# Supplementary Appendix to "Boolean Representations of Preferences under Ambiguity"

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This supplementary appendix is organized as follows. Section S.1 formalizes the uniqueness properties of BEU representations. Section S.2 focuses on the representation obtained by inverting the order of moves of Optimism and Pessimism and uses this to characterize different degrees of ambiguity seeking. Sections S.3 and S.4 present two generalizations of BEU that correspond to relaxations of certainty independence.

## S.1 Uniqueness

For any $\phi \in \mathbb{R}^S$ and $\lambda \in \mathbb{R}$, let $H_{\phi,\lambda} := \{\mu \in \Delta(S) : \mu \cdot \phi \geq \lambda\}$ denote the closed half-space in $\Delta(S)$ that is defined by $\phi$ and $\lambda$. For any belief-set collection $\mathbb{P}$, define its ***half-space closure*** by

$$\overline{\mathbb{P}} := \{H \subseteq \Delta(S) : H \text{ is a closed half-space in } \Delta(S) \text{ and } P \subseteq H \text{ for some } P \in \mathbb{P}\}.$$

**Proposition S.1.1.** *Suppose* $(\mathbb{P}, u)$ *is a BEU representation of* $\succsim$*. Then for any belief-set collection* $\mathbb{P}'$ *and utility* $u'$*,* $(\mathbb{P}', u')$ *is a BEU representation of* $\succsim$ *if and only if* $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ *and* $u \approx u'$*.*

Below we fix the unique functional $I : \mathbb{R}^S \to \mathbb{R}$ associated with $\succsim$, as given by Lemma B.1. We begin with the following lemma:

**Lemma S.1.1.** *Suppose* $(\mathbb{P}, u)$ *is a BEU representation of* $\succsim$*. Then* $\overline{\mathbb{P}} = \{H_{\phi,\lambda} : \phi \in \mathbb{R}^S, \lambda \leq I(\phi)\}$*.*

*Proof.* First, take any $\phi \in \mathbb{R}^S$, $\lambda \in \mathbb{R}$ such that $\lambda \leq I(\phi)$. Since $(\mathbb{P}, u)$ represents $\succsim$, there exists $P \in \mathbb{P}$ such that $\min_{\mu \in P} \mu \cdot \phi = I(\phi)$. Thus, $P \subseteq H_{\phi,I(\phi)} \subseteq H_{\phi,\lambda}$, which implies $H_{\phi,\lambda} \in \overline{\mathbb{P}}$.

Conversely, take any $P \in \overline{\mathbb{P}}$. By definition of $\overline{\mathbb{P}}$, there exist $\phi \in \mathbb{R}^S$, $\lambda \in \mathbb{R}$, and $P' \in \mathbb{P}$ such that $P' \subseteq P = H_{\phi,\lambda}$. Since $(\mathbb{P}, u)$ represents $\succsim$, $I(\phi) \geq \min_{\mu \in P'} \mu \cdot \phi \geq \min_{\mu \in H_{\phi,\lambda}} \phi \cdot \mu$. Hence, $\lambda \leq I(\phi)$. $\square$

**Proof of Proposition S.1.1.** For the "only if" direction, the fact that $\overline{\mathbb{P}} = \overline{\mathbb{P}'}$ is immediate from Lemma S.1.1 and uniqueness of $I$. The proof that $u \approx u'$ is standard.

For the "if" direction, by uniqueness of $I$, it suffices to show that $\max_{P' \in \mathbb{P}'} \min_{\mu \in P'} \mu \cdot \phi = I(\phi)$ for all $\phi \in \mathbb{R}^S$. To show this, observe first that by Lemma S.1.1 and since $\overline{\mathbb{P}} = \overline{\mathbb{P}}'$, there exists $P' \in \mathbb{P}'$ such that $P' \subseteq H_{\phi, I(\phi)}$. This ensures $\min_{\mu \in P'} \mu \cdot \phi \geq I(\phi)$. Suppose next that $\min_{\mu \in P''} \mu \cdot \phi - I(\phi) =: \epsilon > 0$ for some $P'' \in \mathbb{P}'$. Then $H_{\phi, I(\phi)+\epsilon} \supseteq P''$, which implies $H_{\phi, I(\phi)+\epsilon} \in \overline{\mathbb{P}'}$. Since $\overline{\mathbb{P}'} = \overline{\mathbb{P}}$, this contradicts Lemma S.1.1. $\square$

## S.2 Minmax BEU representation

While BEU assumes that Optimism plays first and Pessimism plays second, this is equivalent to a model with the opposite order of moves. We omit all proofs for this section, as they can be obtained as minor modifications of the original proofs for BEU.

