A joint Initiative of Ludwig-Maximilians-Universität and Ifo Institute for Economic Research

Working Papers

WELFARE COMPARISONS: SEQUENTIAL PROCEDURES FOR HETEROGENEOUS POPULATION

Peter J. Lambert
Xavier Ramos*

CESifo Working Paper No. 519

July 2001

CESifo
Center for Economic Studies & Ifo Institute for Economic Research
Poschingerstr. 5, 81679 Munich, Germany
Phone: +49 (89) 9224-1410 - Fax: +49 (89) 9224-1409
e-mail: office@CESifo.de
ISSN 1617-9595

* The authors wish to thank Efe OK and Udo Ebert for their interest and valued comments.

CESifo *Working Paper No. 519*
July 2001

WELFARE COMPARISONS: SEQUENTIAL PROCEDURES FOR HETEROGENEOUS POPULATIONS

Abstract

Some analysts use sequential dominance criteria, and others use equivalence scales in combination with non-sequential dominance tests, to make welfare comparisons of joint distributions of income and needs. In this paper we present a new sequential procedure which copes with situations in which sequential dominance fails. We also demonstrate that the recommendations deriving from the sequential approach are valid for distributions of equivalent income whatever equivalence scale the analyst might adopt. Thus the paper marries together the sequential and equivalizing approaches, seen as alternatives in much previous literature. All results are specified in forms which allow for demographic differences in the populations being compared.

JEL Classification: D63, I31.

Peter J. Lambert
Economics Department
University of York
York YO10 5DD
England
PJLI@York.AC.UK

Xavi Ramos
Departament d'Economia Aplicada
Universitat Autonoma de Barcelona
Edifici B
08193 Bellaterra (Barcelona)
Spain

1. Introduction

The dominance criteria of Atkinson (1970) and Shorrocks (1983) have become well-known, and are now widely used for making welfare comparisons on the basis of income distribution data. These approaches, though, do not take into account the sort of non-income information - such as family size, age, type of housing - which is these days available in plenty in micro data sets, and may be of welfare relevance. Hence the old results have begun to be viewed as of limited usefulness. One could not, for example, use the generalized Lorenz dominance approach to recommend as welfare-improving the transfer of income from single persons to families with children, or to those with special needs such as old age or infirmity.

In response to this perceived shortcoming, Atkinson and Bourguignon (1987) developed their sequential generalized Lorenz dominance criterion, for the comparison of joint distributions of income and needs, the latter assumed to be an ordinal variable, and this criterion has been found broad enough for some operational purposes. There is now a flourishing literature on the sequential approach. One thinks for example of Atkinson's (1990) illustrative account, Jenkins and Lambert's (1993) extension to allow for demographic change, Chambaz and Maurin's (1998) comprehensive survey and introduction of additional sequential procedures, and the exploration of the welfare fundamentals by Ok and Lambert (1999).

Meanwhile, other analysts have been using equivalence scales, to deflate incomes and supposedly render distributions socially homogeneous, and going on to apply the old results. This begs interpretation of the income measure in the Atkinson and Shorrocks theorems, and sole argument in the assumed welfare functions, as equivalent income. However, Ebert (1997) has warned against a too naive approach; if attention is not also paid to the appropriate weighting of income units, the implicit transfer principle is a non-implementable one (of units of living standard), rather than the implementable one involving money transfers from those with higher living standards to those with lower ones. In Ebert (1997, 1999) it is shown how to adapt the generalized Lorenz dominance methodology to this situation using artificial populations of fictional 'equivalent adults', and in Ebert and Moyes (2000) this procedure is pinned down as the only method, in a wide class of possible equivalizing transformations and weighting schemes, which is consistent with normative requirements. Yet there remain conceptual and practical problems in constructing equivalence scales (see *e.g.* Banks and Johnson, 1994).

Fleurbaey *et al.* (2000) articulate a transfer principle that should hold for betwen-group transfers for all equivalence scales in a bounded range, and present the appropriate dominance criterion for welfare functions respecting this principle. The criterion is not sequential. Fleurbaey *et al*. (2000, p. 4) describe it as "a middleway criterion" between Ebert's generalized Lorenz criterion and Atkinson and Bourguignon's sequential generalized Lorenz criterion.

In this paper, we draw together all of the sequential welfare comparison procedures which have been enunciated in the literature, and present a new one which copes with situations in which sequential dominance fails. We then go on to explain how *these same sequential procedures* can be used by those who advocate the use of equivalence scales, to avoid the actual business of choosing a particular equivalence scale, or even a range of values for the equivalence scale deflators. The paper thus makes a contribution to the effort to better understand and marry together these two distinct approaches to making welfare comparisons in the presence of social heterogeneity.

The structure of the paper is as follows. In section 2, we lay out some basic definitions and preliminaries, in terms of which the analysis proceeds. In section 3, simple manipulations are used, involving combinatorics and integration, to obtain the analytical relationships which drive all the subsequent results. In Section 4, we show how the needs dimension can be entered into the social utility-of-income function through certain sets of restrictions, one of which is new. In section 5, the conditions are obtained under which welfare is unambiguously higher for one joint distribution of income and needs than another. Sequential generalized Lorenz dominance emerges as one appropriate criterion. A stronger condition, sequential rank dominance, also emerges, as does a new and much weaker criterion appropriate for cases where generalized Lorenz curves cross somewhere in the sequence. Section 6 contains an example to show that the new criterion is effective: it provides an intuitively agreeable welfare ranking in a simple case where the other criteria both fail. In section 7, the results are respecified to allow for demographic differences. In section 8, we show how the sequential procedures generate recommendations which are valid for distributions of equivalent income, whatever the equivalence scale. Section 9 concludes.

