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AUCTION THEORY FROM AN ALL-PAY VIEW: BUYING BINARY LOTTERIES

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Abstract

An auction is viewed as a process that in equilibrium generates a binary lottery for each bidder, which the bidder "buys" with his bid. This view allows for a simple way to consistently assess differences in bidding behavior over different bidders and different auctions. E.g. all auctions covered by the Revenue Equivalence Theorem are shown to generate lotteries with *identical* probabilities, but different pay-offs. It is then argued, that understanding of (experimentally observed) bidding behavior in auctions is enhanced by drawing on the large literature on choice behavior over lotteries.

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1 Introduction

The theory of contests and tournaments has been developed in specific context, such as rent-seeking, technological competition and labor markets, well outside the confines and with little reference to auction theory. Only recently was it recognized that contests (e.g. wars of attrition) are examples of socalled all-pay auctions, whose analysis can be fully integrated in the main body of auction theory (Amann and Leininger 1995, 1996, Krishna and Morgan, 1997, Mashin, 2000).

A similar statement applies to experimental investigations of contests, which begin with examinations of rent-seeking behavior in various institutional settings (Millner and Pratt 1989, 1991, Shogren and Baik, 1991). Experiments on all-pay auctions, which try to draw on and connect to the large experimental literature on auctions, are very recent (Amann and Leininger, 1997, Barut et al. 1999).
A standard auction like the first or second price auction is formally turned into an all-pay auction by a simple change of the payment rule: instead of requiring only the bid by the successful bidder to be paid, one requires all bids to be paid (by all bidders, irrespective of whether they win or not); still the highest bid only wins. This stylized all-pay structure is present in many economic competitions, which – roughly speaking – require *investment* expenditures; i.e. outlays which are non-retrievably sunk. In the auction paradigma such an outlay is treated as a bid, it has to be paid *unconditionally* (in particular it is not contingent on *winning* like in standard auctions) and hence *all* (participants) *pay*.

The recognition of all-pay auctions as an important part of auction theory and economics (see e.g. Klemperer, 2000) is taken one step further here: we claim that useful insights – both theoretically and behaviorally – can be gained by viewing *all* auctions as *all-pay* auctions. This is done by treating conditional bids in standard auctions as non-conditional, but refundable bids. The refunds are incorporated into the pay-offs of a lottery that, in effect, is offered to each bidder in equilibrium of any auction. The first price all-pay auction serves as a benchmark as the lottery implied by this auction format is very simple and evident: a bidder with valuation v pays with his bid for a lottery, that either pays v (if she wins) or 0 (if she does not win), equilibrium of the auction game only determines the odds of the two outcomes. Analogously, the first-price auction (winner-only-pays) offers to a bidder with valuation v for her bid b a lottery, which either pays v (if she wins) or b (if she does not win). This view allows for a simple consistent way to compare bidding behavior of different bidders in different auctions, that can draw on well-established theory and empirical evidence of choice behavior over lotteries.

The paper is organized as follows: section 1 gives a brief derivation (and full statement) of the revenue equivalence theorem as this forms the basis of much of the later analysis. Section 2 comments on the special status of Vickrey and all-pay auction and presents the "all-pay view", which is developed further in section 4. Section 5 comments on implications for experimental evidence on all-pay auctions and contests.

2 The IPV-Model and Revenue Equivalence

An auctioneer wishes to sell an indivisible object to one of n buyers, $n \geq 2$, indexed $i = 1, \ldots, n$. Each buyer's reservation value, v_i, $i = 1, \ldots, n$, is distributed according to the same distribution function

$$F : [0, \bar{v}] \to [0, 1] \ .$$

The draws of the respective values, v_i, $i = 1, \ldots, n$, are independent of each other, so that from the auctioneer's point of view the random variable $v = (v_1, \ldots, v_n)$ has the distribution function

$$G(v_1, \ldots, v_n) = F(v_1) \cdot F(v_2) \cdots F(v_n)$$

A bidding strategy for any bidder $i = 1, \ldots, n$ can be represented by a function

$$b_i : [0, \bar{v}] \to I\!R_+$$

with $b_i = b_i(v_i)$ denoting the (observed) bid of player i.

A typical auction (and the effects of its particular *rules*) can be represented by two functions *per player*

$$\begin{aligned} &\text{her probability of winning: } p_i(b_1, \ldots, b_n) \quad \text{and} \\ &\text{her (expected) payment: } e_i(b_1, \ldots, b_n) \qquad\qquad i = 1, \ldots, n. \end{aligned}$$

Note, that all of these $2n$ functions depend on the bids submitted by *all* players.
These two functions summarize for each player individually the content of the interactive decision problem present in an auction. They suffice to calculate a bidder's expected pay-off (whom we – for simplicity – portray here as being risk-neutral) as

$$u_i(b_1, \ldots, b_n) = p_i(b_1, \ldots, b_n) \cdot v_i - e_i(b_1, \ldots, b_n) \tag{1}$$

when she knows $b = (b_1, \ldots, b_n)$.