**Theorem S.2.1.** *Preference $\succsim$ satisfies Axioms 1–5 if and only if $\succsim$ admits a minmax BEU representation, i.e., there exists a belief-set collection $\mathbb{Q}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W(f) = \min_{Q \in \mathbb{Q}} \max_{\mu \in Q} \mathbb{E}_\mu[u(f)]$$

*represents $\succsim$.*

Our construction of the maxmin BEU representation considered in the text uses the belief-set collection $\mathbb{P}^* = \mathrm{cl}\{P^*_\phi : \phi \in \mathbb{R}^S\}$ with $P^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \geq I(\phi)\}$. Analogously, it can be shown that the belief-set collection $\mathbb{Q}^* := \mathrm{cl}\{Q^*_\phi : \phi \in \mathbb{R}^S\}$ with $Q^*_\phi := \{\mu \in \partial I(\underline{0}) : \mu \cdot \phi \leq I(\phi)\}$ yields a minmax BEU representation. Paralleling Section 2.3, it is straightforward to show that $C := \partial I(\underline{0})$ again corresponds to the smallest set of priors that is contained in $\overline{\mathrm{co}} \bigcup_{Q \in \mathbb{Q}} Q$ for all minmax BEU representations $\mathbb{Q}$ of $\succsim$, with equality for representation $\mathbb{Q}^*$.

While the different notions of ambiguity aversion are most conveniently characterized using the maxmin BEU representation (cf. Theorem 2), the minmax BEU representation is useful for characterizing their ambiguity-seeking counterparts. Axioms 8 and 9 and Theorem S.2.2 below provide the analogs of Axioms 6 and 7 and Theorem 2, respectively.

**Axiom 8** (Uncertainty Seeking)**.** If $f, g \in \mathcal{F}$ with $f \sim g$, then $\frac{1}{2}f + \frac{1}{2}g \precsim f$.

**Axiom 9** ($k$-Ambiguity Seeking)**.** For all $f_1, ..., f_k \in \mathcal{F}$ with $f_1 \sim f_2 \sim \cdots \sim f_k$ and any $p \in \Delta(Z)$,

$$\frac{1}{k}f_1 + \cdots + \frac{1}{k}f_k = p \Rightarrow p \precsim f_1.$$

We say that $\succsim$ is *absolutely ambiguity-seeking* if there exists a nondegenerate subjective expected utility preference that is more ambiguity-averse than $\succsim$. Analogous to Lemma 1, this is characterized by $\infty$-*ambiguity seeking*, i.e., $k$-ambiguity seeking for all $k$.

**Theorem S.2.2.** *Suppose that $\succsim$ admits a minmax BEU representation $(\mathbb{Q}, u)$. Then:*

1. *$\succsim$ satisfies uncertainty seeking if and only if $\bigcap_{Q \in \mathbb{Q}} Q = C$;*
2. *$\succsim$ is absolutely ambiguity-seeking if and only if $\bigcap_{Q \in \mathbb{Q}} Q \neq \emptyset$;*
3. *$\succsim$ satisfies $k$-ambiguity seeking if and only if $\bigcap_{i=1,\cdots,k} Q_i \neq \emptyset$ for all $Q_1, \cdots, Q_k \in \mathbb{Q}$.*

## S.3 Boolean variational representation

The variational model introduced by Maccheroni, Marinacci, and Rustichini (2006) (henceforth, MMR) relies on the following relaxation of certainty independence, which retains the "location invariance" property of preferences but relaxes the "scale invariance" property; we refer to MMR for a discussion.

