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Abstract

Based on work by S. Malmquist (1953), Caves, Christensen and Diewert (1982) introduced what has been called the Malmquist productivity index. Their theoretical index was given an activity analysis interpretation by Färe, Grosskopf, Lindgren and Roos (1989), who showed how it could be computed using data on inputs and outputs. Färe, Grosskopf, Lindgren and Roos (1989) and Färe and Grosskopf (1994) discovered that under certain conditions the Malmquist index could be ill-defined. This prompted Bjurek (1994) to redefine the index as ratios of Malmquist input and output quantity indexes. In this paper, we investigate the specific conditions under which the Malmquist productivity index is ill-defined and we provide conditions on the data and the reference technologies sufficient for the index to yield appropriate outcomes.

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0. Introduction

In a recent paper, H. Bjurek (1994) introduced a Malmquist total factor productivity index defined as ratios of Malmquist output and input quantity indexes. His motivation for this new index is that the usual Malmquist productivity index may, under variable returns to scale, report an erroneous value. Here we argue that this problem is not important, since the usual Malmquist productivity index should be calculated using a constant returns to scale reference technology, (see Färe, et al. (1994)) in which case the problem raised by Bjurek disappears.

Although the Bjurek formulation reduces the probability that the Malmquist productivity index is ill-defined under a variable returns to scale reference technology, nevertheless, in contrast to his claim, his formulation as well as the constant returns to scale formulation preferred by Färe, et al. may still, in certain circumstances, be ill-defined.

In this note, we identify the two situations in which the Malmquist productivity indexes may be ill-defined. We provide sufficient conditions for these situations not to occur. Finally, we provide restrictions on the production technology to show when the usual and the Bjurek formulation of the Malmquist productivity indexes coincide.

1. Some Background

In the original paper using activity analysis to compute a Malmquist productivity index, Färe, Grosskopf, Lindgren and Roos (1989) showed that the output-oriented formulation may lack solutions if variable returns to scale is imposed on the reference technology. The source of this problem is that the mixed period output sets $P^{t+1}(x^t)$ or $P^t(x^{t+1})$ may be empty[1] in which case there does not exist a scalar θ such that (y^t/θ) or (y^{t+1}/θ) is feasible. Bjurek (1994) solves this problem by not using mixed period output sets in his formulation.

In a later paper, Färe and Grosskopf (1994) provide an example which shows that even if the mixed period input set $L^t(y^{t+1})$ is nonempty, it may be the case that there does not exist a scalar λ such that (x^{t+1}/λ) is feasible. In this case, the Malmquist productivity index may not be well-defined. As we will show, the Bjurek formulation does not solve this last problem.

[1] Inputs are defined by $x = (x_1, \ldots, x_N) \in \Re_+^N$, outputs by $y = (y_1, \ldots, y_M) \in \Re_+^M$ and the output possibility set $P(x)$ consists of all output vectors that can be produced with x.

2. Distance Functions

All Malmquist indexes are defined using distance functions. For example in his original paper, Malmquist (1953) used a ratio of input distance functions to define an input quantity index. Following Shephard (1970), we distinguish between the input and the output distance function. Let $P^t(x^t)$ denote an output set, i.e., the set of all output vectors y^t that can be produced using the input vector x^t, then the output distance function is

$$(2.1) \qquad D_o^t(x^t, y^t) = \inf_\theta\{\theta : (y^t/\theta) \in P^t(x^t)\}.$$

This distance function is the "maximal" feasible radial expansion of y^t. In the case when only one input is used to produce one output under constant returns to scale, then and only then does (2.1) equal:

$$(2.2) \qquad D_o^t(x^t, y^t) = (y^t/x^t)D_o^t(1,1).$$

This condition will be used later in explaining total factor productivity.

The input distance function is defined in terms of the output distance function as

$$(2.3) \qquad D_i^t(y^t, x^t) = \sup_\lambda\{\lambda : D_o^t(x^t/\lambda, y^t) \leq 1\},$$

and one may show that under some weak conditions, see Färe and Primont (1995),

$$(2.4) \qquad D_i^t(y^t, x^t) \geq 1 \text{ if and only if } D_o^t(x^t, y^t) \leq 1.$$

Moreover under constant returns to scale, i.e., whenever $P^t(\lambda x^t) = \lambda P^t(x^t)$, $\lambda > 0$, we have, see Färe (1988)

$$(2.5) \qquad D_i^t(y^t, x^t) = 1/D_o^t(x^t, y^t),$$

which is another condition we will use.

