$

(16a) $E(d \ln X) = g\, dt$ where

(16b) $g = (r - c) + G(k) = (r - c) + G(1+f)$,

(16c) $G(k) = (b-r)k - (\sigma_2^2/2)[(1-k)^2\theta^2 + k^2 + 2k(1-k)\rho\theta]$

$G(1+f) = (b-r)(1+f) - (\sigma_2^2/2)[f^2\theta^2 + (1+f)^2 - 2f(1+f)\,\rho\theta]$

The expected growth rate of net worth E(d ln X) = g dt in (15)(16a-b) is the sum of two terms. The first is (r-c) the interest rate less the constant consumption ratio, and the second is G(k) = G(1+f) a concave function[24] of constant debt/net worth or capital/net worth.

The capital/net worth (debt/net worth) which maximizes the expected growth rate g in (16b), is k**= 1+f** in equation (17a) or (17b).

$$k^{**} = \text{argmax}_{k}\ [(r-c) + G(k)].$$

(17a) $k^{**} = \lambda k_m + \lambda\theta(\theta-\rho)$

$= [(b-r)/(1-\gamma)\ \sigma_2^2(1 + \theta^2 - 2\rho\theta)] + [\theta\ (\theta-\rho)\ \}/\ (1 + \theta^2 - 2\rho\theta)\ ] \geq 0$

$$f^{**} = \text{argmax}_{f}\ [(r-c) + G(1+f)]$$

(17b) $f^{**} = k^{**} - 1 = \lambda k_m + \lambda(\rho\theta - 1)$

The values of f** and k** = 1+f** are exactly the optimal capital/net worth k* and debt/net worth f* derived in (13a) (12a). Hence, the optimal capital or debt which maximizes the expected discounted value of utility also maximizes g, the expected growth rate of net worth, given a constant consumption ratio. **QED**

2.3 Avoidance of Bankruptcy

[24] We always assume that there is a positive variance of the returns on capital and on the net return.

The solvability criterion used in the debt literature discussed in the introduction to our paper can be generalized as a criterion whereby the debt will not lead to bankruptcy. Solvability is then a condition where the net worth $X(t) = K(t) – L(t)$ does not becomes negative.

The general equation for the change in net worth is

$dX(t) = \{b[X(t) + L(t)]dt – r L(t)dt\}- C(t)dt + \{\sigma_2[X(t) + L(t)]dw_2 – \sigma_1 L(t)dw_1\}$.

The first part in braces is the deterministic GNP, based upon the means of the productivity of capital and interest rate. In the “inter-temporal optimization models, consumption smoothing is attempted. The constraint is that the present value of consumption is equal to the expected present value of exogenous income. This approach neglects the second terms in braces, which are Brownian motion terms. The variance of b dt is $\sigma_2^2 dt$ and the variance of r dt is $\sigma_1^2 dt$. Over an infinite horizon the time integral of these variances are infinite. Hence if C(t) is selected on the basis of expectations, there is no feedback control mechanism to prevent bankruptcy $X(t) < 0$ at some date $t > 0$.

Proposition 2 states that if consumption at any time is $C(t) = cX(t)$ a constant fraction $c > 0$ of net worth X(t) at any time, and the debt $L(t) = fX(t)$ is also a constant fraction f (positive or negative) of current net worth, then bankruptcy cannot occur. This feedback control mechanism does not require that $f = f^*$ and $c = c^*$ - that the policies be optimal.

PROPOSITION 2. Given an initial positive net worth $X(0) > 0$, if c and f are constant, the net worth at any subsequent time can never be negative, bankruptcy can never occur regardless of the shocks to the productivity of capital or interest rate.

proof:

The solution of (15) is (18) or (19). Insofar as the initial net worth $X(0) > 0$, the net worth X(t) at any time $t > 0$ will be positive, because each exponential is non-negative.

(18) $\ln X(t)/X(0) = g\, t + [B_1 w_1(t) + B_2 w_2(t)\ ]$

(19) $X(t) = X(0)\ [\exp\,(gt)]\{\exp\,[B_1 w_1(t) + B_2 w_2(t)]\}\ > 0$, for $X(0) > 0$

Given an initial positive net worth, if c and f are constant, the net worth at any subsequent time can never be negative, bankruptcy can never occur regardless of the shocks to the productivity of capital or interest rate. Any shock that affects wealth will lead to an immediate proportionate adjustment of consumption, capital and debt that will preclude net worth from becoming negative. **QED**

3. Vulnerability, Risk-Expected Return Tradeoff and Optimality

An economy is vulnerable to shocks if the shocks to the productivity of capital and interest rate are likely to force consumption to decline below a tolerable level. Social unrest then occurs. First, we give an intuitive description of the risk-expected return trade-off. Second: in section 3.1 we define the expected return and risk per unit of time, for a logarithmic utility function ($\gamma = 0$). As in section 2.2, we continue to consider the controls such that $c(t) = C(t)/X(t) = c$ and $f(t) = L(t)/X(t) = f$ are constants, independent of time. We then derive, in proposition 3, the relation between the debt/net worth and the risk-expected return trade-off.

The intuition is as follows. Consumption over a time interval dt is equation (a). It is GNP plus new borrowing dL(t) less capital formation dK(t).

(a) $C(t)dt = GNP + dL(t) - dK(t)$.

The GNP is equation (b), the productivity of capital times capital $\mathbf{b}(t)K(t)dt$ less interest payments on the debt $\mathbf{r}(t)L(t)dt$, where the stochastic variables are in bold letters. If the country is a debtor then $L(t) > 0$, and if it is a creditor, then $L(t) < 0$. Since capital $K(t) = X(t) + L(t)$, we obtain the second equality..

(b) $GNP = \mathbf{b}(t)K(t)dt - \mathbf{r}(t)L(t)dt = \mathbf{b}X(t)dt + (\mathbf{b}-\mathbf{r})L(t)dt$

Using (b) in (a), consumption is equation (c).