2. Definitions and Preliminaries

We suppose that there are n household ty2pes, differentiated and ranked by needs, type i=1 being judged the neediest in a specific welfare sense to be discussed in section 4. Household money income distributions F and G will be compared, with overall distribution functions F(x) and G(x) where $x \in \mathbf{R}_+$. The type-specific distribution functions associated with F and G will be $F_i(x)$ and $G_i(x)$, and the density functions will be $f_i(x)$ and $g_i(x)$, $1 \leq i \leq n$. The proportion of households belonging to each type i will be assumed the same in F and G for most of the paper, and written p_i. In extending the results to allow for demographic differences, we will later rewrite these proportions p_{iF} and p_{iG}.

Social welfare functions (henceforth SWFs) will be assumed additively separable over money incomes, with different utility functions $U_i(x)$, assumed differentiable, being applied to the different types:

(1) $W_F = \Sigma_{1\le i\le n} p_i \int_0^z U_i(x) f_i(x)dx, \ W_G = \Sigma_{1\le i\le n} p_i \int_0^z U_i(x) g_i(x)dx$

Here z denotes the highest income present in either F or G; z could be an arbitrary income level exceeding this maximum, with no effect on measured welfares. The needs structure will be expressed by conditions relating the utility functions $U_j(x)$ and $U_{j+1}(x)$ of adjacent types, $1\le j\le n-1$. In Ok and Lambert (1999) the assumption of additivity of the SWF across types is relaxed, but we do not pursue that line here. The bundle of utility functions which specify the welfare function, $<U_1,U_2,\ldots U_n>$, will be called a *utility profile*.

3. Signing Welfare Differences

The starting point for the analysis is the formula:

(2) $W_F - W_G = \Sigma_{1\le i\le n} p_i \int_0^z U_i(x)[f_i(x)-g_i(x)]dx$

quantifying the welfare superiority of F over G, which derives from (1). By rewriting this expression in a number of different ways, a range of sequential tests can be identified, in terms of the type-specific distribution functions $F_i(x)$ and $G_i(x)$, $1\le i\le n$, which, if successful, will ensure that W_F-W_G is signed unambiguously positive for appropriate classes of SWFs W (equivalently, for appropriate classes of utility profiles $<U_1,U_2,\ldots U_n>$, for these define the SWFs).

We begin by integrating by parts in (2), and reversing the order of summation and integration:

(3) $W_F - W_G = \Sigma_{1\le i\le n} p_i \int_0^z U_i'(x)[G_i(x)-F_i(x)]dx = \int_0^z \{\Sigma_{1\le i\le n} U_i'(x)[p_i S_i'(x)]\}dx$

where $S_i(x)$ is defined by:

(4) $S_i(x) = \int_0^x [G_i(y)-F_i(y)]dy$.

Next we apply a combinatoric argument[2] to the integrand in the right hand side of (3), and again reverse the order of summation and integration:

(5) $W_F - W_G = \Sigma_{1\le j\le n} \{\int_0^z D_j(x) T_j'(x)dx\}$

where $D_j(x)$ and $T_j(x)$ are defined, for $1\le j\le n$, by:

---

[2] We refer here to the partial summation formula of Abel, according to which, given real numbers v_i and w_i, $1\le i\le n$, $\Sigma_{1\le i\le n} v_i w_i = \Sigma_{1\le j\le n} d_j t_j$, where $d_j = v_j - v_{j+1}$ for $1\le j\le n-1$, $d_n = v_n$, and $t_j = \Sigma_{1\le i\le j} w_i$ for $1\le j\le n$.

(6) $D_j(x) = U_j'(x)-U_{j+1}'(x)$ for $1 \leq j \leq n-1$, $D_n(x) = U_n'(x)$

and:

(7) $T_j(x) = \Sigma_{1\leq i\leq j}\, p_i S_i(x)$

respectively. Integrating by parts again, (5) becomes:

(8) $W_F-W_G = \Sigma_{1\leq j\leq n} \{D_j(z)T_j(z) - \int_0^z D_j'(x)T_j(x)dx\}$

and integrating by parts yet again, it can equivalently be written:

(9) $W_F-W_G = \Sigma_{1\leq j\leq n} \{D_j(z)T_j(z) - D_j'(z)\int_0^z T_j(x)dx + \int_0^z D_j''(y)\{\int_0^y T_j(x)dx\}dy\}$

Expressions (5), (8) and (9) will furnish the conditions for welfare superiority of F over G which form the core of the paper. Distributional conditions will be expressed in terms of the functions $T_j(x)$; normative assumptions about inequality aversion and the needs structure will come through restrictions on the marginal utility differences $D_j(x)$.

4. The Needs Structure

Systematic differences between the utility functions $U_j(x)$ and $U_{j+1}(x)$ of adjacent household types ($1\leq j\leq n-1$) embody the social judgements about needs. Consider this hypothetical question. A new unit of resource is made available (call it $1); among households at a given income level x, to a household of which of the two types j and j+1 would it be socially more advantageous to give the $1? From (6), $D_j(x) = U_j'(x)-U_{j+1}'(x)$ is the additional social merit in awarding the $1 to a household of type j with an income of x, over a household of the next-less-needy type j+1 at the same income level.

If $D_j(x) > 0\ \forall j\leq n-1, \forall x$ then awards are always more efficient in welfare terms when given to a needier, rather than an adjacent less needy, household at each income level. If we impose also the condition $D_n(x) > 0\ \forall x$ then, because $D_n(x) = U_n'(x)$, all utility functions necessarily have positive first derivatives, a condition we might well have imposed *ab initio*. Let $\mathbf{U_1} = \{<U_1,U_2,....\ U_n> : D_j(x) > 0\ \forall j, \forall x\}$, and let $\mathbf{W_1}$ be the class of SWFs based on utility profiles in $\mathbf{U_1}$.

