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# POOLING, PRICING AND TRADING OF RISKS

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## Abstract

Exchange of risks is considered here as a transferable-utility cooperative game. When the concerned agents are risk averse, there is a core imputation given by means of shadow prices on state-dependent claims. Like in finance, a risk can hardly be evaluated merely by its inherent statistical properties (in isolation from other risks). Rather, evaluation depends on the pooled risk and the convolution of individual preferences. Explored below are relations to finance with some emphasis on incompleteness. Included is a process of bilateral trade which converges to a price-supported core allocation.


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# Pooling, Pricing and Trading of Risks

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## 1. Introduction

Borch (1962) considered risks as commodities and explored whether each such object could be priced merely in terms of its own (marginal) distribution or moments. His …ndings were essentially negative: There can hardly exist a linear pricing regime of that sort. Moreover, even if existence were granted, price-taking exchange of risks - say, within a reinsurance market - would not in general produce Pareto e¢cient allocations. And, absent such e¢ciency, there can be no competitive equilibrium. In conclusion Borch (op. cit.) suggested that risk exchange had better be analyzed as a cooperative game.

This paper picks up that suggestion. By reconsidering Borch's approach I am led to analyze a transferable-utility cooperative game, featuring agents who …nd it worthwhile to pool their risks. Given some degree of independence among various risks, their pooling smoothens nature's vagaries: Lucky agents can help unlucky ones; ups somewhere can mitigate downs elsewhere. Given also risk aversion on each part, the advantages (individual and social) of pooling su¢ce, as shown here, to render the core non-empty.

Last but not least important, a core solution can then be computed and implemented by means of a linear price regime. That regime depends on the entire preference pro…le and the aggregate risk - and only on those items. As in competitive equilibrium, individual preferences can be aggregated into those of a single representative …gure, here called the convoluted agent. As in …nance, the premium (the trading price) of any insurance treaty is largely a¤ected by how its indemnity co-varies with the aggregate risk.

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The main novelties of this paper come with risk exchange being viewed not as a competitive economy but rather - and more conveniently - as a cooperative game with side payments. Not surprisingly, the game in question assumes the form of a mutual insurance company. Formally, it fits inside the frames of concave optimization with attending advantages for computation and interpretation. Further, the set-up proposed here invites modelling of trade as repeated, direct exchanges between two new parties each time. These bilateral transactions could proceed by means of predesigned contracts, such instruments here being called insurance treaties. Finally, but more on a technical note, there is tolerance for non-smooth payoffs and room for restrictions on exchange.

## 2. Cooperative Risk Sharing

Ex ante, in face of uncertainty, all concerned parties agree that one and only one state $s \in S$ can come about next period. Thus $S$ is an exhaustive list of mutually exclusive and economically relevant states. For analytical simplicity take $S$ to be finite.[1] For generality, however, it seems prudent to avoid hypotheses about common or subjective beliefs the likelihood of various outcomes. Thus probabilities and expected values will not be mentioned before next section.

Accommodated henceforth is a fixed and finite set $I$ of individual parties. Agent $i \in I$ owns a risk $\mathring{y}^i = (\mathring{y}^i_s)$; the component $\mathring{y}^i_s$ being his claim (indemnity or dividend) in state $s$. One may naturally posit that $\mathring{y}^i_s$ be a real number, referring to money or units of account. Nothing prevents us though, from specifying this item as a vector in some fixed Euclidean space $E$.[2] The advantage of this option is that $\mathring{y}^i_s$ can be construed as a fully specified bundle of various commodities to be delivered in state $s$. So, whether one sees $E$ as the real Euclidean line or as a higher dimensional space, it is - in either case - natural to call $s \mapsto \mathring{y}^i_s \in E$ a profile and write simply $\mathring{y}^i \in Y := E^S$. For reasons that will become clear later, $\mathring{y}^i$ generally must belong to a prescribed, linear subspace $Y$ of $Y$.[3]

The preferences of agent $i$ over consumption profiles in $Y$ are represented by a utility or payoff function $\pi^i(\cdot)$. Thus he can "secure" himself payoff $\pi^i(\mathring{y}^i)$. Assumptions about separability of preferences are made only in the next section. It sometimes simplifies notation to deal also with the disutility $f^i := -\pi^i$ of agent $i$; and then I shall do so.