A Bayesian Nash equilibrium of the bidding game applies, if the chosen bidding functions $\big(b_1^*(v_1), \ldots, b_n^*(v_n)\big)$ satisfy, for each player $i = 1, \ldots, n$, and all $v_i \in [0, \bar{v}]$

$$E_{v_{-i}} u_i\big(b_1^*(v_1), \ldots, b_i^*(v_i), \ldots, b_n^*(v_n)\big) \geq E_{v_{-i}} u_i(b_1^*(v_1), \ldots, b_i, \ldots, b_n^*(v_n))$$

for all $b_i \geq 0$.

Here v_{-i} denotes the vector $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$; i.e. the expectation is taken w.r.t the distribution of *other* players' valuations.
The equilibrium problem becomes tractable – and also special –, if one looks for a symmetric equilibrium, in which all players use the same bidding strategy $b(v) = b_1(v) = \ldots = b_n(v)$.

Then player i's pay-off maximization problem can be written as a function of her own bid only in the reduced form of

$$\max\ u_i(b, v_i)$$

or, in more detail,

$$\max\ \bar{p}_i(b) \cdot v_i - \bar{e}_i(b)$$

whose solution gives $b^*(v_i)$ for any $v_i \in [0, \bar{v}]$.

$\bar{p}_i(b)$ now represents the expected value of the probability of winning for player i when she bids b and $\bar{e}_i$ gives her expected payment. More formally,

$$\bar{p}_i(b) = E_{v_{-i}}\Big(p_i\big(b(v_1), \ldots, b(v_{i-1}), b, b(v_{i+1}), \ldots, b(v_n)\big)\Big)$$

$$\bar{e}_i(b) = E_{v_{-i}}\Big(e_i\big(b(v_1), \ldots, b(v_{i-1}), b, b(v_{i+1}), \ldots, b(v_n)\big)\Big)$$

It is well-known that, if a symmetric equilibrium bidding function exists, it must be monotonic in the player's valuation; i.e. different valuations lead to different bids in equilibrium:

$$v_i > v_i' \quad \Rightarrow \quad b(v_i) > b(v_i')$$

This intuitive property means that $b(\cdot)$ is invertible and hence *behavior* (i.e. a bid) is a sufficient statistic for a player's type or *identity*. Consequently,

$$\begin{aligned} \bar{p}_i(b) = \bar{p}_i\big(b(v)\big) &= \text{Prob }\big(b(v_j) \leq b(v)\big)_{j \neq i} \\ &= \text{Prob }(v_j) \leq v)_{j \neq i} \\ &= F(v)^{n-1} =: \tilde{p}_i(v) = \tilde{p}(v) \end{aligned}$$

Note, that we now for the first time assume, that the highest bidder wins the auction (with certainty).
Accordingly, u_i becomes

$$u_i(b) = u_i\big(b(v)\big) = \tilde{p}(v) \cdot v - \bar{e}_i\big(b(v)\big) =: \tilde{u}_i(v)$$

In equilibrium $b(v)$ is selected optimally, so by the envelope theorem we get

$$\tilde{u}_i'(v) = \tilde{p}_i(v) = \tilde{p}(v)$$

which translates into

$$\begin{aligned} \tilde{u}_i(v) &= \int_0^v u_i'(x)\, dx = \int_0^v \tilde{p}(x)\, dx \\ &= \int_0^v F(x)^{n-1}\, dx =: \tilde{u}(v) \end{aligned}$$

Immediately, expected payment in equilibrium for player i becomes

$$\tilde{e}_i(v) = \tilde{p}(v) \cdot v - \tilde{u}(v) = F(v)^{n-1} \cdot v - \int_0^v F(x)^{n-1}\, dx =: \tilde{e}(v)$$

The last two substitutions show, that we have derived the contents of the celebrated Revenue Equivalence Theorem (RET): in equilibrium a bidder's expected pay-off $\tilde{u}_i(v) = \tilde{u}(v)$, her probability of winning $\tilde{p}_i(v) = \tilde{p}(v)$ and her expected payment $\tilde{e}_i(v) = \tilde{e}(v)$ have all been traced to her *identity* v as the sole determinant. I.e. the particular *rules* of an auction, that guide behavior of a rational bidder of any given identity, are unimportant for the determination of expected revenue for the seller. The latter is simply equal to the expectation of the sum of the expected payments over all bidders. Only the distributional law of bidders' characteristics matters, differences in auction design get *neutralized* by the then different optimal behavior of bidders (in equilibrium). Hence RET amounts to a neutrality theorem regarding auction design, if this design task accepts the basic rule, that the highest bidder wins the auction. For completeness sake, we give a precise statement of the theorem:

Revenue Equivalence Theorem:

> If any of n risk-neutral bidders has a privately known valuation, which is independently drawn from a common, strictly increasing, continuous distribution $F(v)$ on $[0, \bar{v}]$, then any auction mechanism which
>
> a) gives the objects to the bidders with the highest valuations and
>
> b) gives zero surplus to any bidder with the lowest valution
>
> yields the same expected revenue and the same expected payment for a bidder with valuation v (provided each bidder only bids for at most one of the objects).

The stated generalization to auctions of several (identical) objects with unit demand is obvious from the one object case: the chain of arguments just starts out with a different $\tilde{p}(v)$.

What are exected revenues for the seller over all these auctions? By defintion and the expression derived above for $\tilde{e}(v)$ we get

$$R = n \cdot E_v\big(\tilde{e}(v)\big) = n \cdot \int_o^{\bar{v}} \big[v \cdot F(v)^{n-1} - \int_0^{v} F(x)\ dx\big] dv$$

and revenue R is seen to depend on $F(\cdot)$ and n, the number of bidders, irrespective of the particular auction rules.