If in addition $D_j'(x) < 0\ \forall j\leq n-1, \forall x$, then the extra social value in granting a new unit of resource to a needier household at each income level declines with increases in that income level: among rich households, it hardly matters who would get it. This assumption about how needs and income should be related was introduced by Atkinson and Bourguignon (1987). Since $D_n'(x) = U_n''(x)$, adding the condition $D_n'(x) < 0\ \forall x$ ensures concavity of all utility functions, i.e. inequality aversion within types. This is also assumed by Atkinson and Bourguignon. Let $\mathbf{U_2}$ =

$\{<U_1,U_2,.... U_n> \in \mathbf{U_1} : D_j'(x) < 0\ \forall j, \forall x\}$, and let $\mathbf{W_2}$ be the class of SWFs based on utility profiles in $\mathbf{U_2}$.

If also $D_j''(x) > 0\ \forall j \leq n-1, \forall x$, then as between two types j and j+1, it is judged *even better* to give the extra \$1 to a household of type j at *a lower income level*, rather than at a higher one. If we add the condition $D_n''(x) > 0\ \forall x$ then, because $D_n''(x) = U_n'''(x)$, all utility functions now have positive third derivatives, *i.e.* satisfy Kolm's (1976) Principle of Diminishing Transfers for within-group redistributions: small transfers across a fixed income gap are judged better at lower income levels than higher ones. This is consonant with the between-group judgements. Under these assumptions, for each group the social marginal valuation of income decreases at a decreasing rate, as does the difference in social marginal valuation of income between any needy and less needy group. Let $\mathbf{U_3} = \{<U_1,U_2,.... U_n> \in \mathbf{U_2} : D_j''(x) > 0\ \forall j, \forall x\}$, and let $\mathbf{W_3}$ be the class of SWFs based on utility profiles in $\mathbf{U_3}$. The welfare class $\mathbf{W_3}$ has not, to our knowledge, been proferred as normatively interesting before.

5. Distributional Conditions

The central new result of the paper now drops out very simply, along with the existing sequential dominance tests of Atkinson and Bourguignon (1987) which have already been alluded to. Just inspect formulae (5), (8) and (9), for the welfare difference W_F-W_G, and piece together the sign properties assumed of the functions $D_j(x)$ for the three welfare classes $\mathbf{W_1}$, $\mathbf{W_2}$ and $\mathbf{W_3}$ respectively with corresponding appropriate sign properties for the functions $T_j(x)$, to obtain sufficient conditions under which all terms in the right hand sides of (5), (8) and (9) are positive:

Theorem 1

(1) [Atkinson and Bourguignon, 1987, Proposition 2] $W_F \geq W_G\ \forall W \in \mathbf{W_1} \Leftrightarrow T_j'(x) \geq 0\ \forall j, \forall x \in [0,z]$;

(2) [Atkinson and Bourguignon, 1987, Proposition 3] $W_F \geq W_G\ \forall W \in \mathbf{W_2} \Leftrightarrow T_j(x) \geq 0\ \forall j, \forall x \in [0,z]$;

(3) $W_F \geq W_G\ \forall W \in \mathbf{W_3} \Leftrightarrow T_j(z) \geq 0$ and $\int_0^y T_j(x)dx \geq 0\ \forall j, \forall y \in [0,z]$.

The first two of these sequential distributional tests are due to Atkinson and Bourguignon (1987), who also show necessity. For necessity in condition *(3)*, see the appendix. The sequential conditions can, in each case, can be checked in descending order by j, first, when j=1, for the neediest type; second, when j=2, for the two neediest types taken together; and so on, up to the final stage j=n, when all types are merged. We consider each test in turn in a moment.

Consider first what theorem 1 says if the population would be socially homogeneous, with no perceived differences in household needs. Then the case j=1 is the only one, and $T_1(x)$ can be set equal to $\int_0^x [G(y)-F(y)]dy$. The tests in parts *(1)*, *(2)* and *(3)* of theorem 1, on the derivative, level and integral of this function, correspond to those for first, second and third order stochastic dominance in the uncertainty literature (Whitmore, 1970, Brumelle and Vickson, 1975). In the income distribution context, the criterion in *(1)* is known as *rank dominance* (Saposnik, 1981), and that in *(2)* as *generalized Lorenz dominance* (Shorrocks 1983). The former is more demanding than the latter, but has been found almost as successful for international comparisons by Bishop *et al.* (1991). The criterion in *(3)*, which corresponds to Shorrocks and Foster's (1987) transfer sensitivity in the socially homogeneous case (see also the little-known Atkinson, 1973 and Davies and Hoy, 1994 on this), does not translate into a simple dominance criterion, but is satisfied, in particular, when the generalized Lorenz curves for F and G cross once, with F's initially dominant, provided F has the same mean as G and no higher a variance (Dardanoni and Lambert, 1987, Shorrocks and Foster, 1987). In Lambert (1993), sufficient conditions are given for *(3)* to hold when generalized Lorenz curves cross more than once; see also Davies and Hoy (1995).

The same tests can be applied sequentially in the heterogeneous case, in the merged subpopulations comprising the j most needy subgroups, $1 \le j \le n$. This is clear because, if ${}^jF(x)$ and ${}^jG(x)$ are the distribution functions for income in the j^{th} merged subpopulation, so that $[\Sigma_{1\le i\le j} p_i].{}^jF(x) = \Sigma_{1\le i\le j} p_iF_i(x)$ and $[\Sigma_{1\le i\le j} p_i].{}^jG(x) = \Sigma_{1\le i\le j} p_iG_i(x)$, the conditions for first, second and third degree dominance of F over G are in terms of the function $\int_0^x [{}^jG(y)-{}^jF(y)]dy$ which, from (4) and (7), equals $T_j(x)/\Sigma_{1\le i\le j} p_i$. Hence $T_j'(x) \ge 0\ \forall x$ corresponds to rank dominance in this merged subpopulation; $T_j(x) \ge 0\ \forall x$ corresponds to generalized Lorenz dominance; and the mean-variance condition is sufficient for $T_j(z) \ge 0$ and $\int_0^y T_j(x)dx \ge 0\ \forall y$ if the generalized Lorenz curves cross once with ${}^jF$'s (*i.e.* the one induced by F) initially dominant:

Corollary

(1) [Atkinson and Bourguignon, 1987] $W_F \ge W_G\ \forall W \in \mathbf{W_1} \Leftrightarrow$ for each set of the j most needy subgroups, $1 \le j \le n$, ${}^jF$ rank dominates ${}^jG$;

(2) [Atkinson and Bourguignon, 1987] $W_F \ge W_G\ \forall W \in \mathbf{W_2} \Leftrightarrow$ for each set of the j most needy subgroups, $1 \le j \le n$, ${}^jF$ generalized Lorenz dominates ${}^jG$;

(3) If sequential generalized Lorenz dominance as in *(2)* is not satisfied, but at each stage j for which it fails, *(a)* the two generalized Lorenz curves cross once with ${}^jF$'s initially dominant, *(b)* the means are the same, and *(c)* ${}^jF$ has no higher a variance than G, then $W_F \ge W_G\ \forall W \in \mathbf{W_3}$.

The criterion of *(1)* is known as sequential rank dominance and theat of *(2)* as sequential generalized Lorenz dominance. Both were identified by Atkinson and Bourguignon (1987) as equivalent to the tests of parts *(1)* of theorem 1 (see also Atkinson, 1990 on this). The sequential test in *(3)* has not been articulated before. It could be extended to cover cases of multiple generalized Lorenz intersections. All of these tests involve familiar constructions: distribution functions (for rank dominance), generalized Lorenz curves, means and variances. The tests of theorem 1 are in terms of the less familiar functions $T_j(x)$, $1 \leq j \leq n$, but these tests are just as easy to implement as those of the corollary using household survey microdata.

Bourguignon (1989) identifies a dominance criterion that accords with a welfare class we might call $\mathbf{W}_{1½}$, in which the type-specific utility functions have the needs structure $D_j(x) > 0$ $\forall j, \forall x$ of our $\mathbf{W_1}$ and are also concave, as in our $\mathbf{W_2}$. No restrictions are placed on the $D_j'(x)$, $1 \leq j \leq n-1$, for $\mathbf{W}_{1½}$ however. It is easy to see that in $\mathbf{W}_{1½}$, all rich-to-poor transfers from less needy to needier households are approved; this is therefore the case in $\mathbf{W_2}$ and $\mathbf{W_3}$ also. Ebert (2000) characterizes $\mathbf{W}_{1½}$ and $\mathbf{W_2}$ axiomatically by means of transfer principles. Bourguignon's criterion for $\mathbf{W}_{1½}$ is "not easy to evaluate", involving, effectively, "all ordered vectors of [possible] poverty limits" (*ibid*, page 74). His test will succeed in some cases in which sequential rank dominance fails, but never when sequential generalized Lorenz dominance fails. Bourguignon suggests a numerical algorithm for the test.

6. An Example

A simple example shows the potential usefulness of the criterion in part *(3)* of theorem 1. The example we select is one for which the sufficient condition of part *(3)* of the corollary fails. As shown in table 1, suppose that in the (initial) scenario G, three couples have money incomes of 4, 6 and 8, and three single persons each have 6.

Table 1

	couples			singles		
G	4	6	8	6	6	6
F	6	8	10	5	5	5

The change from G to F is brought about by taking \$1 from each single and giving it, with another \$1 from the outside, to each couple. Most analysts would, we believe, favour the \$1 money transfer from each single to each couple. The granting of a further \$1 of resource to each couple is a Pareto improvement in this worse-off group, and would be favoured over the wider population by all welfare functions satisfying the strong Pareto principle. Hence the move from G to F could be seen as a welfare improvement. However, the generalized Lorenz curves for F and G cross, so that sequential rank and generalized Lorenz dominance both fail at the second (final) stage - though of course recording an improvement at the first stage. Bourguignon's test thus also fails. In fact, the generalized Lorenz curves for F and G cross *twice*, see figure 1a: the criterion of the corollary, part *(3)* therefore fails too. But the sequential test of theorem 1, part *(3)*, does not fail: the two T-functions of that test, $T_1(x)$ and $T_2(x)$, satisfy the conditions stated. In fact $T_1(x) \geq 0 \ \forall x$ and, although $T_2(x)$ goes negative in the range $5½ < x < 7$, $T_2(z) \geq 0$ and $\int_{?}^{y} T_2(x)dx \geq 0 \ \forall y$: see figure 1b. Hence $W_F \geq W_G \ \forall W \in \mathbf{W_3}$.

[FIGURE 1 ABOUT HERE]

This example verifies that the test of part *(3)* of theorem 1 is applicable in normatively interesting cases, and that the corresponding test of the corollary may fail in some such cases. It also establishes a firm interest in the welfare class $\mathbf{W_3}$, in which the Principle of Diminishing Transfers and its between-groups counterpart both feature, each advocating that the benefits of distributional changes be directed towards family units at lower rather than higher income levels. $\mathbf{W_3}$ is an appropriate class for welfare analysis, and the T-functions of theorem 1 are an appropriate distributional tool.

7. Demographic Differences

Theorem 1 can be extended to cope with demographic differences between the distributions being compared with an additional normative assumption. Letting p_{iF} and p_{iG} be the proportions of households of type i in distributions F and G respectively, the extension involves simply replacing the function $T_j(x)$ of (7) (see also (4)) by a new one:

(10) $T_j^*(x) = \Sigma_{1\le i\le j} \int_0^x [p_{iG}G_i(y)-p_{iF}F_i(y)]dy$,

provided one accepts that the $U_i(z)$ all converge to the same value as $z \to \infty$.[3] This assumption means that the social decision-maker does not care about the family types of the super-rich. Taken along with the other properties assumed of the utility profiles in our welfare classes, this additional assumption implies $U_1(x) \le U_2(x) \le .. \le U_n(x)\ \forall x$. Denote by $\mathbf{W_1}$*, $\mathbf{W_2}$* and $\mathbf{W_3}$* the subclasses of $\mathbf{W_1}$, $\mathbf{W_2}$ and $\mathbf{W_3}$ in which the additional property holds for all utility profiles.