(c) $C(t)dt = [\mathbf{b}X(t)dt + (\mathbf{b}-\mathbf{r})L(t)dt] - [dL(t) + dK(t)]$

If the expected productivity of capital exceeds the expected interest rate $E(\mathbf{b}-\mathbf{r}) > 0$, then there is an incentive to incur debt. The expected GNP will rise, which will permit a rise in consumption. Similarly if $E(\mathbf{b}-\mathbf{r}) < 0$, there is an incentive to become a creditor; and the expected GNP and consumption will rise.

However, there are risks from the debtor or creditor positions: var (**b – r**)L(t). In the debtor case, given the inability[25] to incur new loans dL(t), the var (b-r)L(t) will force corresponding variations in consumption. In the creditor case $L(t) < 0$, variations in the interest rate on foreign investment decrease the GNP, and adversely affect consumption.

We shall examine the expected return-risk trade-off associated with a foreign debt/net worth $f > -1$, independent of time. There is an efficient region where variations in debt will increase both expected return and risk, and there are inefficient regions where variations in the debt will decrease expected return and increase risk. An economy is vulnerable to shocks in the inefficient regions. These concepts are clarified in section 3.1 below.

3.1 The trade-off between Expected Return and Risk

We derive an "expected return-risk" trade-off of the utility of consumption. As before, utility U(t) is the logarithm of consumption C(t), which is a given proportion c of net worth X(t). Hence utility is equation (20), relative to the values in the initial period. Use equation (18) for the value of log X(t)/X(0).

(20) $U(t) - U(0) = \log C(t)/C(0) = \log X(t)/X(0) = g\, t + B_1 w_1(t) + B_2 w_2(t)$

Expected utility relative to the initial period E[U(t)] - U(0) is the gt term. The expected growth of utility - equation (21)- is growth rate g defined in equation (16b) above. Call g the expected return per unit of time.

(21) **expected return** $= E[U(t) - U(0)] / t = (1/t)E[\log C(t)/C(0)]$

$= (1/t)\, E[\log X(t)/X(0)] = g = (r - c) + G(1+f)$

The expected return (equation 21) is plotted in figure 2 for an arbitrary[26] $c > 0$. Term G(1+f) is a concave function of the debt/net worth ratio, and reaches a maximum at the optimal debt ratio f*. This was proved in proposition 1 above.

[25] In this intuitive part, we treat dL(t) + dK(t) as given. This assumption is not unrealistic in time of negative shocks.
[26] With the logarithmic utility function, the optimum $c = \delta$, the discount rate.

The variance of utility is equation (22), based upon (20)(21) and the definition of B's in equation (15). Call this variance, the "risk" per unit of time.

(22) **risk** = $(1/t)$ var $[U(t) - U(0)] = (1/t)\text{var} \log X(t)/X(0) = (1/t) \text{ var} \log C(t)/C(0)$

$$= (1/t) \text{ E } [B_1w_1(t) + B_2w_2(t)]^2 = [f^2\sigma_1^2 + (1+f)^2\sigma_2^2 - 2f(1+f)\rho\sigma_1\sigma_2]$$

Note that the risk depends upon f but not on c. The intuitive explanation was given at the beginning of section 3 above. The variance of the utility, the risk, is also plotted in figure 2. It is a convex function of the debt.

The minimum value of the risk is obtained when $f = f_o = \lambda(\rho\theta\text{-}1)$ in equation (23). The debt associated with the minimum value of the risk f_o is precisely the intercept term in the optimal debt/net worth $f^* = \lambda k_m + \lambda(\rho\theta\text{-}1)$, in equation (12a).

(23) $f_o = \text{argmin } [f^2\sigma_1^2 + (1+f)^2\sigma_2^2 - 2f(1+f)\rho\sigma_1\sigma_2] = \lambda\ (\rho\theta - 1)$

We interpret risk as a measure of vulnerability of an economy to shocks. According to equation (14) a large unfavorable shock in either the interest rate or the productivity of capital produces a large decline in wealth X(t). For a fixed ratio c = C(t)/X(t) there is a corresponding large drop in consumption C(t). As the debt/net worth f deviates from its minimum at f_o , the variance of the stochastic terms in equation (14) increases, hence also the vulnerability.

We now state the crucial proposition in our paper. It relates optimality to expected return and risk - vulnerability. Figure 2 illustrates the results.

PROPOSITION 3. The efficient region for the debt f (positive for debtor, negative for creditor) lies between f_o and f*, in the following manner. (a) When the Merton point $k_m > 0$, then the optimal debt/net worth f* exceeds the value f_o (b) When the Merton point $k_m < 0$, then the optimal debt/net worth f* is less than the value f_o (c) In the efficient region: as the debt is varied, there is a positive relation between expected return and risk. (d) In the inefficient region: as the debt is varied, there is a negative relation between expected return and raises risk.

Proof. The expected utility of consumption, the "expected return", is maximal at the optimal debt $f^* = \lambda k_m + f_o > 0$, because G(1+f) is maximal at f = f* shown in equation (17b). The risk or vulnerability is minimal at f_o. For positive Merton points $k_m > 0$, the optimal debt f^* exceeds f_o. Therefore for $k_m > 0$, it follows that when $f > f^*$, then $f > f_o$. The region $f > f^*$ is inefficient. See figure 2. The rise in f above f^* reduces expected utility and raises the risk-vulnerability. In the region where $f_o < f < f^*$, the debt is below optimal, there is a trade-off. Expected return and risk will both rise as the debt is brought to the optimal level. Hence for $k_m > 0$, the region $f^* > f > f_o$ is efficient, insofar as increased expected return is obtained at the expense of more risk or vulnerability. Regions $f < f_o$ and $f > f^*$ are inefficient, insofar as a rise in debt can increase expected return and reduce risk or vulnerability.