The integrand in this expression, $\tilde{e}(v)$, allows an economically most important interpretation:
Mathematically, the expression derived for $\tilde{e}(v)$ can be shown to be equal to the expected value of the second highest valuation (among the independently drawn $v_1, \ldots, v_n$) *conditional* on v being the highest valuation.

To see this recall from the theory of order statistics, that the density of the random variable r-lowest value drawn, V_r (i.e. $V_1 < V_2 < \ldots < V_n$), is given by

$$f_r(v) = \frac{n!}{(r-1)!(n-r)!} \cdot F(v)^{r-1} \cdot \big(1 - F(v)\big)^{n-r} \cdot f(v) \qquad r = 1, \ldots, n.$$

Consequently, the conditional density of V_r given the value of $V_s = v$ with $s > r$ is then determined by a sample of size $(s-1)$ only whose elements are all smaller than v; i.e. for $x < v$

$$f_r|_{V_s=v}(x) = \frac{(s-1)!}{(r-1)!(s-r-1)!} \cdot \frac{F(x)^{r-1}\big(F(v) - F(x)\big)^{s-r-1}}{F(v)^{n-1}} \cdot f(x).$$

From this it follows, that with $s = n$ and $r = n-1$

$$\begin{aligned} f_{n-1}|_{V_n=v}(x) &= \frac{(n-1)!}{(n-2)!} \cdot \frac{F(x)^{n-2}\big(F(v) - F(x)\big)^{0}}{F(v)^{n-1}} \cdot f(x) \\ &= (n-1) \cdot \frac{F(x)^{n-2}}{F(v)^{n-1}} \cdot f(x) \end{aligned}$$

which gives

$$
\begin{aligned}
E(V_{n-1}|V_n = v) &= \int_0^v x \cdot f_{n-1}|_{V_n=v}(x)\ dx \\
&= \int_0^v (n-1) \cdot x \cdot F(x)^{n-2} \cdot f(x)\ dx \cdot \frac{1}{F(v)^{n-1}} \\
&= \frac{1}{F(v)^{n-1}} \int_0^v \left(F(x)^{n-1}\right)' \cdot x\ dx \\
&= \frac{1}{F(v)^{n-1}} \left[F(v)^{n-1} \cdot v - \int_0^v F(x)^{n-1}\, dx\right] \\
&\qquad \text{by partial integration} \\
&= v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{n-1} dx
\end{aligned}
$$

The factorial form of this expression after partial integration is identical to the product

$$\frac{1}{\tilde{p}(v)} \cdot \tilde{e}(v)$$

as precisely these terms have been derived above as expressions for $\frac{1}{\tilde{p}(v)}$ and $\tilde{e}(v)$.

Hence in all symmetric auction games covered by RET it is true, that

$$\boxed{\tilde{e}(v) = \tilde{p}(v) \cdot E(V_{n-1} \mid V_n = v)} \qquad (*)$$

A rational bidder with valution v always chooses her bid b in such a way, that – conditional on this bid winning, the probability of which is $\tilde{p}(v)$ – her expected payment is equal to the expected value of the *second* highest valuation among all bidders. The latter is the correct (ex ante) opportunity cost of her winning, which she has to make up for with her payment. This opportunity cost is the valuation of that bidder, who would receive the item if the winner did not participate.

3 Vickrey auction and All-Pay auction

A particular and all-important feature of the Vickrey auction; i.e. the second-price winner-only-pays auction, is that it aligns its pricing rule directly with this always present (ex ante) equilibrium pricing feature of efficient auctions expressed by (*):

conditional on winning its rules *exactly* stipulate payment of the second-highest *bid*.

This immediately reveals the celebrated truth-telling property in (dominant!) equilibrium of the Vickrey auction: with $b_2(v)$ denoting the equilibrium bidding schedule a bidder with valuation $v \in [0, \bar{v}]$ must expect payment of

$$\bar{e}\big(b_2(v)\big) = \bar{p}\big(b_2(v)\big) \cdot E\big(b_2(V_{n-1})|V_n = v\big) \qquad (V)$$

But as shown above $\bar{e}\big(b_2(v)\big) = \tilde{e}(v)$ and $\bar{p}\big(b_2(v)\big) = \tilde{p}(v)$, and hence

$$\tilde{p}(v) \cdot E\big(b_2(V_{n-1}|V_n = v\big) = \tilde{p}(v) \cdot E(V_{n-1}|V_n = v) \qquad \text{for all } v,$$

which implies that $b_2(V_{n-1}) = V_{n-1}$ must hold in order to reconcile (V) with (*) as a special case. (This follows from the continuous relationship between the order statistic V_n and V_{n-1} as a function of v and continuity of b_2.) Thus $b_2(v) = v$, (*) is generally useful for a very economical (and slightly heuristic) derivation of equilibrium bidding strategies. Riley and Samuelson (1981) directly computed from it the solution of the first-price auction: since *only* the winner pays her bid $b = b_1(v)$, it must hold that

$$\tilde{e}(v) = \tilde{p}(v) \cdot b_1(v)$$

with $b_1(\cdot)$ denoting the equilibrium bidding strategy. This gives

$$b_1(v) = E(V_{n-1}|V_n = V) = v - \int_0^v \left(\frac{F(x)}{F(v)}\right)^{n-1} dx$$