Theorem 1*

(1) [Jenkins and Lambert, 1993, Chambaz and Maurin, 1998] $W_F \ge W_G\ \forall W \in \mathbf{W_1^*} \Leftrightarrow T_j^{*\prime}(x) \ge 0\ \forall j, \forall x \in [0,z]$;

(2) [Chambaz and Maurin, 1998] $W_F \ge W_G\ \forall W \in \mathbf{W_2^*} \Leftrightarrow T_j^*(x) \ge 0\ \forall j, \forall x \in [0,z]$;

(3) $W_F \ge W_G\ \forall W \in \mathbf{W_3^*} \Leftrightarrow T_j^*(z) \ge 0$ and $\int_0^y T_j^*(x)dx \ge 0\ \forall j, \forall y \in [0,z]$.

The full, necessary and sufficient results in parts *(1)* and *(2)* of theorem 1* are due to Chambaz and Maurin (1998). Sufficiency in all three cases, *(1)*, *(2)* and *(3)* is readily seen by tracking through the mathematics of section 3 again, and is demonstrated explicitly in Jenkins and Lambert (1993) for part *(2)*. See the appendix for the proof of necessity in case *(3)*. The modified criteria are not, however, expressible in terms of rank and generalized Lorenz dominance: the corollary to theorem 1 does not extend.

8. Equivalization

An alternative approach to the sequential one for making social welfare comparisons in heterogeneous populations is to select an equivalence scale, deflate household incomes accordingly, and apply the tests that are appropriate for socially homogeneous populations to the pooled equivalent incomes of all households. However, not all analysts agree with the normative use of equivalence scales (Fisher 1987, Coulter *et al.* 1992), and even among those who do, there is plenty of "room for disagreement" about what the appropriate equivalence scale should be (Atkinson 1992, p. 43). Authors such as Buhmann *et al.* (1988) and Cutler and Katz

[3] Alternatively one can posit the existence of an income level z* such that $U_1(z) = U_2(z) = ... = U_n(z)$ for all $z \ge z^*$ and all admissible utility profiles (see Jenkins and Lambert, 1993, but also Moyes, 1999).

(1992) have suggested parameterization as a way to take account of a range of judgements about the needs of families of different sizes. As we shall see, the sequential tests in our theorem 1 and corollary allow the analyst who believes in equivalence scales to side-step the issue of choosing one, and make welfare comparisons that are robust to changes in equivalence scale relativities.

An equivalence scale is a set of income deflators m_i, $1 \le i \le n$, ordered to express the needs structure:

(11) $m_1 > m_2 > ... > m_n$

We will consider the class of SWFs defined over equivalent incomes which is advocated by Ebert (1997, 1999), and indeed is obligatory if the normative approach of Ebert and Moyes (2000) is endorsed. This involves a fictitious population of "equivalent adults", m_i such beings occupying each houshold of type i. If there are no demographic differences between F and G, the Ebert SWFs can be written:

(12) $E_F = \sum p_i m_i \int_0^z U(x/m_i) f_i(x)dx$, $E_G = \sum p_i m_i \int_0^z U(x/m_i) g_i(x)dx$

where $U(\alpha)$ is increasing and concave in equivalent income α. (We attend to demographic differences shortly).

Let $\mathbf{E_1}$ be the class of Ebert SWFs as in (12). Now define two subclasses, $\mathbf{E_2} = \{E \in \mathbf{E_1} : U''(\alpha) + \alpha U'''(\alpha) > 0 \ \forall\alpha\}$ and $\mathbf{E_3} = \{E \in \mathbf{E_2} : 2U'''(\alpha) + \alpha U''''(\alpha) < 0 \ \forall\alpha\}$, for reasons which will become apparent. Both of these include the subclass $\mathbf{E_A}$ for which the utility-of-equivalent-income function takes the Atkinson (1970) form $U(\alpha) = \alpha^{1-e}/(1-e)$ where $0 < e \neq 1$ measures inequality aversion. The class $\mathbf{E_A}$, extended to include the inequality-neutral utility function for which e=0, has been characterized axiomatically by Ebert (1995).

We may identify the contribition $m_i U(x/m_i)$ in the Ebert SWF with the contribution $U_i(x)$ to aggregate welfare according to (1), so that $<m_1U(x/m_1), m_2U(x/m_2), .. m_nU(x/m_n)>$ becomes a utility profile for the preceding welfare analysis. Then a little manipulation shows that:

(13) $\mathbf{E_A} \subset \mathbf{E_k} \subset \mathbf{W_k}$, $k = 1,2,3.$

as shown in the appendix (and in fact $\mathbf{E_1} \subseteq \mathbf{W_{1½}}$). The message in (13) is clear. Distributional analysts in favour of equivalizing may try sequential methods using money income distributions before beginning the restrictive business of choosing an equivalence scale and resorting to mythical populations:

Theorem 2

(1) Under the conditions of either theorem 1, part *(1)* or the corollary, part *(1)*, $E_F \geq E_G$ for all equivalence scales and all $E \in \mathbf{E_1} \supset \mathbf{E_A}$

(2) Under the conditions of either theorem 1, part *(2)*, or the corollary, part *(2)*, $E_F \geq E_G$ for all equivalence scales and all $E \in \mathbf{E_2} \supset \mathbf{E_A}$

(3) Under the conditions of either theorem 1, part *(3)* or the corollary, part *(3)*, $E_F \geq E_G$ for all equivalence scales and all $E \in \mathbf{E_3} \supset \mathbf{E_A}$

Bourguignon's algorithm (which is non-sequential) is in fact necessary and sufficient for dominance for the class $\mathbf{E_1}$ (*ibid,* p. 76). Fleurbaey *et al.*'s (2000) criterion, which is not sequential either, is necessary and sufficient for $\mathbf{E_1}$-dominance for the case when equivalence scale relativities are bounded; the algorithm they specify for the case n = 2 reduces to Bourguignon's in case the equivalence scale bounds are removed. It is an open question to find conditions on money income distributions which are equivalent to $\mathbf{E_2}$-, $\mathbf{E_3}$- or $\mathbf{E_A}$-dominance for all equivalence scales. The question is not easy, since with every change of equivalence scale there is a change in the (artificial) population across which welfare in alternative distributions of income are being compared. The Bourguignon/Fleurbaey *et al.* approach may offer some scope here.[4] If one's normative interest is captured by the Atkinson-type SWFs of the subclass $\mathbf{E_A}$, then any of our sequential tests, the weakest being the one permitting single generalized Lorenz intersections, will suffice to ensure dominance whatever the equivalence scale.