The argument is symmetrical for negative Merton points, $k_m < 0$. The minimum risk value f_o is the debtor ($f > 0$) or creditor ($f < 0$) position that is optimal if the Merton point is zero, that is when the expected net return $(b-r) = 0$. When $k_m < 0$, the optimal debt f^* is less than f_o, the minimum risk value. Suppose that $f_o = \lambda(\rho\theta - 1) < 0$, the minimum risk occurs if the country is a creditor. Then the optimal position is that the country should become more of a creditor and take on more risk. There are two inefficient regions and one efficient region. The efficient region is: $f^* < f < f_o$, and the two inefficient regions are outside that range. **QED**

The <u>efficient</u> regions are summarized.

When $k_m > 0$, $f_o < f < f^*$. When $k_m < 0$, $f^* < f < f_o$

f^* = optimal debt/net worth f_o = minimum risk debt/net worth

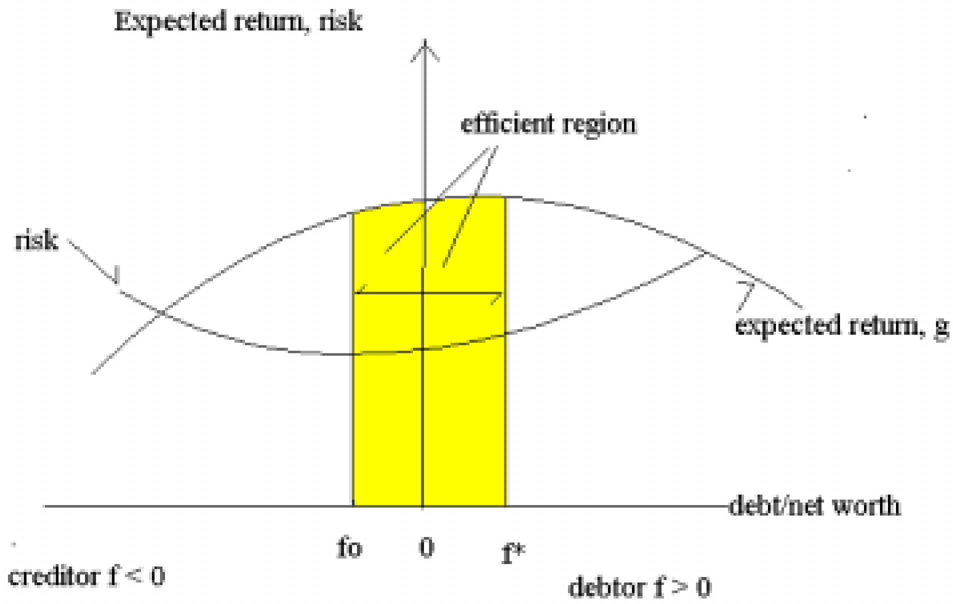

Figure 2. Expected return, risk tradeoff. Efficient region f* > f > fo. km > 0

4 The current account deficit when optimal policies are used

Some economists have argued[27] that continued current account deficits, high growth rates and fixed exchange rates raise the probability of a crisis. In the current section we use our analysis based upon stochastic optimal control to answer the following questions: When optimal policies are followed, what is the expected current account? When is there an excessive current account deficit, in the sense that the current account deficit is greater than should occur when optimal policies are followed?

The current account deficit is the change in the debt dL over a given time interval. From equations (4)-(6), the actual current account deficit as a fraction of net worth is equation (24). It is equal to absorption/net worth less GNP/net worth over time interval dt,. The first term is absorption - equal to consumption plus investment - as a fraction of net worth. The second term is expected GNP, equal

[27] Corsetti et al (1998).

to expected GDP less expected interest payments on the debt, all as fractions of net worth. The third term contains the shocks to GNP arising from the Brownian motion of the interest rate and the productivity of capital. An expected current account deficit results when absorption exceeds expected GNP.

(24) $dL/X = (c + I/X)\, dt - (bk - rf)\, dt + (\sigma_1 f dw_1 - \sigma_2 k dw_2)$

$c = C/X$, $f = L/X$, $k=K/X$, $k - f = 1$

This equation is definitional. It does not specify whether or not optimal policies are being followed. We now derive the current account deficit when consumption, debt and capital are optimal. In the optimality case, the current account deficit/net worth $dL(t)/X(t) = d(f^*X(t))/X(t)$ is equation (25a), which is the product of the optimal debt f* and the growth of net worth.

(25a) $dL/X = f^*\, dX/X$, (12a) $f^* = \lambda k_m + \lambda(\rho\theta - 1)$

The optimal debt/net worth $f(t) = f^*$ is equation (12) or (12a), graphed in figure 1. The debt is adjusted immediately and costlessly to achieve the optimal ratio. The adjustment is done by selling or buying capital in exchange for debt. The term f^*dX/X is the current account deficit over an interval resulting from the growth process, given the optimal debt Term dX/X is derived from equation (14a). As expected net worth grows when optimal policies are followed - equations (11a) (12a) - the expected optimal debt should grow at the same rate. Then the "optimal" expected current account deficit/net worth over a period dt, is equation (25). The derivation is as follows. Let asterisks denote the quantities when optimal policies are followed.

$E[dL^*/X^*] = f^*\, E(dX^*/X^*) = f^*A^*\, dt$, where $A^* = (b - \delta) + (b - r)f^*$ is from equation (14a). Then the expected optimal current account deficit is (25).

(25) $[E(dL^*/dt)/X^*] = [(b-\delta)f^* + (b-r)f^{*2}]$, $f^* = \lambda\, k_m + \lambda\, (\rho\theta - 1) \geq -1$

We prove the following proposition concerning the optimal expected current account.