**Axiom 10** (Weak Certainty Independence)**.** For any $f, g \in \mathcal{F}$, $p, q \in \Delta(Z)$, and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha)p \succsim \alpha g + (1-\alpha)p \Longrightarrow \alpha f + (1-\alpha)q \succsim \alpha g + (1-\alpha)q.$$

We now show that dropping uncertainty aversion from MMR's axioms corresponds to adding a maximization stage into the variational model. A ***cost collection*** is a collection of functions $c : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ such that each $c \in \mathbb{C}$ is convex and $\mathbb{C}$ is ***grounded*** (i.e., $\max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} c(\mu) = 0$).

**Theorem S.3.1.** *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 10 if and only if $\succsim$ admits a* ***Boolean variational (BV)*** *representation, i.e., there exists a cost collection $\mathbb{C}$ and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W_{BV}(f) := \max_{c \in \mathbb{C}} \min_{\mu \in \Delta(S)} \mathbb{E}_\mu[u(f)] + c(\mu) \tag{20}$$

*is well-defined and represents $\succsim$.*

We note that our characterization of the set of relevant priors under BEU generalizes to the Boolean variational model. Specifically, let $\mathrm{dom}(c) := \{\mu : c(\mu) \in \mathbb{R}\}$ denote the effective domain of any cost function. Then there exists a unique closed, convex set $C$ such that $C \subseteq \overline{\text{co}}\left(\bigcup_{c\in\mathbb{C}} \text{dom}(c)\right)$ for all Boolean variational representations of $\succsim$, with equality for the representation $\mathbb{C}^*$ we construct in the proof of Theorem S.3.1 below. Moreover, it can again be shown that $C$ is the Bewley set of the unambigous preference $\succsim^*$. The argument relies on the observation that $C = \overline{\text{co}}\left(\bigcup_{\phi\in\text{int}U} \partial I(\phi)\right)$, where $I$ is the utility act functional obtained in the proof of Theorem S.3.1 and $U$ its domain. Details are available on request.

### S.3.1 Proof of Theorem S.3.1

We will invoke the following result from MMR:

**Lemma S.3.1** (Lemma 28 in MMR)**.** *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 10 if and only if there exists a nonconstant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a normalized niveloid $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Recall that functional $I : U \to \mathbb{R}$ is a *niveloid* if $I(\phi) - I(\psi) \leq \max_s(\phi_s - \psi_s)$ for all $\phi, \psi \in U$. Lemma 25 in MMR shows that $I$ is a niveloid if and only if it is monotonic and constant-additive.

Based on this result, the necessity direction of Theorem S.3.1 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and Axiom 10. Let $I$, $u$, and $U$ be as given by Lemma S.3.1. Since $I$ is a niveloid, it is 1-Lipschitz. Hence, Lemma A.1 yields a subset $\hat{U} \subseteq \text{int}U$ with $U \setminus \hat{U}$ of Lebesgue measure 0 such that $I$ is differentiable on $\hat{U}$. Define $\mu_\psi := \nabla I(\psi)$ and $w_\psi := I(\psi) - \nabla I(\psi) \cdot \psi$ for each $\psi \in \hat{U}$. By Lemma A.4 and the fact that niveloids are monotonic and constant-additive, $\mu_\psi \in \Delta(S)$ for all $\psi \in \hat{U}$. For each $\psi \in U$, define

$$D_\psi := \{(\mu, w) \in \Delta(S) \times \mathbb{R} : \mu \cdot \psi + w \geq I(\psi)\} \cap \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in \hat{U}\},$$

and let $\mathbb{D} := \{D_\psi : \psi \in U\}$. The following lemma implies that each $D_\psi$ is nonempty; note also that it is closed, convex, and bounded below.

**Lemma S.3.2.** *For every $\phi, \psi \in U$, $\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi)$ with equality if $\phi = \psi$.*

*Proof.* First, consider any $\phi, \psi \in \hat{U}$. Let $K_\psi := \{\xi \in \hat{U} : \mu_\xi \cdot \psi + w_\xi \geq I(\psi)\}$ be as in Lemma A.6. Note that $D_\psi = \overline{\text{co}}\{(\mu_\xi, w_\xi) : \xi \in K_\psi\}$, so that

$$\inf_{\xi\in K_\psi} \mu_\xi \cdot \phi + w_\xi = \min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w,$$

where the minimum is attained as $D_\psi$ is closed and bounded below. Thus, Lemma A.6 implies that

$$\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi), \tag{21}$$

where (21) holds with equality if $\psi = \phi$ by definition of $D_\psi$.