Theorem 2 can be extended to cases in which the demographics differ in F and G. This involves a restriction to subclasses $\mathbf{E_k}$***** of the $\mathbf{E_k}$ (k = 1,2,3) in which the utility of equivalent income function is such that mU(z/m) becomes invariant to changes in m as $z \to \infty$. For this, $U(\alpha)$ and $\alpha U'(\alpha)$ must approach the same limit as $\alpha \to \infty$, a significant restriction. The sequential tests of theorem 1* then yield unambiguous recommendations for the subclasses $\mathbf{E_1}$*****, $\mathbf{E_2}$***** and $\mathbf{E_3}$***** and all equivalence scales, but these subclasses do not contain $\mathbf{E_A}$. Fleurbaey *et al.* (2000) consider this scenario and adapt their algorithm accordingly.

[4] Fleurbaey *et al.* (2000), in their conclusions, suggest "to start again all the analysis of this paper in order to have the dominance criterion obtained degenerating to Atkinson and Bourguignon's one" when the bounds on equivalence scales are removed (rather than to Bourguignon's, as now). The Atkinson and Bourguignon criterion is the one in part (2) of theorem 2. Thus a way forward might be to combine bounded equivalence scales with the restriction $U''(\alpha) + \alpha U'''(\alpha) > 0 \ \forall\alpha$ defining the class $\mathbf{E_2}$. The third derivative condition $U'''(\alpha) > 0 \ \forall\alpha$, which is implied by this restriction, is interesting in its own right but its implications have yet to be explored.

9. Conclusions

The dominance criteria of Atkinson (1970) and Shorrocks (1983), for making welfare comparisons between populations on the basis of income distribution data, have been adapted in existing literature to take non-income information into account. One adaptation is to use sequential procedures (Atkinson and Bourguignon, 1987); another is to use an equivalence scale, create artificial populations of 'equivalent adults' and continue as before (Ebert, 1997). In this paper, we have presented a new sequential procedure which copes with situations in which sequential dominance fails. We have also shown how to modify it to allow for demographic differences. Finally, we have explained how the sequential procedures can be used to generate recommendations which are valid for distributions of equivalent income, whatever the equivalence scale. The paper thus serves both to survey and extend the sequential literature, and to marry it with the literature on equivalization, which has until recently been seen as entirely distinct.

Appendix

Proof of necessity in theorem 1, part (3).

Fix a number $\lambda > 1$ and let a(x) be any real-valued function which is positive over $[0, \lambda z]$. Let $b(x) = \int_{\lambda z}^{x} a(t)dt$, $c(x) = \int_{\lambda z}^{x} b(t)dt$, and $d(x) = \int_{\lambda z}^{x} c(t)dt$. Then $b(x) < 0$, $c(x) > 0$ and $d(x) < 0$ over [0,z]. Also $b'(x) = a(x)$, $c'(x) = b(x)$ and $d'(x) = c(x)$ $\forall x$.

Now let $\alpha_1, \alpha_2, \ldots \alpha_n$ be given positive real numbers and define $U_i(x) = \sum_{i\leq k\leq n} \alpha_k .d(x)$, so that $U_i'(x) = \sum_{i\leq k\leq n} \alpha_k .c(x)$, $U_i''(x) = \sum_{i\leq k\leq n} \alpha_k .b(x)$ and $U_i'''(x) = \sum_{i\leq k\leq n} \alpha_k .a(x)$. Thus $U_i'(x) > 0$, $U_i''(x) < 0$ and $U_i'''(x) > 0$ $\forall x \in [0,z]$.

Observe that $D_j(x) = \alpha_j\, c(x) > 0$, $D_j'(x) = \alpha_j\, b(x) < 0$ and $D_j''(x) = \alpha_j\, a(x) > 0$ for all $x \in [0,z]$. Thus the utility profile $<U_1,U_2,\ldots. U_n>$ belongs to $\mathbf{U_3}$. This is so for each number $\lambda > 1$, each choice of function $a(x) > 0$ on $[0,\lambda z]$ and all choices of positive real numbers $\alpha_1, \alpha_2, \ldots \alpha_n$. Hence, substituting in (9), we know that a necessary condition for welfare dominance of F over G for the class $\mathbf{W_3}$ is that

(A) $\quad W_F - W_G = \Sigma_{1\leq j\leq n}\, \alpha_j \left\{c(z)T_j(z) - b(z)\int_0^z T_j(x)dx + \int_0^z a(y)\{\int_0^y T_j(x)dx\}dy\right\} \geq 0$

for each $\lambda > 1$, each function $a(x) > 0$ on $[0,\lambda z]$ and all positive reals $\alpha_1, \alpha_2, \ldots \alpha_n$.