PROPOSITION 4: If the Merton point $k_m > (1 - \rho\theta) > 0$, and the expected interest rate exceeds the discount rate $r > \delta$, then a permanent expected current account deficit/net worth is optimal. If the Merton point $k_m < (1 - \rho\theta)$, and $r > \delta$, then a permanent expected current account surplus is optimal.

proof: The proof follows from equation (25). The graph of equation (25), is a parabola (see figure 3). There are two roots. One is the origin $f_1 = 0$. The second is $f_2 = -(b-\delta)/(b-r)$. When the Merton point $k_m = (b-r)/(1-\gamma)\sigma_2^2$ is positive, then $(b-r) > 0$, the expected productivity of capital exceeds the expected interest rate. When $(b > r)$ and and $(r > \delta)$, then $(b > \delta)$, the second root f_2 is negative and less than -1. Figure 3 is drawn for this case, but the algebraic treatment is general.

The optimal debt f* will be positive - the country is a debtor - when the Merton point exceeds $(1 - \rho\theta)$. At the origin, the slope is $(b-\delta) > 0$. The expected current account deficit/net worth is a quadratic function rising for for $f > 0$ as described in figure 3. Therefore for all $k_m > (1 - \rho\theta)$ and $(r - \delta) > 0$, it is optimal to have expected current account deficits. A debtor country will be at point $f^* = 0C > 0$ and have expected current account deficit net worth $E(dL)/dt \;/\; X = CC' > 0$. In creditor countries, $k_m < (1 - \rho\theta)$ and the ratio of optimal debt/net worth $f < 0$ such as point 0D. The optimum expected current account surplus/net worth is DD'. **QED**

Figure 3 or equation (25) is our benchmark for the expected ratio of the current account /net worth when optimal policies are followed. A high Merton point will tend to imply a high optimal debt/net worth (eqn. 12a), and together with a high $(r-\delta)$ there will be a high optimal expected growth rate of net worth. At any time, however, the Brownian motion terms in (14) produce current accounts/net worth, which deviate from their expected values. It follows that permanent current account deficits/net worth do not imply that non-optimal policies are being followed.

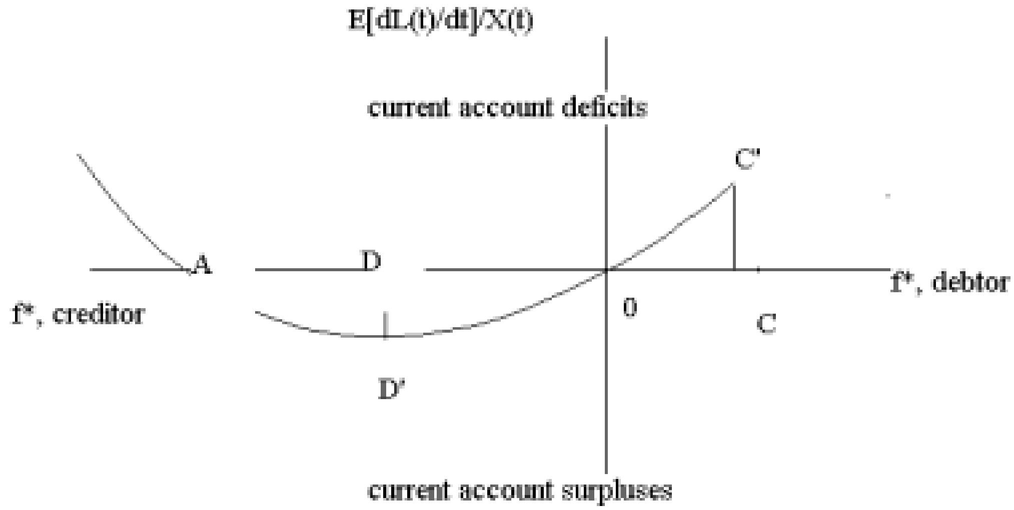

Figure 3. Expected current account/net worth, related to optimal debt f*

5. Crises and Risk: Integration of Theory and Historical Experience

The recent literature[28] retrospectively describes the chronology and origins of currency (balance of payments) and banking (financial) crises[29]. Both types of crises have been preceded by a multitude of weak and deteriorating objective economic fundamentals. There were very few crises where the economic fundamentals were sound[30]. This suggests that it would be difficult to claim that crises arise from "self-fulfilling prophesies"[31].

[28]See Kaminsky and Reinhart (1999) for an excellent analysis of the empirical regularities and sources and scope of problems concerning the onset of currency and banking crises. See also International Monetary Fund: WEO, May 1998: 81-82, Kaminsky, Lizondo and Reinhart (1997); World Bank/International Monetary Fund/World Trade Organization Conference on Capital Flows, Financial Crises and Policies, World Bank, Washington DC April 15-16, 1999.

[29] Currency crises were measured as exchange market pressure: a weighted average of the percentage decline in reserves and the percentage depreciation of a currency. Banking crises were measured in terms of events.

[30] Kaminsky and Reinhart (1999:491)

[31] The NATREX model - Stein, Allen, (1997), Stein (1999) demonstrates that exchange rate movements and currency crises result from movements in objectively measured economic real fundamentals. The NATREX model derives a moving equilibrium real exchange rate that is compatible with internal and external equilibrium. This rate is a function of objectively measured real "fundamentals" denoted Z(t), a vector of social consumption/GDP and the productivity of

We conclude our paper by relating the historical experience of crises to our theoretical analysis. Excessive debt is the level of debt in the inefficient region: $f > f^*$. In this region the economy is vulnerable to shocks from the productivity of capital, the interest rate and the negative correlation between them. Crises are highly likely. By reducing the debt towards the optimal f^*, the expected return can rise and the risk or vulnerability can be decreased.

5.1 Historical Experience

In the period after financial markets were liberalized, there was an interaction between currency and banking crises. There are several aspects of the crises, generated by internal and external factors. In general, there is below normal growth, which may arise as a result of a decline in the growth of exports. The latter results from a worsening of the terms of trade, or an overvalued exchange rate. A rising cost of credit further diminishes the growth of the economy.