3. Malmquist Productivity Indexes

In the case when one input is used to produce one output, total factor productivity is defined by

$$(3.1) \qquad \text{TFP} = \frac{y^{t+1}/x^{t+1}}{y^t/x^t}.$$

This productivity measure can be transformed into a t-period Malmquist productivity index as defined by Caves, Christensen and Diewert (1982). In particular under constant returns to scale, we get

$$(3.2) \qquad \text{TFP} = \frac{y^{t+1}/x^{t+1}}{y^t/x^t} = \frac{D_o^t(1,1)y^{t+1}/x^{t+1}}{D_o^t(1,1)y^t/x^t} = \frac{D_o^t(x^{t+1}, y^{t+1})}{D_o^t(x^t, y^t)},$$

which is the Caves, Christensen and Diewert (1982) formulation. Next, Färe and Grosskopf (forthcoming) prove that the two expressions (3.1) and

$$(3.3) \qquad \frac{D_o^t(x^{t+1}, y^{t+1})}{D_o^t(x^t, y^t)}$$

coincide if and only if the technology exhibits constant returns to scale. Thus if we want the t-period Malmquist productivity index (3.3) to take the form (3.1) we need to impose constant returns to scale. A similar argument applies to show that constant returns to scale is necessary and sufficient for the $t+1$-period index

$$(3.4) \qquad \frac{D_o^{t+1}(x^{t+1}, y^{t+1})}{D_o^{t+1}(x^t, y^t)}$$

to collapse into (3.1) for the single-input, single-output case.

In the general case with N inputs and M outputs, we define the Malmquist productivity index as in Färe, Grosskopf, Lindgren and Roos (1989)

$$(3.5) \qquad M_o = \left[\frac{D_o^{t+1}(x^{t+1}, y^{t+1})}{D_o^{t+1}(x^t, y^t)} \frac{D_o^t(x^{t+1}, y^{t+1})}{D_o^t(x^t, y^t)}\right]^{\frac{1}{2}}$$

with the technology exhibiting constant returns to scale.

As we mentioned earlier, Bjurek (1994) defines a Malmquist productivity index as ratios of output and input quantity indexes. In particular, his formulation in terms of distance function is

$$(3.6) \qquad MB = \left[\frac{D_o^{t+1}(x^{t+1}, y^{t+1})/D_o^{t+1}(x^{t+1}, y^t)}{D_i^{t+1}(y^{t+1}, x^{t+1})/D_i^{t+1}(y^{t+1}, x^t)} \cdot \frac{D_o^t(x^t, y^{t+1})/D_o^t(x^t, y^t)}{D_i^t(y^t, x^{t+1})/D_i^t(y^t, x^t)}\right]^{\frac{1}{2}}$$

which is the geometric mean of a $t+1$-period and a t-period ratio of an output and an input quantity index.

We showed above that under constant returns to scale, the t and $t+1$ period indexes and hence the Malmquist index (3.5) take the form (3.1) provided only one input is used to produce a single output. This condition is fulfilled by (3.6) without the condition of constant returns to scale. The reason for this is that the input distance function is always homogeneous of degree +1 in inputs and the output distance function is always homogeneous of the same degree in outputs.

It is of interest to show under which conditions the two formulations (3.5) and (3.6) coincide. To address this question, we introduce the notion of inverse homotheticity (see Färe and Primont (1995) for details). The technology is inverse homothetic if the output distance function takes the form

(3.7) $D_o^t(x^t, y^t) = D_o^t(1, y^t)/\Im(D_o^t(1, x^t)),$

where $\Im(\bullet)$ is an increasing, invertable function. Under constant returns to $\Im(\bullet)$ becomes the identity function and (3.7) can be written as

(3.8) $D_o^t(x^t, y^t) = D_o^t(1, y^t)/D_i^t(1, x^t).$

It follows that under constant returns to scale, if the technology is inverse homothetic, then the two Malmquist indexes (3.5) and (3.6) coincide.

4. The Main Results

The problem that may occur in using either of the Malmquist indexes (3.5) or (3.6) is that the mixed period distance functions may take the value 0 or $+\infty$. In either case the indexes break down. Thus in order for the indexes to be valid, the distance functions must be larger than 0 and finite. To guarantee this, conditions must be imposed both on the data and on the reference technologies.

We start by assuming that all data satisfies the nonnegativity conditions

$$\text{(i) } x \geq 0,\ x \neq 0,\ \text{(ii) } y \geq 0,\ y \neq 0,$$

which says that at least one input is used and at least one output is produced. Next we identify two cases when both indexes fail. The first is illustrated by Figure 1, and is from Färe and Grosskopf (1994).