An important hypothesis is now in order: Speaking of payoff (instead of utility) it presumably is cardinal and transferable among agents. Consequently, any coalition $C$ of agents - that is, any nonempty community $C \subseteq I$ - could pool their claims and

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[1] The subsequent arguments can accommodate an infinite measure space $S$ together with a Hilbert space $Y$ of square-integrable profiles $y \in E^S$. Of particular theoretical interest are non-atomic measure spaces; see [1], [11].

[2] In fact, any topological vector space $E$ would do provided it be locally convex and Hausdorff. One can construe $\mathring{y}^i$ as a consumption profile to which agent $i$ is entitled. This viewpoint fits to finance, and it opens up for a dynamic perspective.

[3] The particular instance $Y = Y$ is later referred to as complete.

undertake transfers among themselves. Motivation for such an enterprise stems from the fact that $C$ may, at least in principle, ensure itself an aggregate payoff

$$\pi^C(\bar{y}^C) := \sup\left\{ \sum_{i\in C} \pi^i(y^i) \;\middle|\; \sum_{i\in C} y^i = \sum_{i\in C} \bar{y}^i =: \bar{y}^C;\ \text{all } y^i \in Y \right\} \tag{1}$$

$\geq \sum_{i\in C} \pi^i(\bar{y}^i)$. Clearly, (1) models pooling and friction-free redistribution of perfectly divisible risks, $\bar{y}^C$ being the total claim held by coalition $C$. To induce participation of everybody in a large, common pool - that is, to stimulate formation of the grand coalition $C = I$ - one need a viable scheme for payoff sharing. For its acceptance a proposed scheme had better be efficient, incentive compatible, and "equitable". Any core imputation fills that bill. This solution concept, most popular in cooperative game theory, amounts here to specify a compensation schedule $c = (c^i) \in \mathbb{R}^I$ which supports

$$\left.\begin{array}{lll} \text{Pareto efficiency:} & \sum_{i\in I} c^i = \pi^I(\bar{y}^I) & \text{and ensures} \\ \text{no blocking:} & \sum_{i\in C} c^i \geq \pi^C(\bar{y}^C) & \text{for all } C \subset I. \end{array}\right\} \tag{2}$$

Is such a scheme $c$ of side payments available? Can a core solution $c$ be exhibited, computed and interpreted? Yes, if agents are risk averse, indeed it can!

The argument goes in terms of "price regimes" and standard Lagrangians. To introduce and conveniently handle these objects, equip the space $\mathbb{Y}$ with a fixed inner product denoted simply by juxtaposition $y'y$ of the two vectors. Modulo that product $\mathbb{Y}$ permits a decomposition $Y \oplus N$ into the direct sum of two orthogonal spaces; $N$ being the normal complement to the given subspace $Y$.

Correspondingly, let the dual space $\mathbb{Y}^*$, which comprises all continuous linear functionals on $\mathbb{Y}$ (and no more), be decomposed orthogonally as a corresponding direct sum $N^\perp \oplus Y^\perp =: Y^* \oplus N^*$. This means that any element in $\mathbb{Y}^*$ comes a unique sum $y^* + n^*$ such that $(y^* + n^*)(y + n) = y^*y + n^*n$. Note that each $n^* \in N^* := Y^\perp$, when considered as a price, renders all bundles in $Y$ freely available. In geometrical terms, $n^* \in N^*$ iff it stands "orthogonally" (normally) onto $Y$, this meaning that $n^*y = 0$, $\forall y \in Y$.

Given various functions $f : \mathbb{Y} \to \mathbb{R} \cup \{\pm\infty\}$ it is expedient to consider their customary convex conjugates:

$$y^* + n^* \in \mathbb{Y}^* \mapsto f^*(y^* + n^*) := \sup\left\{ y^*y + n^*n - f(y+n) \mid y \in Y; n \in N \right\}. \tag{3}$$

After these preparations associate to problem (1) the Lagrangian

$$L^C = L^C(y; n; y^*; n^*) := \sum_{i\in C} \left\{ \pi^i(y^i + n^i) + y^*(y^i - \bar{y}^i) + n^{i*}n^i \right\}.$$

Here $y = (y^i) \in Y^I$, $n = (n^i) \in N^I$ and $n^* = (n^{i*}) \in N^{I*}$.[4] It follows from (3) that

$$\sup_{y;n} L^C(y; n; y^*; n^*) = \sum_{i\in C} \left\{ f^{i*}(y^* + n^{i*}) - y^*\bar{y}^i \right\}.$$

[4] In omitting arguments and writing only $L^C$ it is tacitly understood that $y^i \in Y$, $n^i \in N$ are so-called primal variables whereas $y^* \in Y^*$, $n^{i*} \in N^*$ are corresponding dual ones.