First: There are unsustainable internal macroeconomic policies. Expansionary monetary and fiscal policies increase absorption relative to GDP and produce "excessive" debt. There is "over-investment" in real assets, which drives equity and real estate prices to "unsustainable" levels. The eventual tightening of policies to contain inflation and promote the adjustment of the external debt positions leads to a slowdown of the economy, declining net worth and collateral, and rising levels of non-performing loans that threaten bank solvency.

Second: The external factors which produce crises arise from: overvalued exchange rates, declines in the terms of trade and rises in world interest rates. In the emerging market economies: (a) An overvalued exchange rate or decline in the terms of trade (ratio of export/import prices) decrease the growth of exports which decreases the growth of GDP. These factors deteriorate the quality of the

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capital at home and abroad. The NATREX is denoted as R[Z(t)]. This gives economic content to the "trend" used in the empirical literature. We define misalignment as the deviation of the actual real exchange rate R(t) from the NATREX. We show that misalignment R(t) – R[Z(t)] leads to changes in the nominal exchange rate. Crises are expected to be successful only if there is loan portfolios of banks and impair the capacity of domestic firms to service their debts. (b) With of the integration of world capital markets, movements in interest rates in the major industrialized countries are important to the emerging economies.. Sustained declines in world interest rates have induced capital flows to the emerging markets, as world investors seek higher yields. An abrupt rise in the interest rates in industrial countries raises the cost of borrowing, and impairs the ability of the emerging market countries to service the shorter-term debt. The higher interest rates also adversely affect economic activity, reduce the quality of loan portfolios and weaken the banking system. Investors, domestic as well as foreign, become apprehensive about the value of the currency and attempt to convert domestic assets into foreign assets. The exchange rate tends to depreciate and/or international reserves decline.

Third: Domestic interest rates rise as a result of an attempt by the government to maintain the exchange rate peg. Credit becomes more difficult to obtain. Moreover, the expectation of the depreciation of the currency raises interest rates further. The cost in domestic currency of servicing the foreign debt increases. There are more non-performing loans which further undermine the viability of the financial system. In this manner, the currency crisis aggravates the banking crisis, and the banking crisis reduces the growth rate and aggravates the currency crisis.

Fourth: Contagion effects[32] may occur when a core of countries have a common lender or very close trade relationships with each other. If the common lender bank is confronted with a marked rise in non-performing loans in several countries, then it may attempt to reduce overall risk by lending less, and on less favorable terms, to the other countries. Close trade relationships produce another possible channel for "contagion".

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misalignment. For a discussion of what is known concerning equilibrium exchange rates see MacDonald and Stein (1999, ch.1).

[32] See Reinhart and Kaminsky (1999) in World Bank/IMF/World Trade Organization .

5.2 The Current Literature

The various strands in the literature view crises in terms of equations (1) and (1a) above, repeated here[33]. We defined h(t) = L(t)/Y(t) as the ratio of the foreign debt L(t) to the gross domestic product Y(t) accruing to the residents of the country.

(1) $dh(t)/dt = [\ r(t) - g(t)]\ h(t) - [1 - c(t) - i(t)]$; $\quad g(t) = Y'(K(t),t)\ i(t)$

(1a) $dh(t)/dt = A(t)\ h(t) - B(t)$ $\quad A = [\ r(t) - g(t)],\ \ B = [1 - c(t) - i(t)]$

The boom period is characterized as follows. The "excessive" consumption c(t) – that is, non-productive expenditure – raises absorption, produces negative trade balances B(t) and increases the growth of the debt. The government subsidized interest rates raise the investment ratio i(t) and produce the same effects.

The pre-crisis period is characterized as follows. An overvalued exchange rate decreases the growth of exports, reduces the trade balance B(t), lowers the growth rate of GDP and raises the foreign debt. The rise in world interest rates and the restrictive monetary policies raise domestic interest rates raise r(t) which reduce the growth rate g(t). These factors raise A(t) = [r(t) – g(t)].

The fragility of the banking system concerns the probability that there will be a banking crisis when there are the shocks to the productivity of capital and the real interest rate. The interaction between the fragility of the banking system and the growth rate is subsumed under the negative correlation between the growth rate and the interest rate. Shocks which raise the interest rate raise both the level and the variance of A(t) = [r(t) - g(t)].

The stochastic variables have Brownian motion terms, so that they are unpredictable. They are not stationary processes. The variance increases with the horizon. The standard "intertemporal optimization" approach in economics is conditional upon knowing the expected present value of GDP over an infinite horizon. With Brownian motion disturbances, this is unknowable. Think of

[33] There is a different notation in equations (1a) from that used in the rest of our paper. In each case, we clarify the meaning of the variables.

"knowledge" as the ratio of the mean to the variance. This ratio - "knowledge" – of the mean goes to zero as the length of the horizon goes to infinity. Hence when the disturbances are Brownian motion, the usual standard "intertemporal optimization" approach is not operational.

The recent World Bank/International Monetary Fund/World Trade Organization conference (1999) stressed that there still is an unfulfilled need for an early warning signal (EWS) that a country will face a currency or balance of payments crisis over a given time horizon[34]. We explained at the beginning of this paper why the standard analysis of the vulnerability of an economy to excessive debt, based upon equation (1), is inadequate. We know that the available measures of expectations by market participants - interest rates and forward exchange rates - display a poor record in anticipating crises. Market expectations of currency crises typically do not rise until very shortly before the crisis[35]. We want an EWS to do better than the market anticipations.

Some writers regard current account deficits as EWS. However, proposition 4 proves that it may very well be optimal to have permanent and large current account deficits.

The literature cited above has not been successful in finding reliable early warning signals. The only good indicator was the real exchange rate relative to its trend.[36] Similarly, the analysis of "contagion" effects concluded that the financial contagion effects have some predictive power, but the results are not reliable.

5.3 The Implications of the Stochastic Optimal Control Approach

Our approach towards the foreign debt and crises is different in several respects. The following summarizes our contribution. First, the stochastic

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[34] We are drawing upon the survey paper by Borenzstein et al.(1999) for an evaluation of the state of the art, and the unanswered questions.