; i.e. a bidder "shades" her bid below her valuation. This technique was further exploited by Wolfstetter (1995) to derive the solution $b_r(v)$ for the r-th price auction (see below). Obviously, bidders bid *less* than their true valuation in the first-price auction, because the opportunity cost of a bid is "overstated" by its pricing rule. As Wolfstetter (1995) shows, they bid *more* than their valuation in equilibrium of the n-th price auctions for $n \geq 3$, because then the opportunity cost is "understated" by the pricing rule. Only the Vickrey auction leads them to bid their true valuations, because its pricing rule is precisely in line with opportunity cost. To see this from (*) denote by $b_r(v)$ the equilibrium bidding schedule of the r-th price auction. Then

$$\bar{e}\big(b_r(v)\big) = \bar{p}\big(b_r(v)\big) \cdot E\big(b_r(V_{n-(r-1)})|V_n = v\big),$$

since only the winnner pays an amount equal to the expected value of the bid of the bidder with the r-th highest valuation (monotonicity of b_r) given the highest valuation is equal to the valuation of the player considered. The RET and (*) then imply that

$$E\big(b_r(V_{n-(r-1)})|V_n = v\big) = E(V_{n-1}|V_n = v) \qquad \text{for all} \quad v \in [0, \bar{v}].$$

$V_n > V_{n-1}$ resp. $V_{n-(r-1)} < V_{n-1}$, $r > 2$, now means that $b_1(V_1) < V_1$ (underbidding) resp. $b_r(V_{n(r-1)} > V_{n-(r-1)}$ (overbidding) in order to satisfy the above equation.

The Vickrey auction has for this reason often served as a "benchmark" for judging bidding behavior in auctions (by assessing and explaining "deviations" from truth-telling). This status of the Vickrey auction derives from *efficiency* considerations: an auction is efficient, if the buyer with the highest valuation gets the object in equilibrium of the auction game (here we assume absence of externalities in order to legitimately equate private and social benefits of the resulting allocation)[1]. Since bidding one's valuation is a (dominant) equilibrium in the second-price auction, efficiency is a most obvios property of it. But note, that *all* auctions covered by RET – by assumption of the theorem – satisfy efficiency. Also note, that it is not true – but often falsely attributed to the Vickrey auction – that truth-telling constitutes "non-strategic", sincere behavior in contrast to "bid-shading" (for strategic; i.e. "opportunistic" reasons) in other auction formats. The point is, of course, that the same strategic equilibrium considerations that lead to bid shading in other formats lead to truth-telling in the second-price format. In the following we propose a new 'reference auction' or benchmark, namely the first-price all-pay auction, henceforth FPAPA, and show that this gives a new access to comparative assessments of bidding behavior in different auctions.

In the FPAPA *all* bids submitted by the bidders – including the losing ones – have to be paid for by the respective bidders and – as before – the highest

[1] Jehiel and Moldovanu (1996), Jehiel, Moldovanu and Stacchetti (1996) and Caillaud and Jehiel (1998) consider the case with externalities among bidders.

bidder wins the prize. Bidding in this auction represents an *unconditional* commitment to pay the amount bid and therefore renders expected payment (of a bidder) into a *deterministic* variable (from *her* point of view as a decision maker). Her bid – and hence her (expected) payment – only depends on her *own* information and not on information of other bidders. The FPAPA is the only auction game $[p_i(b_1, \ldots, b_n), e_i(b_1, \ldots, b_n)]_{i=1}^n$ as defined in the beginning with the property, that other players' information only enters into a bidder's decision calculus via $p_i(\cdot)$, her probability of winning, but not via $e_i(\cdot)$, her expected payment (as $e_i(b_1, \ldots, b_n) = e_i(b_i) = e(b)$ in this case). More precisely, $e(b) = b$ and thus (*) immediately gives

$$\begin{aligned}
\bar{b}_1(v) &= : \tilde{e}(v) = \tilde{p}(v) \cdot E(V_{n-1} \mid V_n = v) \\
&= F(v)^{n-1} \cdot \left[v - \int_0^v \left(\frac{F(x)}{F(v)} \right)^{n-1} dx \right] \\
&= F(v)^{n-1} \cdot v - \int_0^v F(x)^{n-1} \, dx
\end{aligned}$$

For analysis of the FPAPA under various informational assumptions see Baye, Kovenock and Vries (1996), Amann and Leininger (1995, 1996) and Krishna and Morgan (1997).