Declare now the pair $(y^*; n^*) \in Y^* \times N^{I*}$ a shadow price regime iff

$$\pi^I(\bar{y}^I) \geq \sum_{i \in I} \left\{ f^{i*}(y^* + n^{i*}) - y^* \bar{y}^i \right\}. \tag{4}$$

**Theorem 1.** (Shadow prices on risks generate core solutions) For any shadow price regime $(y^*; n^*)$ the payment scheme

$$c^i := f^{i*}(y^* + n^i) - y^* \bar{y}^i \tag{5}$$

belongs to the core. That is, it satisfies (2).

*Proof.* For each coalition $C$ and any dual pair $(y^*; n^*)$ one has

$$\sum_{i \in C} \left\{ f^i(y^* + n^{i*}) - y^* \bar{y}^i \right\} \geq \inf_{y^*; n^*} \sup_{y; n} L^C \geq \sup_{y; n} \inf_{y^*; n^*} L^C = \pi^C(\bar{y}^C). \tag{6}$$

Thus, invoking definition (5), the "no blocking constraints" in (2) are all easily satisfied. But Pareto efficiency follows straightforwardly as well because

$$\pi^I(\bar{y}^I) \geq \sum_{i \in I} \left\{ f^i(y^* + n^{i*}) - y^* \bar{y}^i \right\} = \sum_{i \in I} c^i \geq \pi^I(\bar{y}^I).$$

The left inequality was assumed in (4), and the right one derives for the instance $C = I$ from (6). □

The competitive nature - and the decentralizing impact - of a shadow price regime is speaking: If offered additional payment $y^*(y^i - \bar{y}^i) + n^{i*} n^i$ for replacing $\bar{y}^i + 0$ by $y^i + n^i$, agent $i$ would make a choice that perfectly fits problem (1) when $C = I$.

Theorem 1 says that $(y^*; n^*)$ is a shadow price regime iff it realizes the saddle value $\min \sup L^I = \sup \inf L^I$. Put differently: what comes into play is a lop-sided min-max result. But existence of saddle points/values cannot generally be guaranteed unless some versions of compactness, continuity, and convexity are in vigor. Ignoring compactness for a while, we are - as usual in microeconomics - left with concerns about continuity and convexity of preferences. To get away from these, simply assume that all payoffs $\pi^i(\cdot)$ are upper semi-continuous, concave functions, from $Y$ into $\mathbb{R} \cup \{-\infty\}$.[5] So, in particular, each agent is risk averse.

To appreciate the concavity assumption - and to understand the nature of shadow price regimes - consider a representative agent, here called the convoluted agent, who enjoys payoff

$$\pi^I(y^I; n) := \sup \left\{ \sum_{i \in I} \pi^i(y^i + n^i) \;\middle|\; \sum_{i \in I} y^i = y^I;\ y^i \in Y \right\}.$$

[5] The extreme value $-\infty$ serves to account for restrictions and spare explicit mention of these. Quasi-concavity of each $\pi^i$ would suffice for many arguments here.

The two arguments which affect that synthetic or fictive fellow are first, an aggregate risk $y^I \in Y$; and second, a profile $n = (n^i)$ of vectors normal to $Y$. Clearly, $\pi^I(y^I; n)$ equals the payoff that would accrue to the grand coalition $I$ if allowed to replace $\bar{y}^I$ by $y^I$ and, at the same time, offer each member $i$ a normal component $n^i$. The bivariate function $\pi^I(\cdot;\cdot)$ thus defined inherits concavity from the terms $\pi^i$. Consequently, the convoluted agent is risk averse as well. Note, after having accepted apologies for slight abuse of notation, that $\pi^I(\bar{y}^I; 0) = \pi^I(\bar{y}^I)$ where the latter (right hand) value already was defined in (2).