[35] Kaminsky and Reinhart (1999: 485).

[36] Kaminsky, Lizondo and Reinhart (1997). The NATREX model - Stein, Allen, (1997), Stein (1999) - gives economic content to the "trend" used in the empirical literature. We define misalignment as the deviation of the actual real exchange rate R(t) from the NATREX. We show that misalignment R(t) – R[Z(t)] leads to changes in the nominal exchange rate. Crises are expected to be successful only if there is misalignment.

optimal control approach, which we use, is predicated upon a system where we are unable to anticipate the future. It is based upon dynamic programming with a stabilizing feedback control mechanism. Bankruptcy cannot occur, regardless of the shocks[37].

Second: our optimality criterion is the expected discounted value utility of consumption over an infinite horizon, subject to the constrained laws of motion. Our optimal debt/net worth f* and consumption/net worth are derived from the expected utility maximization using stochastic optimal control.

Third: we view a crisis in terms of vulnerability. Our measure of vulnerability is the variance of the utility of consumption per unit of time. A high variance indicates a great likelihood of a crisis. Since we use a logarithmic utility function, vulnerability - or risk - is measured as the variance per unit of time of the logarithm of consumption.

Fourth: we derive a frontier between the expected return and vulnerability. "Expected return" is measured as the expected growth rate of utility. Vulnerability is the variance per unit of time of the logarithm of consumption.

Fifth: a foreign debt is called "excessive" if a reduction can reduce vulnerability without sacrificing expected return. Suppose that the expected productivity of capital exceeds the expected interest rate, $k_m > 0$ as graphed in figure 2. When the debt/net worth exceeds the optimum debt/net worth f*, a reduction can increase the expected utility and decrease vulnerability.

Sixth: our derived equation for the optimal foreign debt/net worth, denoted f*, has the following important properties. (a) It is the debt that maximizes the expected value of the utility of consumption over an infinite horizon, given the constrained law of motion of net worth. (b) For any given consumption/net worth ratio $c > 0$, the optimal debt/net worth f* also maximizes the expected growth of net worth.

[37] Proposition 2 states these conditions. A crucial assumption if that the debt and consumption are controls that can be varied instantaneously and costlessly.

Many studies[38] claim that weaknesses in the financial sector were at the root of the Asian crises. With private securities markets (bonds, equity) underdeveloped until the 1990's, corporations relied heavily upon the banking system for financing. By and large, external borrowing by the corporate sector, intermediated mainly through the banking system, was the main vehicle by which foreign funds were mobilized. One of the features of the Southeast Asian crisis is the large size and critical role played by the corporate sector's foreign debt. The very high and rapidly growing debt/equity ratio in the Southeast Asian economies indicate that both the banking and corporate sectors were becoming increasingly vulnerable to adverse shocks.

In our framework, the above argument is seen through our concept of "vulnerability: which is directly related to the difference between the actual and the optimal debt/net worth. We provide an objective measure of excessive debt/net worth. The optimal debt $f^* = \lambda k_m + \lambda(\rho\theta - 1)$, contains all of the elements featured in the description of a debt crisis, and integrates the real shocks with the fragility of the banking system. The term $k_m = (b - r)/(1-\gamma)\sigma_2^2$ is referred to as the Merton point. It is the mean productivity of capital less the mean real interest rate, and $(1-\gamma)\ \sigma_2^2$ is the variance of the productivity of capital times[39] relative risk aversion. The intercept term $\lambda(\rho\theta - 1)$ denoted f_o is the debt/net worth that minimizes the risk measured as the variance of utility per unit of time.

The interaction of the real and financial shocks is the correlation coefficient ρ. The fragility of the financial system is aggravated by a correlation $\rho < 0$ which has been the case in the Emerging Market countries. When the productivity of capital is shocked below its mean, there are more non-performing loans, bank failures - due to the high debt/equity ratios of banks - and capital flight. The capital outflow drains the reserves. The tightening of credit is reflected in a rise in interest rates. Moreover, if there is an attempt to maintain the exchange rate in face of the

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[38] See for example: International Monetary Fund, World Economic Outlook (WEO) May 1998, pp. 6, 85-105.

[39] We generally have worked with the case where $\gamma = 0$, the log utility function.

shocks, interest rates are raised further and the correlation ρ becomes even more negative. The conclusion is that: the optimal debt/net worth of an economy with $(\rho\theta - 1) < 0$ is low, compared to one where interest rates are positively correlated $(\rho > 0)$ with the productivity of capital as in the US.

We have used the techniques of stochastic optimal control to respond to the question: <u>when is it rational for market participants to anticipate a crisis</u>? On the basis of proposition 3, we know that when the debt f(t) exceeds the optimal debt f* the economy is very vulnerable to shocks, and the probability of a crisis is high. By reducing the debt to the optimal level, the economy can simultaneously increase the growth and utility of consumption and reduce the risk, the variance of the utility of consumption.

There is an important area of research, which has been neglected here. It concerns the relation of the optimal debt $f^* = \lambda k_m + \lambda(\rho\theta - 1)$ to "risk aversion" $(1-\gamma)$ that is contained in the Merton point $k_m = (b - r)/(1-\gamma)\sigma_2^2$. In Merton's analysis the agent must select the ratio of risky assets/net worth and safe assets/net worth. The k_m is the optimum ratio of risky assets/net worth.

The portfolio chosen by the agent has no effect upon the interest rate on the safe asset. In our case, there is no safe asset. The debt is financed at an uncertain interest rate over its lifetime, and the return on capital is also stochastic[40]. Suppose that there are two countries A and B with (b-r) > 0: the expected productivity of capital exceeds the expected interest rate. Assume that country A has a low coefficient of risk aversion $(1-\gamma)$ close to zero and country B has a higher coefficient $(1-\gamma)$ close to unity. Country A would optimally incur an infinite amount a debt to finance capital formation, and B would incur a finite amount. Would the international capital lenders be willing to finance country A's proposed capital formation? We would expect that the international lenders have their coefficients of risk aversion $(1-\gamma^*)$ and that they would charge countries A and B different interest rates depending upon the amount of debt they have incurred.