Next, consider any $\phi, \psi \in U$. Take sequences $\phi_n \to \phi$, $\psi_n \to \psi$ such that $\phi_n, \psi_n \in \hat{U}$ for each $n$, where we choose $\phi_n = \psi_n$ if $\phi = \psi$. For each $n$, the previous paragraph yields some $(\mu_n, w_n) \in D_{\psi_n}$ such that $\mu_n \cdot \phi_n + w_n = \min_{(\mu,w)\in D_{\psi_n}} \mu \cdot \phi_n + w \leq I(\phi_n)$, with equality if $\phi = \psi$. Thus, for each $n$, we have $I(\psi_n) - \mu_n \cdot \psi_n \leq w_n \leq I(\phi_n) - \mu_n \cdot \phi_n$. Since $\phi_n \to \phi$, $\psi_n \to \psi$, and $I$ is continuous, this implies that sequence $(w_n)$ is bounded. Thus, up to restricting to a suitable subsequence, we can assume that $(\mu_n, w_n) \to (\mu_\infty, w_\infty)$ for some $(\mu_\infty, w_\infty) \in \Delta(S) \times \mathbb{R}$. Then $(\mu_\infty, w_\infty) \in D_\psi$ and $\mu_\infty \cdot \phi + w_\infty \leq I(\phi)$ by continuity of $I$, with equality if $\phi = \psi$. Thus, $\min_{(\mu,w)\in D_\psi} \mu \cdot \phi + w = \inf_{(\mu,w)\in D_\psi} \mu \cdot \phi + w \leq I(\phi)$, with equality if $\phi = \psi$, where the minimum is attained since $D_\psi$ is closed and bounded below. $\square$

Finally, we obtain a Boolean variational representation of $\succsim$ as follows. For each $D \in \mathbb{D}$, define $c_D : \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by $c_D(\mu) := \inf\{w \in \mathbb{R} : (\mu, w) \in D\}$ for each $\mu \in \Delta(S)$, where by convention the infimum of the empty set is $\infty$. Note that $c_D$ is convex for all $D$ by convexity of $D$. Moreover, for all $\phi \in U$, $\min_{(\mu,w)\in D} \mu \cdot \phi + w = \min_{\mu\in\Delta(S)} \mu \cdot \phi + c_D(\mu)$. Thus, Lemma S.3.2 implies

$$I(\phi) = \max_{D\in\mathbb{D}} \min_{\mu\in\Delta(S)} \mu \cdot \phi + c_D(\mu) \tag{22}$$

for all $\phi \in U$. Since $I$ is normalized, applying (22) to any constant vector $\underline{a} \in U$, yields $I(\underline{a}) = a + \max_{D\in\mathbb{D}} \min_{\mu\in\Delta(S)} c_D(\mu) = a$. Thus, collection $(c_D)_{D\in\mathbb{D}}$ is grounded. Hence, $\mathbb{C}^* := \{c_D : D \in \mathbb{D}\}$ is a cost collection and $(\mathbb{C}^*, u)$ is a BV representation of $\succsim$ by Lemma S.3.1. $\square$

## S.4 Rational Boolean representation

Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011) (henceforth, CMMM) maintain uncertainty aversion, but further relax independence to hold only for objective lotteries:

**Axiom 11** (Risk Independence)**.** For any $p, q, r \in \Delta(Z)$ and $\alpha \in (0, 1)$,

$$p \succsim q \Longrightarrow \alpha p + (1-\alpha) r \succsim \alpha q + (1-\alpha) r.$$

Dropping uncertainty aversion from CMMM's axioms yields the following Boolean generalization of their representation:

**Theorem S.4.1.** *Preference $\succsim$ satisfies Axioms 1–4 and Axiom 11 if and only if $\succsim$ admits a* ***rational Boolean (RB)*** *representation, i.e., there exists a collection $(G_t)_{t\in T}$ of quasiconvex functions $G_t : \mathbb{R} \times \Delta(S) \to \mathbb{R} \cup \{\infty\}$ that are increasing in their first argument and grounded*[28]

[28] That is, $\max_{t\in T} \inf_{\mu\in\Delta(S)} G_t(a, \mu) = a$ for all $a$.