Compactness was briefly mentioned here above to ensure existence of a shadow price regime. It turns out that if

$$\pi^I(\cdot;\cdot) \text{ is finite-valued in a neighborhood of } (\bar{y}^I; 0); \qquad (7)$$

then that concern is taken care of. In fact, qualification (7) yields a "neo-classical", marginalistic interpretation of shadow prices. For the statement recall that $g \in Y^*$ if called a supergradient of a proper function $f : Y \to \mathbb{R} \cup \{-\infty\}$ at the point $y$ iff $f(y') \le f(y) + g(y' - y)$ for all $y' \in Y$. On such occasions one writes $g \in \partial f(y)$. The following result derives now directly from convex analysis [7], [9]:

**Proposition 1.** (Existence and characterization of shadow price regimes)
- Under qualification (7) there exists a super-gradient $(y^*; n^*) \in \partial\pi^I(\bar{y}^I; 0)$; and each such item constitutes a shadow price regime.
- Conversely, any shadow price regime $(y^*; n^*)$ must be a supergradient for the convoluted agent $\pi^I(\cdot;\cdot)$ at $(\bar{y}^I; 0)$.
- In sum, a shadow price regime $(y^*; n^*)$ - and a corresponding core solution (5) can be defined - iff $\pi^I(\cdot;\cdot)$ is superdifferentiable at $(\bar{y}^I; 0)$.
- If $(y^*; n^*)$ is a shadow price regime, and $(y^i)$ solves problem (2) for the grand coalition, it holds for each $i$ that

$$y^* + n^{i*} \in \partial\pi^i(y^i). \quad \Box \qquad (8)$$

Inclusions (8) tell that all agents use the same $y^* \in Y^*$ to price choices within the feasible space $Y$.[6] That is, up to idiosyncratic normal components $n^{i*}, i \in I$, they agree on one price in $Y$. This reflects that in the "market game" [10], restricted to $Y$, all desirable exchanges have been made. An infeasible exchange, one whose normal component $n$ does not vanish, may very well yield valuations $n^{i*}n$ which vary across agents. If however, potential exchanges constitute a **complete space**, that is, if $Y = Y$, then clearly, $n^* = 0$, and things become simpler. In that instance $y^*$ is briefly named a shadow price.

**Corollary 1.** (Shadow prices under completeness) Suppose $Y = Y$. Then:
- If the function $\pi^I(\cdot)$ defined in (2) is finite-valued in a neighborhood of $\bar{y}^I$, it is

[6] Smooth versions of (8) are prominent in models of incomplete financial markets; see [8].

super-differentiable at $\bar{y}^I$; and any super-gradient $y^* \in \partial\pi^I(\bar{y}^I)$ constitutes a shadow price (with $n^* = 0$).

- Conversely, any shadow price $y^*$ must be a supergradient for the convoluted agent $\pi^I(\cdot)$ at $\bar{y}^I$.
- In sum, a shadow price $y^*$ - and a corresponding core solution (5) can be defined with $n^* = 0$ - iff $\pi^I(\cdot)$ is superdifferentiable at $\bar{y}^I$.
- If $y^*$ is a shadow price, and $(y^i)$ solves problem (2) for the grand coalition, it holds for each $i$ that
$$y^* \in \partial\pi^i(y^i). \quad \Box$$

It is often natural to assume $\pi^i$ monotone in each variable. So, typically $y^*$ is a positive price regime. For illustration of Theorem 1 suppose individual payoff is a marginal function

$$\pi^i(y^i) := \sup\left\{\Pi^i(x^i, y^i) \;\middle|\; x^i \in X^i\right\}, \tag{9}$$

stemming from a bivariate proper, concave objective $\Pi^i(\cdot,\cdot)$ defined over a Euclidean vector space $X^i \times Y$. Coalition $C \subseteq I$ could then achieve

$$\pi^C(\bar{y}^C) := \sup\left\{\sum_{i\in C}\Pi^i(x^i, y^i) \;\middle|\; \sum_{i\in C} y^i = \bar{y}^C;\ x^i \in X^i;\ y^i \in Y\right\}.$$

Let here $L^C(x, y, n, y^*, n^*) := \sum_{i\in C}\left[\Pi^i(x^i, y^i + n^i) + y^*(y^i - \bar{y}^i) + n^{i*}n^i\right]$ and $\hat{f}^i := -\Pi^i$. Note that $\sup_{x,y,n} L^C = \sum_{i\in C}\left\{\hat{f}^{i*}(0, y^* + n^{i*}) - y^*\bar{y}^i\right\}$. Verbatim imitation of the proof of Theorem 1 yields:

**Proposition 2.** (Core solutions in terms of non-reduced payoff functions) Given reduced payoff functions like (9), suppose $\pi^I(\bar{y}^I) \geq \sum_{i\in I}\left\{\hat{f}^{i*}(0, y^* + n^{i*}) - y^*\bar{y}^i\right\}$ for some price regime $(y^*, n^*)$. Then, by offering agent $i$ compensation $c^i := \hat{f}^{i*}(0, y^* + n^{i*}) - y^*\bar{y}^i$, we get a core solution. $\Box$

## 3. Common Predictions and Separable Preferences

Assume in addition here that everybody holds the same opinion about the likelihood of various outcomes $s \in S$. Formally, there is a common probability distribution $p = (p_s)$ over $S$; the numbers $p_s$ being strictly positive with sum 1. Each linear functional on $Y = E^S$ can now be represented in terms of the statistically motivated, probabilistic inner product $y'y := \sum_{s\in S} p_s y'_s y_s$. Such representation is particularly useful for the important instance where preferences are of von Neumann- Morgenstern separable type. Then ex ante payoff

$$\pi^i(y^i) := E\pi^i_{\cdot}(y^i_{\cdot}) = \sum_{s\in S} p_s\pi^i_s(y^i_s) \tag{10}$$

equals the expected value of its ex post state-dependent counterpart, and (8) amounts to have $y^*_s + n^{i*}_s \in \partial\pi^i_s(y^i_s)$ for each $s$. Given such separable format (10), if coalition C undertook pooling ex post, after $s$ has been unveiled, it would there obtain over-all payoff

$$\pi_s^C(\bar{y}_s^C) := \sup\left\{ \sum_{i \in C} \pi_s^i(y_s^i) \;\middle|\; \sum_{i \in C} y_s^i = \sum_{i \in C} \bar{y}_s^i =: \bar{y}_s^C \right\}. \tag{11}$$

Thus one may speak about contingent, state-dependent cooperation, implemented after the fact. Like above, a compensation scheme $c_s = (c_s^i) \in \mathbb{R}^I$ belongs to the core of the state-$s$ cooperative game iff

$$\begin{array}{lll} \text{Pareto efficiency prevails:} & \sum_{i \in I} c_s^i = \pi_s^I(\bar{y}_s^I) & \text{and} \\ \text{there is no blocking:} & \sum_{i \in C} c_s^i \geq \pi_s^C(\bar{y}_s^C) & \text{for all } C \subset I. \end{array}$$

Also like before, if $y_s^*$ is a Lagrange multiplier - that is, if $y_s^*$ satisfies the Kuhn-Tucker conditions - of problem (11) for $C = I$, then by writing $f_s^i := -\pi_s^i$ and providing compensation

$$c_s^i := f_s^{i*}(y_s^*) - y_s^* \bar{y}_s^i$$

to agent $i$, one obtains a state-$s$ ex post core solution.

Opportunistic behavior of this sort - where agents prefer to wait and see - will, when feasible, not generally fit with (2). The simple reason is, of course, that in passing from (2) to (11) all constraints $y^i \in Y$ were dropped or ignored. When relieved of his constraint, agent $i$ receives compensation $\bar{c}^i := \sum_{s \in S} p_s c_s^i$ in the mean. Comparing with (5) it holds that $\sum_{i \in I} \bar{c}^i \geq \sum_{i \in I} c^i$. If $Y$ is a strict subspace of $\mathbb{Y}$, the last inequality tends to be strict. Equality holds however, under completeness:

**Theorem 2.** (Completeness of the market and time consistency of cooperation) *Suppose claims can be traded in a complete space; that is, suppose $Y = \mathbb{Y}$. Then any shadow price $y^* = (y_s^*)$ supports an over-all ex ante core solution $c^i := f^{i*}(y^*) - y^* \bar{y}^i$ as well as an ex post core solution $c_s^i := f_s^{i*}(y_s^*) - y_s^* \bar{y}_s^i$ in each state $s$. It holds that $c^i = \sum_{s \in S} p_s c_s^i$. And it does not matter whether these cooperative treaties were written before or after the state has been unveiled.* □

## 4. Bilateral Exchange of Risks

Construction (1) invites some pressing questions. Namely, when $C = I$, who undertakes the optimization - and how? Further, since efficient solution requires revelation of true preferences, can the solution procedure - or at least the outcome - be implemented? May either fall victim to strategic communication?