Therefore, the "optimal debt" f* and the interest rate at which the country can borrow must be interrelated. In the present paper, we have shown the power of the stochastic optimal control approach on the basis of a prototype model. The more realistic complications will be covered in a subsequent paper.

[40] A creditor country faces the risk of variations in the interest rate on foreign loans. Hence there is risk on any debt $f > 0$ or creditor position $f < 0$.

APPENDIX A

In this appendix, we give conditions on the model parameters such that equation (10) has a solution with A > 0. Then we sketch a formal derivation of equation (10) using the dynamic programming principle. In case $\sigma_1 = 0$ (where only the productivity is random) our problem is equivalent to the classical Merton optimal portfolio model with a single risky asset. An argument which proves that $V(X) = (A/\gamma)X^\gamma$ is indeed the value function, and that the controls are indeed optimal, is given in Fleming-Soner (1992: 174). See also Fleming-Rishel (1975: 160) for the corresponding finite time horizon result. The verification argument when both σ_1 and σ_2 can be non-zero in our paper is entirely similar.

<u>Conditions for A > 0</u>

In equation (10)

$\max_c [\, (1/\gamma A)c^\gamma - c] = [(1-\gamma)/\gamma]\, A^{1/(\gamma-1)}$

The maximum occurs at $c = C/X = c^* = A^{1/(\gamma-1)}$ as in equation (11). The max over $f \geq -1$ in equation (10) is at $f = f^*$, which satisfies (12) if $f^* > 1$. When $f^* > -1$, a calculation making use of expression (12a) for f^*, gives

$(b - r) f^* + (1/2)(\gamma - 1)[\, f^{*2}\sigma_1^2 + (1+f^*)^2\sigma_2^2 - 2(1+f^*)f^*\rho\sigma_1\sigma_2 \,]$

$= \lambda\, \sigma_2^2(1-\gamma)f^{*2} + \sigma^2 f^{*2} + \sigma_2^2$

where λ and σ^2 are defined in Box 1.

Thus, equation (10) becomes (after multiplying by γ)

(A1) $\delta = b\gamma + (1-\gamma)\, A^{1/(\gamma-1)} + \gamma[\lambda\sigma_2^2(1-\gamma) + \sigma^2)f^{*2} + \sigma_2^2\,]$

To have a solution with A > 0, we must have

(A2) $\delta > b\gamma + \gamma[\lambda\sigma_2^2(1-\gamma) + \sigma^2)f^{*2} + \sigma_2^2]$

When $\gamma < 0$, this imposes no restriction on the discount factor $\delta > 0$. However, for $0 < \gamma < 1$, the inequality (A2) imposes a positive lower bound on δ. (The case $f^* = -1$ is similar).

When $\gamma = 0$, $U(C) = \ln C$ and $V(X) = A \ln X + B$. From equation (10), $A = 1/\delta$. Moreover, $c^* = 1/A = \delta$ as in equation (11a).

Derivation of equation (10)

The dynamic programming principle states that, for each finite time $S > 0$ and where $S \geq t \geq 0$

(A3) $V(X) = \max E\{ \int_{S>t>0} (1/\gamma)C(t)^{\gamma} e^{-\delta t} dt + V(X(S)e^{-\delta S}\}$, $S \geq t \geq 0$

where the max is taken among the admissible controls on the time interval 0 to S. We recall that $X(0) = X$. Then over the interval $S > t > 0$

(A4) $V(X(S))e^{-\delta S} - V(X) = \int_{S>t>0} d[V(X(t))e^{-\delta t}]\, dt = \int [-\delta V(X(t)) + d\, V(X(t))]e^{-\delta t}\, dt$

From equation (8a) and the Ito differential rule

(A5) $dV(X(t)) = V_x(X(t))dX(t) + (1/2)V_{xx}(X(t)[L(t)^2\sigma_1^2 + (X(t)+L(t))^2\sigma_2^2 - 2L(t)(X(t)+L(t))\rho\sigma_1\sigma_2]dt$

We take expectations in (A4) and recall that expectations of the stochastic integral terms are zero.

(A6) $E\,[V(X(S))]e^{-\delta S} - V(X) = E\,\{\int_{0<t<S} [-\delta V(X(t)) + V_x(X(t))((b-r)(X(t) + L(t)) + rX(t) - C(t)) + (1/2)V_{xx}(X(t))(L(t)^2\sigma_1^2 + (X(t)+L(t))^2\sigma_2^2 - 2L(t)(X(t)+L(t))\rho\sigma_1\sigma_2]dt$

We add to each side of (A6) $E[\int_{S>t>0} (1/\gamma)C(t)^{\gamma} e^{-\delta t} dt]$.

According to (A3), the result is non-positive, and is zero when optimal controls are chosen. We then let $S \Rightarrow 0$, and consider only controls which are nearly constant on the interval 0 to S.

This completes the formal derivation of equation (10a). By taking the maximum over C, L in (10a), optimal policies $C^*(X)$, $L^*(X)$ as functions of wealth X are obtained.

When $V(X) = (A/\gamma)X^{\gamma}$ is substituted in (10a) with $c = C/X$, $f = L/X$, equation (10a) is obtained after dividing by X. The optimal policies are $C^*(X) = c^*X$, $L^*(X) = f^*X$ where constants c^* and f^* are as in equations (11), (12), provided that $f^* > -1$.

APPENDIX B

THE PRODUCTIVITY OF CAPITAL

The productivity of capital Y(t)/K(t) = **b**(t) in equation (6) is a crucial stochastic variable. We assumed that **b** dt ~ N(b dt, σ_2^2dt). The US data is presented here to show the strength and weaknesses of this assumption.