From a decision-theoretic point of view bidding in an all-pay auction is akin to an insurance or gambling problem: the bidder makes an outlay with certainty (her bid) to "buy" or claim a chance (amount of probability) of winning, a situation well-examined and understood (see e.g. Hirshleifer and Riley, 1984). There is, however, a crucial difference: in a standard insurance or gambling problem the transfers of ressources across different contingent states occurs against the background of (exogenously) *fixed* probabilities. Here, in the interactive decision context of the FPAPA game the actions of all bidders *endogenously* determine the probabilities. More specifically, with the *unconditional* bid $\bar{b}_1(v)$ – like any other unconditional bid she could contemplate – the bidder de facto buys a lottery, that either pays v or 0. In *equilibrium* the lottery pays

v	with probability	$\tilde{p}(v) = F(v)^{n-1}$
0	with probability	$\left(1 - \tilde{p}(v)\right) = 1 - F(v)^{n-1}$

For this lottery the (risk-neutral) bidder does not pay $\tilde{p}\cdot v$, its expected value, but – due to the strategic interaction with the finitely many other bidders – somewhat less, namely $\tilde{p} \cdot E(V_{n-1} \mid V_n)$. (For $n \to \infty$; i.e. a "large" number of independent draws, this expression is seen to approach $\tilde{p} \cdot v$). This, in the case of the FPAPA very *natural* interpretation of buying a lottery at a certain price, is now shown to facilitate a general alternative explanation of bidding behavior in auctions. We start out from the observation that the odds $\tilde{p}(v) : (1-\tilde{p}(v))$ for winning/not winning are *invariant* over all auctions covered by the RET.

4 Bidding behavior from an all-pay viewpoint

Recall from above that in equilibrium of *all* auction games (covered by the Revenue Equivalence Theorem) a bidder with valuation $v \in [0, \bar{v}]$ faces the same odds of

$$p(v) = F(v)^{n-1} \qquad \text{for winning} \quad \text{and}$$

$$1 - \tilde{p}(v) \qquad \text{for not winning}$$

the item. At these *identical* odds every auction gane offers a *different* lottery in equilibrium. The nature and conditionality of different payment rules only leads to different *net* payments of the implied lotteries. We now systematically determine these "lotteries" and assess and compare their respective values to a risk-neutral bidder.
So think of a bid submitted by a player as a payment (purchasing price) for a lottery. Note then, that whereas in the FPAPA only probabilities of contingencies (in the acquired lottery) are determined endogenously by players' actions, it is in general probabilities *and* pay-offs in contingencies that are subject to joint determination.

E.g. in the first-price auction (winner-only-pays), FPA, the (equilibrium) bid, b, is expended for a lottery, that pays v with probability $\tilde{p}(v)$ and b with probability $\big(1 - \tilde{p}(v)\big)$. The pay-off in the case of *not* winning just reflects the fact, that in this contingency the bid is *refunded*. Obviously, the lottery a bidder "gets" in equilibrium of FPA dominates the one she gets in FPAPA as

the latter one also pays v for winning, but 0 for losing; accordingly her bid in the former is much higher than in the latter. The lower the lottery return in the contingency 'not winning', the higher the *commitment* of a player to her payment in a sunk cost sense. From this viewpoint bidding behavior across different auction games is easily understood:
Let the number of bidders, n, and the joint distribution $G(v_1, \ldots, v_n)$ of their valuations be given. For any auction (covered by RET) it is true, that

> a bidder with (private) valuation v, who bids b (according to a monotonic equilibrium bidding function), buys a lottery p:
>
> with probability $\tilde{p}(v)$, receive $v + a_1$
>
> with probability $(1 - \tilde{p}(v))$, receive $a_2 \cdot b - a_3$.

Note, that only the parameters a_1, a_2 and a_3 in the pay-offs depend on the specific auction rules, but not the respective odds. More precisely, it is the *payment rule* of an auction that, if different from the FPAPA-rule "each bidder pays her bid", *distorts* the pay-off structure from $(v, 0)$ in the (winning, not winning)- contingencies to $(v+a_1, a_2 \cdot b - a_3)$, with $a_1, a_3 \in [0, b]$ and $a_2 \in [0, 1]$, in a systematic way; i.e. $a_1 = a_2 = a_3 = 0$ in the FPAPA. Table I shows these lotteries for the r-th price auction, RPA, and the r-th price all-pay auction, RPAPA, for a given bid $b = b(v)$, that results from any common monotonic bidding function $b(\cdot)$. A common $b(\cdot)$ is sufficient to determine the odds at $p : (1 - p)$ with $p = \tilde{p}(v)$ for any $v \in [0, \bar{v}]$.

Table I: Lotteries offered by different auctions (in equilibrium)

bid	$b = b(v)$	
nature of bid	refundable (winner-only-pays)	
odds	p	$1 - p$
first-price auction	v	b
second price auction	$v + (b - E(B_{n-1} \| B_n = b))$	b
r-th price auction	$v + (b - E(B_{n-(r-1)} \| B_n = b))$	b

bid	$b = b(v)$	
nature of bid	non-refundable (all-pay)	
odds	p	$1 - p$
first-price auction	v	0
second price auction	$v + (b - E(B_{n-1} \vert B_n = b))$	0
r-th price auction	$v + (b - E(B_{n-(r-1)} \vert B_n = b))$	0