I shall divorce these issues and address first how a center or a consultant, who holds all necessary information, might take up the computational task. Suppose henceforth that problem (1) has at least one optimal solution for $C = I$. Denote by $P_Y$ the orthogonal projection of vectors in $\mathbb{Y}$ onto the subspace $Y$. Let $\{\gamma_k\}$ be a numerical sequence of so-called step sizes, selected a priori subject to

$$\gamma_k \geq 0, \quad \sum_{k=0}^{\infty} \gamma_k = +\infty, \quad \text{and} \quad \sum_{k=0}^{\infty} \gamma_k^2 < +\infty.$$

The computing center, or the said consultant, could proceed by *gradient projection* described as follows:

- Start at stage $k := 0$ with step size $\gamma := \gamma_0$ and choices $y^i \in Y, i \in I$, determined by history, guesswork or accident. It should hold though that $\sum_{i \in I} y^i = \bar{y}^I$.
- Select for each agent $i$ a marginal payoff vector $m^i \in \partial\pi^i(y^i)$ and project it onto the subspace $Y$; that is, let $\hat{m}^i = P_Y m^i$. Let thereafter $\bar{m} := \sum_{i \in I} \hat{m}^i / |I|$ be the uniform average of those projections.
- Update for each $i$ his choice by the rule

$$y^i \leftarrow y^i + \gamma(\hat{m}^i - \bar{m}). \tag{12}$$

- Move to next stage $k \leftarrow k + 1$ with new step size $\gamma \leftarrow \gamma_k$.
- Continue to Select until convergence. $\Box$

**Proposition 3** The described procedure of iterated gradient projection converges to an optimal of problem (1) for the grand coalition.

*Proof.* Given $y^i \in Y$ for all $i \in I$, and also $\sum_{i \in I} y^i = \bar{y}^I$, the projection of the large vector $[y^i + \gamma m^i]_{i \in I} \in Y^I$ onto the affine subspace $\left\{ (y^i) \in Y^I \mid \sum_{i \in I} y^i = \bar{y}^I \right\}$ equals $[y^i + \gamma(\hat{m}^i - \bar{m})]_{i \in I}$. Thus (12) is the method of (super) gradient projection applied to the separable objective $\sum_{i \in I} \pi^i(y^i)$. Convergence now follows from received theory; see [5]. $\Box$

After convergence to an optimal profile $(y^i)$, pick a common $y^* \in \cap_{i \in I} P_Y[\partial\pi^i(y^i)]$, and for each agent $i$, a normal $n^{i*} \in \partial\pi^i(y^i) - y^*$ which satisfies (8). In particular, if each payoff $\pi^i(\cdot)$ is differentiable at $y^i$, it holds that $y^* = P_Y[\partial\pi^i(y^i)]$ and $n^{i*} = \partial\pi^i(y^i) - y^*$.

For greater realism the centralized projection algorithm, just described, had better be replaced by an iterative, non-coordinated procedure driven by the agents themselves. Next I outline one possible avenue along which they could travel. It involves *repeated bilateral exchanges of risk* and goes as follows:

- Start at stage $k := 0$ with step size $\gamma := \gamma_0$ and choices $y^i \in Y, i \in I$, determined by history, guesswork or accident. It should hold that $\sum_{i \in I} y^i = \bar{y}^I$.
- Choose two agents $i, i'$ according to the uniform distribution (i.e. in equi-probable manner).
- Select marginal payoffs $m^i \in \partial\pi^i(y^i)$, $m^{i'} \in \partial\pi^{i'}(y^{i'})$ and project these objects onto the subspace $Y$; that is, let $\hat{m}^i = P_Y m^i$ and $\hat{m}^{i'} = P_Y m^{i'}$.
- Update the choices by bilateral exchange

$$y^i \leftarrow y^i + \gamma(\hat{m}^i - \hat{m}^{i'}) \text{ and } y^{i'} \leftarrow y^{i'} - \gamma(\hat{m}^i - \hat{m}^{i'}) \tag{13}$$