Now as $\lambda \to 1$, $b(z) \to 0$ and $c(z) \to 0$. Letting $\lambda \to 1$ in (A), we must have

(B) $\quad \Sigma_{1\leq j\leq n}\, \alpha_j \int_0^z a(y)\{\int_0^y T_j(x)dx\}dy \geq 0$

for all positive reals $\alpha_1, \alpha_2, \ldots \alpha_n$ and all functions $a(x) > 0$ on [0,z]. In turn this means that for each j,

(C) $\quad \int_0^z a(y)\{\int_0^y T_j(x)dx\}dy \geq 0$

for all functions $a(x) > 0$ on [0,z] (otherwise, for a contradiction, let $\alpha_i \to 0$ in (B) for $i \neq j$). This in turn means that

(D) $\quad \int_0^y T_j(x)dx \geq 0.$

for each $y \in [0,z]$; otherwise, if $\int_0^y T_j(x)dx < 0$ on some interval $[u,v] \subseteq [0,z]$, we could choose a sequence of positive-valued functions $a_k(x)$ which approach zero outside [u,v] as $k \to \infty$, staying strictly positive somewhere within [u,v], and then (C) would be contradicted.

To show that $T_j(z) \geq 0$ $\forall j$ is also necessary for welfare dominance of F over G for the class $\mathbf{W_3}$, redefine a(x), b(x), c(x) and d(x) as follows. Let N be a positive integer, $a(x) = 1/N$, $b(x) = (x-\lambda z)/N$, $c(x) = 1 + \{(x-\lambda z)^2/2N\}$ and $d(x) = \{(x-\lambda z)^3/6N\} + (x-\lambda z)$. As before, $b(x) < 0$, $c(x) > 0$ and $d(x) < 0$ over [0,z] and $b'(x) = a(x)$, $c'(x) = b(x)$ and $d'(x) = c(x)$ $\forall x$. Defining the utility profile $<U_1,U_2,\ldots. U_n>$ exactly as before, (A) still holds. This time let $N \to \infty$ in (A). Since a(x)

$\rightarrow 0$, b(x) $\rightarrow 0$ and c(x) $\rightarrow 1$ for all x as $N \rightarrow \infty$, we may conclude that

(E) $\Sigma_{1\leq j\leq n}\,\alpha_j\,T_j(z) \geq 0$

for all positive reals $\alpha_1, \alpha_2, \ldots \alpha_n$. This in turn forces

(F) $T_j(z) \geq 0$

for each j, using a similar argument to the preceding one.

Necessity in theorem 1, part (3).*

In the presence of distributional change, $T_j(x)$ is replaced in (9) by $T_j^*(x)$ and an additional restriction is required on utility profiles. Redefining $U_i(x) = \sum_{i\leq k\leq n} \alpha_k .[d(x) - d(z)]$ in the above, this restriction is met, since $U_i(z) = 0\ \forall i$, but nothing else is materially affected; the same proofs go through for the functions $T_j^*(x)$ and the welfare class $\mathbf{W_3^*}$ as for the functions for $T_j(x)$ and the welfare class $\mathbf{W_3}$.

Proof that $\boldsymbol{E_k} \subset \boldsymbol{W_k}$, k = 1,2,3, as in (13).

Setting $U_i(x) = m_i U(x/m_i)$, the marginal utility difference $D_j(x) = U'(x/m_j)-U'(x/m_{j+1})$ is positive for all equivalence scales satisfying (11) since U is concave; also U_i is concave for the same reason; hence $\mathbf{E_1} \subset \mathbf{W_{1½}} \subset \mathbf{W_1}$. We have $D_j'(x) < 0\ \forall x$ for all equivalence scales if and only if U''(x/m)/m decreases with m given x, and $D_j''(x) > 0\ \forall x$ for all equivalence scales if and only if $U'''(x/m)/m^2$ increases with m given x. The derivatives of these two functions with respect to m are $-[U''(\alpha) + \alpha U'''(\alpha)]/m^2$ and $-[2U'''(\alpha) + \alpha U''''(\alpha)]/m^3$ respectively, where $\alpha = x/m$. It follows that $\mathbf{E_2} \subset \mathbf{W_2}$ and $\mathbf{E_3} \subset \mathbf{W_3}$.

References

Atkinson, A.B. (1970). On the measurement of inequality. *Journal of Economic Theory*, vol. 2, pp. 244-263.

Atkinson, A.B. (1973). Further notes on the measurement of inequality. *Mimeo*.

Atkinson, A.B. (1990). Public economics and the economic public. *European Economic Review*, vol. 34, pp. 225-248.

Atkinson, A.B. (1992). Measuring inequality and differing social judgements. *Research on Economic Inequality*, vol. 3, pp. 29-56.

Atkinson, A.B. and F. Bourguignon (1987). Income distribution and differences in needs. Chapter 12 in G.R. Feiwel (ed.) *Arrow and the Foundations of the Theory of Economic Policy*. London: Macmillan.

Banks, J. and P. Johnson (1994). Equivalence scale relativities revisited. *Economic Journal*, vol. 104, pp. 883-890.

Bishop, J.A., J.P. Formby and P.D. Thistle (1991). Rank dominance and international comparisons of income distribution. *European Economic Review*, vol. 35, pp. 1399- 1409.

Bourguignon, F. (1989). Family size and social utility: income distribution dominance criteria. *Journal of Econometrics*, vol. 42, pp. 67-80.

Brumelle, S.L. and R.G. Vickson (1975). A unified approach to stochastic dominance. In W.T. Ziemba and R.G. Vickson (eds.) Stochastic Optimization Models in Finance. New York: Academic Press.

Buhmann, B., L. Rainwater, G. Schmauss and T.M. Smeeding (1988). Equivalence scales, well-being, inequality and poverty: sensitivity estimates across 10 countries using the LIS database. *Review of Income and Wealth*, vol. 34, pp. 115-142.

Chambaz, C. and E. Maurin (1998). Atkinson and Bourguignon's dominance criteria: extended and applied to the measurement of poverty in France. *Review of Income and Wealth*, vol. 44, pp. 497-513.

Coulter, F.A., Cowell, F.A. and Jenkins, S.P. 1992: Differences in needs and the assessment of income distributions. *Bulletin of Economic Research*, 44, 77-124.

Cutler, D.M. and L. Katz (1992). Rising inequality? Changes in the distribution of income and consumption in the 1980's. *American Economic Review*, vol. 82, pp. 546-551.