In this case, we have a nonempty input requirement set $L^t(y^\tau)$, $\tau = t$ or $t+1$ and an input vector x^{t+1} that is not a member of the set. As the figure is drawn it is clear that there does not exist a scalar $\lambda > 0$ such that $(x^{t+1}/\lambda)\epsilon L^t(y^\tau)$, $\tau = 1$ or $t+1$. In this case, the input distance function equals zero, i.e.,

(4.1) $D_i^t(y^\tau, x^{t+1}) = 0, \tau = t \text{ or } t+1.$

In the Bjurek formulation, this mean that

(4.2) $D_i^t(y^t, x^{t+1}) = 0,$

and hence $MB = 0$. In the Färe, Grosskopf, Lindgren and Roos formulation, under constant returns to scale,

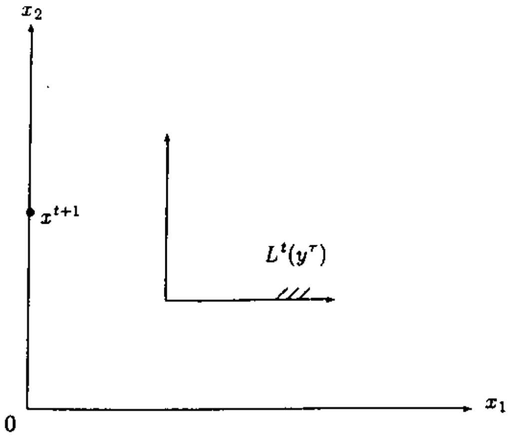

Figure 1:

(4.3) $D_o^t(y^{t+1}, x^{t+1}) = 1/D_i^t(y^{t+1}, x^{t+1}) = +\infty,$

which causes computational problems in (3.5). The problem described by (4.1) is avoided if (a) inputs are freely disposable, i.e., $x' \geq x \in L(y) \Rightarrow x' \in L(y)$ and (b) if $x^{t+1} > 0$. A formal proof follows directly from Färe and Jansson (1975).

Figure 2 illustrates the second case when the two indexes fail.

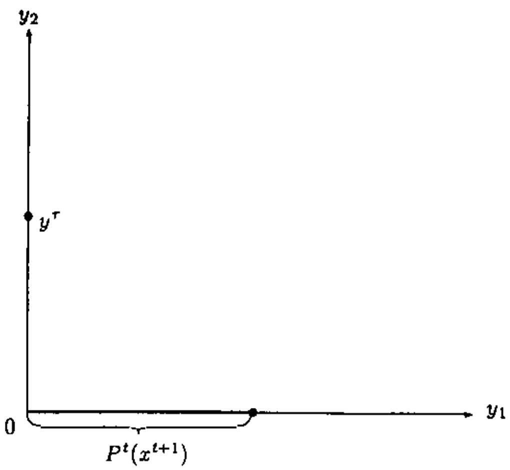

Figure 2:

The output set $P^t(x^{t+1})$ does not contain any output with positive second component, y_2. The output vector y^τ, $\tau = 1$ or $t+1$ has a zero y_1 component and clearly there does not exist a positive scalar θ such that $(y^\tau/\theta) \in P^t(x^{t+1})$. In this case, the distance function equals $+\infty$, i.e.,

$$D_o^t(x^{t+1}, y^\tau) = +\infty, \quad \tau = t \text{ or } t+1. \tag{4.4}$$

In the Bjurek case, we have

$$D_o^t(x^{t+1}, y^t) = +\infty \tag{4.5}$$

and in the other case, we have

$$D_o^t(x^{t+1}, y^{t+1}) = +\infty. \tag{4.6}$$

Each of these conditions invalidates the corresponding index.

The problem described by (4.4) is avoided if (c) outputs are strongly disposable, i.e., $y' \leq y \in P(x) \Rightarrow y' \in P(x)$ and if (d) there exists a $y \in P^t(x^{t+1})$ such that $y > 0$, i.e., $y_m > 0$, $m = 1, \ldots, M$. The proof of this statement is left to the reader.

Next we illustrate the situation addressed by Bjurek (1994), see Figure 3.

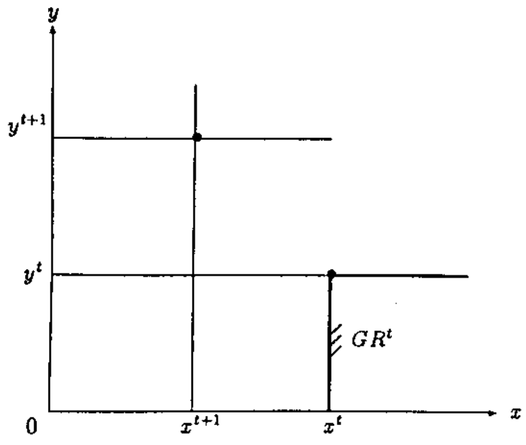

Figure 3:

The graph of the technology at period t is denoted by GR^t and it is defined in terms of the output set as

$$GR^t = \{(x^t, y^t) : y^t \in P^t(x^t), x^t \in \Re^M_+\}. \tag{4.7}$$

Now, the mixed period output set, namely

$$P^t(x^{t+1}), \tag{4.8}$$

of the figure is empty and hence no scaling θ of y^{t+1} can make $(x^{t+1}, \theta y^{t+1})$ feasible. Similarly the
mixed period input set

$$L^t(y^{t+1}) \tag{4.9}$$

is empty. Note though that both $P^{t+1}(x^t)$ and $L^t(y^t)$ are nonempty, which is how Bjurek "solves" the second problem. Here we establish conditions under which $P^t(x^{t+1})$ and $L^t(y^{t+1})$ are nonempty (similar conditions can be derived for $P^{t+1}(x^t)$ and $L^{t+1}(y^t)$, we leave those for the reader). We start by examining the output set, and we note that

$$y^t \in P^t(x^t) \Leftrightarrow x^t \in L^t(y^t). \tag{4.10}$$

If $(x^{t+1} > 0)$, then there exists a positive scalar $\lambda > 0$ such that

$$(x^{t+1}/\lambda) \geqq x^t, \tag{4.11}$$

thus if inputs are strongly disposable then

$$(x^{t+1}/\lambda) \in L^t(y^t), \tag{4.12}$$

hence by constant returns to scale

$$x^{t+1} \in L^t(\lambda y^t) \tag{4.13}$$

therefore $\lambda y^t \in P^t(x^{t+1})$ and $P^t(x^{t+1})$ is nonempty. In sum, if x^{t+1} is strictly positive, if inputs are freely diposable and if the technology exhibits constant returns to scale then $P^t(x^{t+1})$ is nonempty.

Regarding the input set, we use (4.10) and assume that $y^t > 0$. There exists a positive scalar $\theta > 0$ such that

$$(y^{t+1}/\theta) \leq y^t, \tag{4.14}$$

thus if outputs are strongly disposable,

$$(y^{t+1}/\theta) \in P^t(x^t) \tag{4.15}$$

and by constant returns to scale

$$y^{t+1} \in P^t(\theta x^t), \tag{4.16}$$

therefore $(\theta x^t) \in L^t(y^{t+1})$ and the mixed period output set $L^t(y^{t+1})$ is nonempty.

We conclude this section by noting that since the usual Malmquist productivity index (3.5) should be computed under constant returns to scale, the conditions for it and the Bjurek formulation (3.6) to be valid are the same. A sufficient condition is that all observations of inputs and outputs are strictly positive, and that inputs and outputs are freely disposable.

5. Comments

We would like to end this paper with some general comments that apply to the two productivity indexes discussed above. Productivity or productivity change is a measure of how an economic unit has changed its performance. This implies that the measure should compare technologies, which FGLR does 'directly' and Bjurek does indirectly by comparing ratios of output to input indexes (average product) over time, and the Bjurek formulation does this by introducing unobserved input-output vectors. To see this, for simplicity, assume that (x^t, y^t) and (x^{t+1}, y^{t+1}) are efficient, i.e., $D_i^{t+1}(y^{t+1}, x^{t+1}) = D_o^{t+1}(x^{t+1}, y^{t+1}) = D_o^t(x^t, y^t) = D_i^t(y^t, x^t) = 1$. The two indexes can then be written as

$$(5.1) \qquad M_o = (D_o^t(x^{t+1}, y^{t+1}) / D_o^{t+1}(x^t, y^t))^{\frac{1}{2}}$$

and

$$(5.2) \qquad MB = \left[\frac{D_i^{t+1}(y^{t+1}, x^t)}{D_o^{t+1}(x^{t+1}, y^t)} \frac{D_o^t(x^t, y^{t+1})}{D_i^t(y^t, x^{t+1})}\right]^{\frac{1}{2}}$$

respectively. From (5.1), we see that the observed input-output vectors (x^t, y^t) and (x^{t+1}, y^{t+1}) are evaluated relative to the technologies from period $t+1$ and t respectively, i.e., (5.1) uses observed inputs and outputs to evaluate the change in productivity. From (5.2) we see that (x^t, y^{t+1}) and (x^{t+1}, y^t) are used in the evaluation of productivity. Although, x^t, y^t, x^{t+1} and y^{t+1} are all observable, the 'mixed' pairs are not. They are an artificial construct that is required if one insists on constructing separate Malmquist quantity indexes of output and input to measure productivity. Comparison of (5.1) and (5.2) also illustrates that the Bjurek approach is more computationally intensive, requiring computation of twice as many distance functions to arrive at a measure of productivity change.

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