- Move to next stage $k \leftarrow k + 1$ with new step size $\gamma \leftarrow \gamma_k$.
- Continue to Choose two agents until convergence. $\Box$

**Theorem 3.** Repeated bilateral exchanges of risks lead to an optimal solution of problem (1) for the grand coalition.

**Proof.** Let the set $\Omega$ consist of all unordered pairs $\{i, i'\}$ of distinct agents $i, i' \in I$. The event $\omega = \{i, i'\}$ means that agents $i, i'$ are offered the opportunity to trade between themselves. Quite naturally, such offers should be egalitarian. So, endow $\Omega$ with the uniform probability measure; that is, each unordered pair is selected with equal probability $Pr := 2/[|I|(|I| - 1)]$ where $|I|$ is the number of agents.

For every $\omega = \{i, i'\} \in \Omega$ and bundle $y = (y^i)$ define $\pi(\omega, y) := \pi^i(y^i) + \pi^{i'}(y^{i'})$. Let $E$ stand for expectation taken with respect to $\omega$, using the uniform probability $Pr$. Note that $E\pi(\omega, y) = 2Pr \sum_{i \in I} \pi^i(y^i)$. Thus, in problem (1), for the case $C = I$ one may equally well maximize the objective $E\pi(\omega, y)$; and so will be done here.

The modified but equivalent objective $E\pi(\omega, y)$ invites use of stochastic gradient techniques [5]. To see precisely how, it is convenient for any $m^i \in Y$ to let $\mathbf{m}^i$ denote the vector in $Y^I$ which has $m^i$ in component $i$ and zero elsewhere. With this notation observe that $\partial\pi(\omega, y)$ is composed of vectors $\mathbf{m}^i + \mathbf{m}^{i'}$ with $m^i \in \partial\pi^i(y^i)$ for $i$, and similarly for $i'$. So, when projecting $\partial\pi(\omega, y)$ onto $\mathbb{Y}^I$ one gets sums $\hat{\mathbf{m}}^i + \hat{\mathbf{m}}^{i'}$ with $\hat{m}^i \in P_{\mathbb{Y}}\partial\pi^i(y^i)$ for $i$, and similarly for $i'$. Finally, projecting once again, this time onto the affine subspace $A := \left\{(y^i) \in \mathbb{Y}^I : \sum_{i \in I} y^i = \bar{y}^I\right\}$ generates $\hat{m}^i - \hat{m}^{i'}$ in component $i$, the opposite vector $\hat{m}^{i'} - \hat{m}^i$ in component $i'$, and zero elsewhere. After so much bookkeeping it follows that the standard stochastic projected gradient procedure, namely

$$y \leftarrow P_A[y + \gamma \mathbf{m}], \quad \mathbf{m} \in \partial\pi(\omega, y)$$

is nothing else than (13). Convergence now follows from an appeal to known results; see [5]. □

## 5. Trade of Insurance Treaties

It is time to justify why only risks residing in a subspace $\mathbb{Y} \subset Y$ are traded. Clearly, if any exchange in $Y$ were possible, we would be in the standard setting of a barter economy.

For a modified and more realistic setting, one which better fits and justifies the existence of insurance (as well as finance), suppose exchange is mediated only via a finite set $J$ of so-called instruments, briefly named insurance treaties. By such a treaty $j \in J$ is here understood a contract that promises to pay its holder a specified indemnity (coverage or dividend) $d_{sj}$ if state $s \in S$ comes about. Suppose treaties are perfectly divisible, traded with no quantity restrictions and victims to no transaction costs.

As customary, for the sake of simple exposition, only two time periods are considered, namely: now and next period. In other words: all treaties expire after one appropriately defined time step.

By a portfolio (of treaties) is meant a vector $x = (x_j) \in X := \mathbb{R}^J$, saying precisely how much is held of various contracts. Note that portfolio $x$ yields indemnity $y_s = \sum_{j \in J} d_{sj} x_j$ in state $s$. So, letting $D = [d_{sj}]$ denote the $S \times J$ indemnity matrix, portfolio $x$ entitles its holder to payoff profile $y = Dx$.