B_r denotes the order statistic for the r-th *lowest* bid; i.e. $E_r\big(B_{n-(r-1)}|B_n = b\big)$ is the expected value of the r-th *highest* bid given the highest is b.
Note, that we do not aim for exceptional realism when assuming for reasons of simplicity that in the RPAPA all bidders pay their bids except the highest bidder, who wins and pays the r th highest bid (which is the (n-(r-1))-th lowest bid), $r = 1, \ldots, n$. This may seem strange in that a loser may have to pay more than the winner! However, one could easily replace this simple rule by stipulating that all bidders pay their bids except for the (r-1) highest bidders, who pay the r-th highest bid. Now a loser would never have to pay more than the winner and could except a positive payment from the lottery, if losing, namely the average difference between her bid and the r-th highest bid in case her bid is among the (r-1) highest without actually being the highest. This is expected to happen with probability $\big(1 - F_{B_{n-(r-1)}}(b)\big)$ with $F_{B_{n-(r-1)}}$ denoting the distribution function of the 'r-th highest bid' - order statistic. This derivation just serves to illustrate, that *any* auction game can be viewed this way. The socalled 'sad loser' - auction is an interesting case in point: here the highest bidder wins the object and the lowest bidder – and *only* the lowest one – has to pay her bid; this auction features refundable as well as non-refundable bids. The pay-offs offered at odds $(p, 1-p)$ are in this case $(v + b, (1 - w) \cdot b)$ with $w = F(v)^{n-1} + \big(1 - F(v)\big)^{n-1}$, which makes $(1 - w)$ the probability of neither winning nor sadly losing the auction for a bidder of valuation v.(The sad-loser auction, too, is covered by RET.)

Let l_r denote the lottery offered by the RPA and let $\bar{l}_r$ denote the one offered by RPAPA. It is immediate from Table I:

Proposition 1: Any risk-neutral bidder i, $i = 1, \ldots, n$, endowed with a common bidding function $b(\cdot)$ would rank, for any $v \in [0, \bar{v}]$, the lotteries l_r and $\bar{e}_r$ as follows:

$$\begin{array}{ccccccc} l_1 & < & l_2 & < & \ldots & < & l_n \\ \vee & & \vee & & & & \vee \\ \bar{l}_1 & < & \bar{l}_2 & < & \ldots & < & \bar{l}_n \end{array}$$

I.e. l_n is the most attractive one and $\bar{l}_1$ the least attractive one, l_1 (=FPA) and l_2 (=Vickrey auction) are intermediate cases.

Of course, in equilibrium bidders – if bidding in different auction games – use *different* common bidding schedules. Yet by the RET they always expect the same (net) pay-off. Consequently Proposition 1 implies that

Corollary 2: Let $F(v)$ be given and denote by $b_r(v)$ resp. $\bar{b}_r(v)$ the equilibrium bids of a player with valuation v in RPA resp. RPAPA. Then the following holds for all $v \in [0, \bar{v}]$:

$$\begin{array}{ccccccc} b_1 & < & b_2 & < & \ldots & < & b_n \\ \vee & & \vee & & & & \vee \\ \bar{b}_1 & < & \bar{b}_2 & < & \ldots & < & \bar{b}_n \end{array}$$

Note that the "lowest" ranking is occupied by the first-price all-pay auction, FPAPA, in which expected payment only depends on a bidder's *own* information. We claim

Lemma 3: The FPAPA exhibits the lowest maximal equilibrium payment for a bidder of type v, $v \in [0, \bar{v}]$, among all auction games covered by RET.

Proof: Laffont and Roberts (1996) as well as Maskin (2000) show, that for any auction game A, for which a bidder's payment does not depend on her type *exclusively* (but also on wether she wins or loses), there exists an equivalent all-pay auction game $\bar{A}$, that induces the same expected payment and allocation at a lower maximal payment paid by each type of bidder. It is easy to see that among those FPAPA with $e_i(b_1, \ldots, b_n) = b_i$ gives the lowest (maximal) payments.

This property of the all-pay mechanism features prominently, when bidders are budget-constrained (see e.g. Laffont and Robert (1996), Che and Gale 1995) and Maskin (2000)): low equilibrium payments reduce the prospect that those constraints become binding and lower their impact, if they are binding. FPAPA may even be an optimal auction from the seller's point of view under those circumstances (Laffont and Robert, 1996).

A useful implication of our analysis so far is, that FPAPA is strategically equivalent to FPA. More precisely,

Proposition 4: With independent private values the first-price all-pay auction, FPAPA, and the first-price auction, FPA, are *strategically equivalent* in the following sense:
If $b_1(v)$ is an equilibrium bidding strategy of the FPA then
$\bar{b}_1(v) = \tilde{p}(v) \cdot b_1(v)$ is an equilibrium bidding strategy of the FPAPA.

Proof: Obvious from (*) as $b_1(v) = E(V_{n-1}|V_n = v)$ and $\bar{b}_1(v) = \tilde{e}(v)$.

Proposition 4 says that a player who places a *conditional* bid of $b_1(v)$ in equilibrium of the FPA submits an *unconditional* bid of $\tilde{p}_1(v) \cdot b_1(v)$, with $\tilde{p}(v)$ exactly describing her probability of winning, in the FPAPA. A little further reflection leads to the intuitive insight, that

$$b_1(v) = \int_0^v x \cdot f_{n-1}|_{V_n=v}(x)\ dx \qquad \text{and}$$

$$\bar{b}_1(v) = \int_0^v x \cdot f_{n-1}(x)\ dx \qquad \text{must hold.}$$

Here $f_{n-1}\ |_{V_n=v}\ (x)$ denotes the conditional density of the second-highest valuation, whereas $f_{n-1}(x)$ denotes the (unconditional) density of the highest valuation of the players other than the player holding valuation v (these are (n-1) bidders). We then readily confirm that the two densities just differ by a factor $\tilde{p}(v) = F(v)^{n-1}$ as

$$\begin{aligned} f_{n-1}|_{V_n=v}(x) &= (n-1) \cdot \frac{F(x)^{n-2}}{F(v)^{n-1}} \cdot f(x) \qquad \text{(see above) and} \\ f_{n-1}(x) &= (n-1) \cdot F(x)^{n-2} \cdot f(x) \end{aligned}$$

(just differentiate the distribution function of the maximum of (n-1) i.i.d. random variables distributed according to $F(x)$, which is $F(x)^{n-1}$).