Dardanoni, V. and Lambert, P.J. (1987). Welfare rankings of income distributions: a rôle for the variance and some insights for tax reform. *Social Choice and Welfare*, vol. 5, pp. 1-17.

Davies, J. and M. Hoy (1994). The normative significance of using third-degree stochastic dominance in comparing income distributions. *Journal of Economic Theory*, vol. 64, pp. 520-530.

Davies, J. and M. Hoy (1995). Making inequality comparisons when Lorenz curves intersect. *American Economic Review*, vol. 85, pp. 980-986.

Ebert, U. (1995). Income inequality and differences in household size. *Mathematical Social Sciences*, vol. 30, pp. 37-55.

Ebert, U. (1997). Social welfare when needs differ: an axiomatic approach. *Economica*, vol. 64, pp. 233-244.

Ebert, U. (1999). Using equivalent income of equivalent adults to rank income distributions. *Social Choice and Welfare*, vol. 16, pp. 233-258.

Ebert, U. (2000). Sequential generalized Lorenz dominance and transfer principles. *Bulletin of Economic Research*, forthcoming.

Ebert, U. and P. Moyes (2000). Adjusting income for needs: can one avoid equivalence scales? *Discussion Paper No. V-204-00*, University of Oldenberg.

Fisher, F.M. (1987). Household equivalence scales and interpersonal comparisons. *Review of Economic Studies*, vol. 54, pp. 519-524.

Fleurbaey, M., C. Hagneré and A. Trannoy (2000). Welfare comparisons with bounded equivalence scales. *Journal of Economic Theory*, forthcoming.

Jenkins, S.P. and P.J. Lambert (1993). Ranking income distributions when needs differ. *Review of Income and Wealth*, vol. 39, pp. 337-356.

Kolm S.-C. (1976). Unequal inequalities, I and II. *Journal of Economic Theory*, vol. 12, p. 416-442 and vol. 13, pp. 82-111.

Lambert, P.J. (1993). *The Distribution and Redistribution of Income: A Mathematical Analysis* (second edition). Manchester: University Press.

Moyes, P. (1999). Comparison of heterogeneous distributions and dominance criteria. *Economie et Prévision*, No. 138-139, pp. 125-146.

Ok, E.A. and P.J. Lambert (1999). On evaluating social welfare by sequential generalized Lorenz dominance. *Economics Letters*, vol. 63, pp. 45-53.

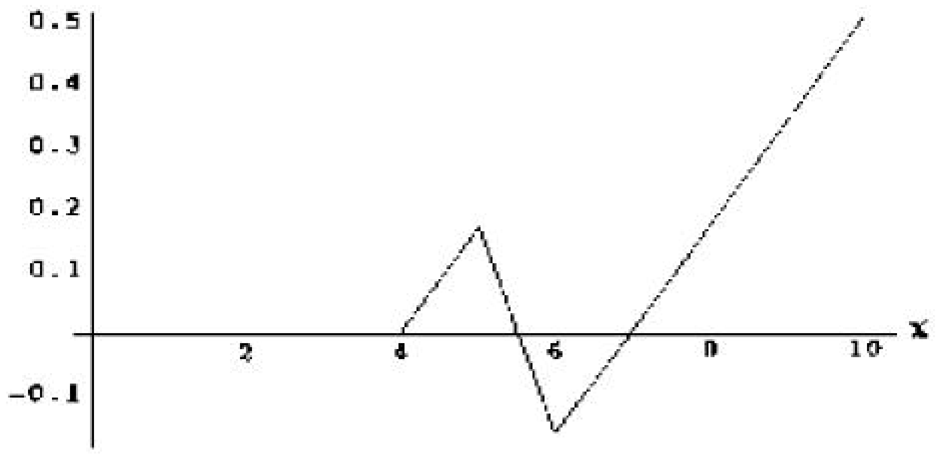
Saposnik, R. (1981). Rank dominance in income distribution. *Public Choice*, vol. 36, pp. 147-151.

Shorrocks, A.F. (1983). Ranking income distributions. *Economica*, vol. 50, pp. 1-17.

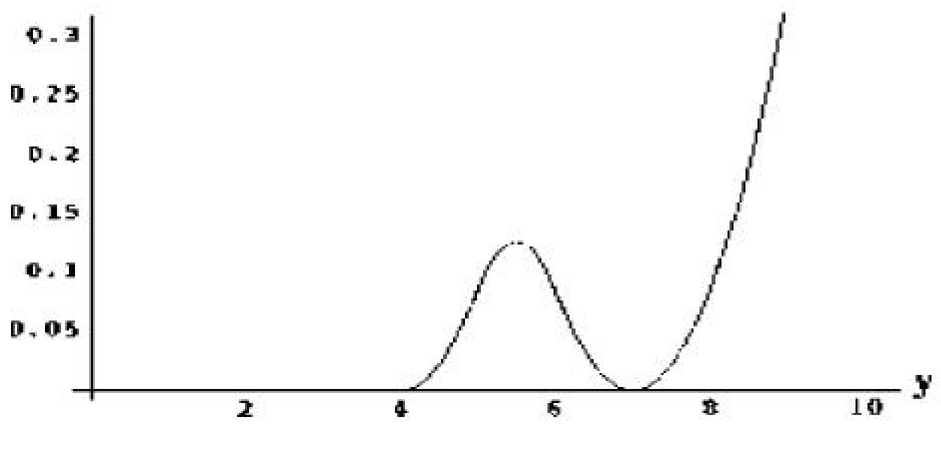
Shorrocks A.F. and J. Foster (1987). Transfer sensitive inequality measures. *Review of Economic Studies*, vol. 54, pp. 485-497.

Whitmore, G.A. (1970). Third degree stochastic dominance. American Economic Review, vol. 24, pp. 617-628.

Figure 1 : The example



a: the function $T_2(x)$



b: the function $\int_0^y T_2(x)dx$