Correspondingly, let $Y := \operatorname{Im} D := DX := \{Dx \mid x \in X := \mathbb{R}^J\}$ denote the image space under the indemnity matrix $D$. Clearly, $Y$ consists of feasible indemnity schedules and is a subspace of $\mathcal{Y} = E^S$, possibly a strict one. Note that schedules in $\mathcal{Y} \setminus Y$ cannot be synthesized - whence are not tradable - via the given instruments. A profile $y \in Y$ will be realized by any portfolio $x \in X$ which solves $Dx = y$. At least one such $x$ exists by the definition of $Y$. Uniqueness of $x$ follows iff $D : X \to Y$ is one-to-one. Then necessarily $|J| = \operatorname{rank}(D) = \dim Y$. In particular, when $Y = E^S$, there must be as many treaties as there are states.

Suppose now that agent $i$ already holds portfolio $\bar{x}^i$, generating risk $\bar{y}^i := D\bar{x}^i$. Coalition $C$ can achieve

$$\pi^C(\bar{y}^C) = \sup\left\{\sum_{i \in C} \pi^i(y^i) \;\middle|\; \sum_{i \in C} y^i = \bar{y}^C,\ y^i = Dx^i, x^i \in X\right\}$$
$$= \sup\left\{\sum_{i \in C} \pi^i(Dx^i) \;\middle|\; \sum_{i \in C} Dx^i = D\bar{x}^C, x^i \in X\right\}.$$

(This case fits the frames of Proposition 2.) Let $D^*$ denote the transpose matrix. Prices $y^*$ on risks are transported back to prices $x^* = D^* y^*$ on portfolios by the rule $x^*_j = \sum_{s \in S} d_{sj} y^*_s$, and we get

**Proposition 4.** (Shadow prices on treaties generate core solutions) For any shadow price regime $(y^*, n^*)$ the payment scheme

$$c^i := f^{i*}(D^* y^* + n^i) - y^*(D\bar{x}^i) = f^{i*}(D^* y^* + n^i) - (D^* y^*)\bar{x}^i \tag{14}$$

belongs to the core. That is, it satisfies (2).

Clearly, $i$ could have access to a particular set $J^i$ of treaties, defined by an $S \times J^i$ matrix $D^i$. If so, (14) would remain a core solution with $D^i$ instead of $D$. Agent $i$ might also have handy a technology by which his effort $e^i$ produces a profile $E^i(e^i) \in Y$. Then, if $i$ has non-reduced, concave payoff $\hat{\pi}^i(e^i, y^i)$, coalition $C$ gets reduced payoff

$$\pi^C(\bar{y}^C) = \sup\left\{\sum_{i \in C} \hat{\pi}^i(e^i, D^i x^i + E^i(e^i)) \;\middle|\; \sum_{i \in C} \left[D^i x^i + E^i(e^i)\right] = \bar{y}^C\right\}.$$

When however, only agent $i$ knows $e^i$ or $E^i(\cdot)$, there are problems (with hidden actions or types), these making the prospects for efficient cooperation appear less good.[7]

[7] [3] and [4] deal with core solutions under asymmetric information.

I end this section by considering repeated bilateral exchanges of portfolios. It could go as follows:

- Start at stage $k := 0$ with step size $\gamma := \gamma_0$ and choices $x^i \in X, i \in I$, determined by history, guesswork or accident.
- Choose two agents $i, i'$ according to the uniform distribution (i.e. in equi-probable manner).
- Select marginal payoffs $m^i \in \partial\pi^i(Dx^i)$, $m^{i'} \in \partial\pi^{i'}(Dx^{i'})$ and let $x^{i*} = D^* m^i$ and $x^{i'*} = D^* m^{i'*}$.
- Update the choices by bilateral exchange

$$x^i \leftarrow x^i + \gamma(x^{i*} - x^{i'*}) \quad \text{and} \quad x^{i'} \leftarrow x^{i'} - \gamma(x^{i*} - x^{i'*})$$

- Move to next stage $k \leftarrow k+1$ with new step size $\gamma \leftarrow \gamma_k$.
- Continue to Choose two agents until convergence. $\square$

**Theorem 4.** *Repeated bilateral exchanges of portfolios lead to an optimal solution of problem (1) for the grand coalition.*

## 6. Concluding Remarks

Given wide-spread risk aversion, or at least risk neutrality, the cooperative incentives become so strong and well distributed that the grand coalition can safely form. Its formation means that all risks are pooled and that benefits be shared in ways not blocked by any subgroup. However, when preferences are not convex, the price-based compensation scheme (5) is likely to reside out-of-core; see [6].

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