Remark:
It is not true for $r > 1$, that RPA and RPAPA are strategically equivalent. To see this consider the expressions for the expected payment of a bidder who bids according to $b_r(v)$ in RPA resp. $\bar{b}_r(v)$ in RPAPA, which read

$$\begin{aligned} e\big(b_r(v)\big) &= p(b_r(v)) \cdot E\big(b_r(V_{n-(r-1)})|V_n = v\big) \qquad \text{and} \\ e\big(\bar{b}_r(v)\big) &= p(\bar{b}_r(v)) \cdot E\big(\bar{b}_r(V_{n-(r-1)})|V_n = v\big) + \big(1 - p(\bar{b}_r(v))\big) \cdot \bar{b}_r(v). \end{aligned}$$

RET yields that $e\big(b_r(v)\big) = e\big(\bar{b}_r(v)\big) = \tilde{e}(v)$, for all $v \in [0, \bar{v}]$, and $p\big(b_r(v)\big) = p\big(\bar{b}_r(v)\big) = \tilde{p}(v)$, hence (*) implies that

$$\begin{aligned} &\text{(RPA)} \qquad E\big(b_r(V_{n-(r-1)})|V_n = v\big) = E(V_{n-1}|V_n = v) \qquad \text{and} \\ &\text{(RPAPA)} \qquad E\big(\bar{b}_r(V_{n-(r-1)})|V_n = v\big) + \frac{1 - \tilde{p}(v)}{\tilde{p}(v)} \cdot \bar{b}_r(v) = E(V_{n-1}|V_n = v) \end{aligned}$$

Consequently, the relationship between bids in RPA and RPAPA is governed by the equation

$$E\big(b_r(V_{n-(r-1)})|V_n = v\big) = E\big(\bar{b}_r(V_{n-(r-1)})|V_n = v\big) + \frac{1 - \tilde{p}(v)}{\tilde{p}(v)} \cdot \bar{b}_r(v) \quad (E)$$

for all $v \in [0, \bar{v}]$

Note that the expectation operators on both sides of the equation are applied to the same (conditional) distribution. For $r = 1$ – and only in this case – this distribution collapses into the single point v:

$$\begin{aligned} E\big(b_r(V_n)|V_n = v\big) &= E\big(\bar{b}_r(V_n)|V_n = v\big) + \frac{1-\tilde{p}(v)}{\tilde{p}(v)} \cdot \bar{b}_r(v) \\ \text{becomes} \qquad b_r(v) &= \bar{b}_r(v) + \frac{1-\tilde{p}(v)}{\tilde{p}(v)} \cdot \bar{b}_r(v) \\ &= \frac{1}{\tilde{p}(v)} \cdot \bar{b}_r(v) \end{aligned}$$

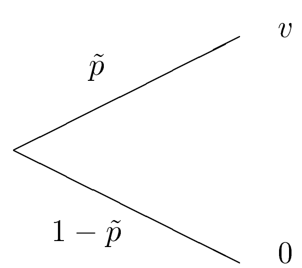
For $r > 1$ no such reduction is possible, instead, setting $b_r(v) = a(v) \cdot \bar{b}_r(v)$ and inserting this into (E) yields

$$a(v) = 1 + \frac{1-\tilde{p}(v)}{\tilde{p}(v)} \cdot \frac{\bar{b}_r(v)}{E\big(\bar{b}_r(V_{n-(r-1)})|V_n = v\big)}$$

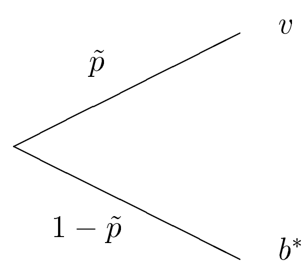
and we see, that knowing the bid of type v in the RPAPA does not suffice to infer her bid in the RPA, one has to know the bidding schedule $\bar{b}_r(\cdot)$.

5 Applying the all-pay view

In the light of Table I the stategic equivalence of FPAPA and FPA (Proposition 4) can be put as follows:
equilibrium of FPAPA offers to a bidder with valuation v the lottery $\bar{l}_1$:



equilibrium of FPA offers to the same bidder (under the same circumstances regarding the probabilistic law of valuations) the lotery l_1:



with $b^* = b(v)$.