*and a nonconstant affine utility $u : \Delta(Z) \to \mathbb{R}$ such that*

$$W_{RB}(f) := \max_{t\in T} \inf_{\mu\in\Delta(S)} G_t(\mathbb{E}_\mu[u(f)], \mu) \tag{23}$$

*is well-defined, continuous, and represents $\succsim$.*

### S.4.1 Proof of Theorem S.4.1

The following result follows from a minor modification of the proof of Lemma 57 in CMMM:

**Lemma S.4.1.** *Preference $\succsim$ satisfies Axioms 1–4 and 11 if and only if there exists a non-constant affine function $u : \Delta(Z) \to \mathbb{R}$ with $U := (u(\Delta(Z)))^S$ and a monotonic, normalized and continuous functional $I : U \to \mathbb{R}$ such that $I \circ u$ represents $\succsim$.*

Based on this result, the necessity direction of Theorem S.4.1 is standard. We now prove the sufficiency direction. Suppose $\succsim$ satisfies Axioms 1–4 and 11. Let $I$, $u$, and $U$ be as given by Lemma S.4.1.

Define $D_\psi := \{(\mu, I(\psi) - \mu\cdot\psi) \in \mathbb{R}^S_+ \times \mathbb{R} : \mu \in \mathbb{R}^S_+\}$ for each $\psi \in U$. Note that $D_\psi$ is nonempty and convex. Let $I_\psi(\phi) := \inf_{(\mu,w)\in D_\psi} \mu\cdot\phi + w$ for each $\phi, \psi \in U$.

Take any $\phi, \psi \in U$. Observe that

$$I_\psi(\phi) = \inf_{\alpha>0, s\in S} I(\psi) + \alpha(\phi_s - \psi_s) = \begin{cases} I(\psi) \text{ if } \phi \geq \psi \\ -\infty \text{ if } \phi \ngeq \psi \end{cases}$$

Thus, $I(\phi) \geq I_\psi(\phi)$ by monotonicity of $I$, with equality if $\phi = \psi$. That is, for each $\phi \in U$,

$$I(\phi) = \max_{\psi\in U} I_\psi(\phi). \tag{24}$$

For each $\psi \in U$, define a function $G_\psi : \mathbb{R} \times \Delta(S) \to \mathbb{R} \cup \{\infty\}$ by

$$G_\psi(t, \mu) = \sup\{I_\psi(\xi) : \xi \in U, \xi\cdot\mu \leq t\}$$

for each $(t, \mu)$. The map is quasi-convex (Lemma 31 in CMMM) and increasing in $t$.

**Lemma S.4.2.** $I_\psi(\phi) = \inf_{\mu\in\Delta(S)} G_\psi(\mu\cdot\phi, \mu)$ *for each $\phi, \psi \in U$.*

*Proof.* Observe that RHS $= \inf_{\mu\in\Delta(S)} \sup\{I_\psi(\xi) : \xi\cdot\mu \leq \phi\cdot\mu\}$. To see that LHS $\leq$ RHS, observe that $I_\psi(\phi) \leq \sup\{I_\psi(\xi) : \xi\cdot\mu \leq \phi\cdot\mu\}$ holds for any $\mu \in \Delta(S)$.

To see that LHS $\geq$ RHS, note first that if $\phi \geq \psi$ then LHS $= I(\psi)$ and RHS $\in \{I(\psi), -\infty\}$, so the inequality clearly holds. If $\phi \ngeq \psi$ then $\phi_s < \psi_s$ for some $s \in S$.

Thus, by taking $\mu = \delta_s$, any $\xi$ with $\xi \cdot \mu \leq \phi \cdot \mu$ satisfies $\xi_s \leq \phi_s$, which implies $\xi \not\geq \psi$, whence $I_\psi(\xi) = -\infty$. $\square$

Setting $T = U$, Lemma S.4.2 and (24) ensure that $W_{RB}$ given by (23) represents $\succsim$ and is continuous. Finally, to check groundedness, note that since $I$ is normalized, we have $a = I(\underline{a}) = \max_{\psi \in U} \inf_{\mu \in \Delta(S)} G_\psi(a, \mu)$ for any $a \in \mathbb{R}$. $\square$