We measure the productivity of capital in marginal terms: b(t) = [dY(t)/dt] / [dK(t)/dt] = g(t)/i(t), where g(t) = (1/Y(t)) dY(t)/dt = growth rate of GDP, and i(t) = (1/Y(t))dK(t)/dt is the ratio of investment (capital formation) to GDP. The graph below plots this measure of b(t) and refers to it as OUTINV (output growth /investment ratio). The data cover the period 1959:1 – 1997:2. We also append the basic characteristics of the series.

We assumed that the productivity of capital **b** dt has a mean b dt and a variance σ_2^2dt. The figure below shows that the situation is somewhat more complicated. The productivity of capital is stationary, that is mean reverting. This is seen from a unit root test[41], using 4-quarter lags.

ADF Test Statistic	-3.676835	
	1% Critical Value*3.4767	-
	5% Critical Value	-2.8815
	10% Critical Value	-2.5773

*MacKinnon critical values for rejection of hypothesis of a unit root.

*MacKinnon critical values for rejection of hypothesis of a unit root.

For the US, the interrelations among: the productivity of capital, the interest rate, the net return and the unemployment rate are displayed below. The productivity of capital and interest rate are positively correlated, but not very highly. The productivity of capital is negatively related to the unemployment rate UNRATE.

Correlations among b = output/capital, r = real long term interest rate, b – r = net return, UNRATE = unemployment rate

[41] The unit root test concerns whether the change in a variable Δx(t) = x(t) – x(t-1) is iid or whether it depends also upon its previous value x(t-1). The regression is:
Δx(t) = mx(t-1) + a + Σb(j)ΔX(t-j) + e(t). Parameter a is a constant and e(t) is a random variable. If the variable is iid, then m is not significantly different from zero. If the ADF is significantly different from zero, we reject the hypothesis of a unit root, that m = 0.

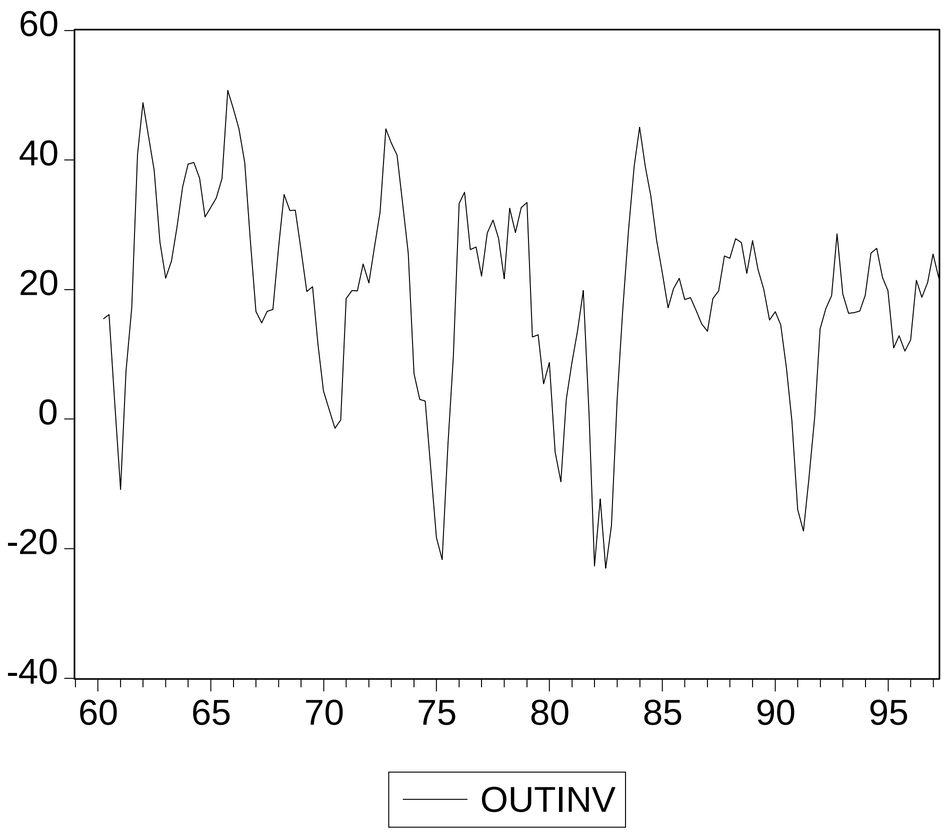
OUTINV (b)	USRLT (r)	NETRET (b-r)	UNRATE	
1.000000	0.241963	0.987625	-0.376497	OUTINV
0.241963	1.000000	0.086797	0.149009	USRLT
0.987625	0.086797	1.000000	-0.410648	NETRET
-0.376497	0.149009	-0.410648	1.000000	UNRATE

As explained in the text, we use the Brownian motion assumption for the productivity of capital instead of the mean reversion assumption, because the latter raises the order of the system and we cannot solve the system without a computer. At this stage we want to under the basic analytical processes, before we simulate.

The graph of OUTINV and the unit root test above show that the constant mean output/capital ratio is a sensible assumption. This assumption is in the spirit of the "endogenous technical change" models, but differs from the standard smooth production function where output/capital is a smooth concave function of the capital/labor ratio.

One of several ways to rationalize the production function (6a)-(6c) is to assume that there is a Leontief production function:Y(t) = min [b(t)K(t), a(t)N(t)], where K(t) is capital and N(t) is labor. The technical change is labor augmenting: da(t)/dt > 0, and b(t) is given by (6c). Let capital be the constraining input, or assume that b(t)K(t) = a(t)N(t). This gives us Y(t) = b(t)K(t). Capital productivity is b(t). Labor productivity is Y(t)/L(t) = a(t) which grows at rate (1/a(t)da(t)/dt.

OUTINV = GROWTH RATE/(INVESTMENT/GDP)
= g(t)/i(t) = coefficient b(t) in text



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