As a risk-neutral bidder (with valuation v) can expect the same expected pay-off from participation in each auction (RET), he should in effect be indifferent between buying either of the two lotteries. I.e. if he prices (i.e. buys) lottery $\bar{l}_1$ at $\bar{b}$ (with his bid in FPAPA) then he should price l_1 at $b = \frac{1}{\tilde{p}} \cdot \bar{b}$ (and consequently bid this in FPA), because

$$\tilde{p} \cdot v - \bar{b} = \tilde{p} \cdot v + (1-\tilde{p}) \cdot b^* \quad \text{if and only if} \quad b = \frac{1}{\tilde{p}} \cdot b^*.$$

Conversely, paying b^* for l_1 – as is done with fair accurancy on average in FPA-experiments (see e.g. Davis and Holt (1993), chapter 8, or Kagel and Roth, 1995, chapter 8) – should result in paying $b = \tilde{p} \cdot b^*$ for $\bar{l}_1$. This simple "discounting" with the probability of winning, $\tilde{p}$, is definitely not observed in FPAPA-experiments (see Amann and Leininger (1997) and Barut, Kovenock and Noussair (1999)).

Amann and Leininger (1997) report substantial overbidding in the (single-unit) FPAPA with revenues between 1.6 and 2.5 times the revenue of the FPA experiments; i.e. revenue equivalence is clearly rejected. Barut et al. (1999) show that with multiple units this marked difference between FPAPA and FPA disappears, they become *empirically* revenue equivalent. The bidding pattern over bidders with different valuations is in rough accordance with symmetric Bayesian equilibrium in the case of FPA, but not in the case of FPAPA, neither in the single-unit case of Amann and Leininger (1997) nor in the multi-unit case of Barut et al. (1999). Bidding in the FPAPA follows a dichotomous pattern in the following sense: for high valuations bidding is frequently almost as close to the valuation as in the respective FPA (and therefore much too high compared to Bayesian equilibrium); for low and medium valuations it is frequently close to zero or zero. Bidders either "go" for the unit or "stay out" of the competition. In the single-unit case of Amann and Leininger (1997) two bidders on average go for the item, which generates the overexpenditures mentioned above. In the multiple-unit case of Barut et al. this dichotomy in bidding behavior is also present in a most pronounced way. But the consequences in terms of overexpenditures get – statistically – completely washed out by the fact, that now more units are available. Whereas Amann and Leininger find, that with 6 bidders for a single unit revenue in FPAPA is about twice the revenue of FPA, Barut et al. conclude with 6 bidders for *two* units that FPA and FPAPA are revenue-equivalent. In both studies FPA performs much more efficient in allocating the unit(s) than FPAPA, so even in the revenue-equivalent case of Barut et al. bidders prefer to participate in FPA rather than FPAPA.

Applying the all-pay view of auctions as lotteries may help to explain this kind of behavior: there is a substantial literature on choice over binary lotteries, in which several regularities in choice behavior – some of them inconsistent with the axioms of expected utility theory – have been observed (see e.g. Holt and Davis (1993), chapter 8 or Kagel and Roth (1995), chapter 8). This large body of literature has – at least to the author's knowledge – not been linked to behavioral results of – equally large – literature on auction experiments. (At least not in a direct way, there is a literature on the use of lotteries in order to induce certain risk-preferences in auctions (see e.g. Selten, Sadrieh and Abbink, 1999), which has an entirely different focus.) The present approach to view auctions as purchases of (equilibrium) lotteries by bidders suggests, that this could beneficially be done. E.g. if we interpret the bids in FPA and FPAPA as the values attached to the equilibrium lotteries by bidders, then revenue dominance of FPAPA over FPA means, that subjects prefer the former to the latter. However, when asked which auction game they prefer to play, they choose the latter. This is akin to the phenomenon of "preference reversal" (Grether and Plott, 1979) observed in *individual* choice over lotteries. It is not clear, what causes this behavior in the "auction lotteries": standard explanations of "preference reversal" like involvement of prospect theory do not work as the probabilities of winning are identical in both lotteries. One possibility is to draw on findings of Selten, Sadrieh and Abbink (1999). Their result on choice and valuation of binary lotteries led them to postulate a "background-risk effect": the higher the variance in lottery pay-offs, the more pronounced is the presence of uncertainty in the perception of the decision-makers. As a consequence they become more risk-sensitive; i.e. more attention is paid to the presence of uncertainty, which results in less stable, almost erratic behavior (Selten et al. coin the term "capriciousness"), that changes by leaps and bounds between risk-loving and risk-averse. The overall result is a larger derivation from expected value maximization. Uncertainty is much more prominent in the lottery $\bar{l}_1$ bought in FPAPA than it is in l_1 bought in FPA. The larger variance in pay-offs and the possibility of suffering a loss (ex post) in FPAPA clearly triggers risk-sensitivity in bidders perception. At the same time the presence of uncertainty in l_1 of FPA may get repressed by an impression of being " in control" of the variance in pay-offs, because it depends on the chosen bid. This prevents bidders from realizing that application of the simple linear discounting rule suggested by Proposition 4 is in their best interest.

6 Conclusion

We have viewed an auction as a process that generates (in equilibrium) a binary lottery for each bidder, which the bidder " buys" and codetermines with his bid. The first-price all-pay auction is shown to occupy a prominent role in this view: it offers a "benchmark" lottery, in which pay-offs are *independent* of bidders' behavior. Variation of the auction format leads to variations in these lotteries. A particular informative class of variations is the one across the set of auctions covered by the revenue equivalence theorem. They all generate lotteries with probabilities identical to those of the all-pay auction lottery, so that differences between them can be understood by assessing deviations in their pay-offs from the benchmark lottery.

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