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Abstract

We uncover new volatile sentiment-driven equilibria in a canonical dynamic economy with nominal rigidities. Self-fulfilling fluctuations emerge because, at the zero lower bound, output is demand-determined, and demand is unanchored. If agents conjecture higher asset price volatility, then wealth, aggregate demand, and hence production all become more sensitive to shocks, justifying the higher price uncertainty. Different from the existing literature, our multiplicity is unrelated to inflation or investment. Certain unconventional policies (e.g., asset purchases) can help, but they must be very aggressive and highly credible; otherwise, unconventional policies may actually increase uncertainty rather than reduce it.

JEL Codes: E00, E12, E30, E40, G01.

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1 Introduction

In the textbook New Keynesian model, output is determined by aggregate demand, and current demand depends on future demand. This dependence opens the door to multiple equilibria. Monetary policy can restore determinacy by following an active rule that strongly reacts to output and inflation deviations (e.g., the Taylor principle). However, multiplicity can re-emerge if policy constraints like the zero lower bound (ZLB) can interfere with the active policy rule.

In influential work, Benhabib et al. (2001a) showed that the ZLB can lead to "deflationary trap" equilibria. In their model, the interest rate falls to the ZLB because of low inflation expectations, which become self-fulfilling because policy is stuck at the ZLB and the recession lowers inflation. More recently, Benigno and Fornaro (2018) showed there can also be "stagnation trap" equilibria. In their model, the interest rate falls to the ZLB because of low growth expectations, which become self-fulfilling because policy is stuck at the ZLB and the recession lowers R&D investment.

Set against this background, the present paper describes "volatility trap" equilibria. In these equilibria, the interest rate falls to the ZLB because of a sudden rise in asset-price volatility that raises the risk premium. This volatility becomes self-fulfilling because the policy is stuck at the ZLB where aggregate demand is unanchored. In particular, aggregate demand depends on asset prices, and asset prices can fluctuate subject to mild valuation requirements about the relationship between current prices and future prices.

We develop these insights in a standard New Keynesian model: markets are complete, the representative agent is fully rational, but prices are sticky and monetary policy is constrained by the ZLB. To distinguish ourselves from the earlier literature on indeterminacies in this class of models, we study a stylized setting where prices are fully rigid and where aggregate capital growth is exogenous. Thus, nothing about our equilibrium design relies on inflation dynamics (Benhabib et al., 2001a,b) or investment dynamics (Benigno and Fornaro, 2018). Another key difference between our paper and the earlier literature is our focus on risk and risk premia as a driving force, and as such, we study the fully nonlinear version of our model. This becomes easier in continuous time, which is why we adopt a continuous-time risk-centric articulation of the New Keynesian setting, due to Caballero and Simsek (2020c).

Now, we describe the intuition behind our equilibria in more detail. Consider a sudden wave of *fear*, meaning agents perceive higher volatility going forward. Fearful agents engage in precautionary savings, putting downward pressure on the interest rate. If the fear is sufficiently strong, the economy is pushed to the ZLB. The interest rate can no longer clear the bond market, so instead aggregate wealth falls. The resulting drop in aggregate demand also lowers output, through a reduction in capacity utilization, since production is demand-determined at the ZLB. Given the drop in output, and the binding ZLB constraint, the conjectured rise in volatility can be self-justified. In particular, demand becomes unanchored: it can rise back to its efficient level or it can decline further, and nothing besides coordination pins down which will occur.

Agents' beliefs about this entire fear-driven sequence will be justified, so long as they lead to stable long-run behavior. It turns out that volatility, which raises risk premia, is precisely what generates stability. In a fear-driven recession, required returns must be satisfied by an expected future appreciation in asset prices (since asset dividends are low after the output drop), and this positive drift pushes the economy back towards recovery in expectation. In other words, a fearful regime is partly self-fulfilled by an expected future decline in volatility, i.e., an improvement in conditions. This highlights an intuitive distinction between our fear-driven equilibria and the multiplicity of non-stochastic equilibria of our model (and the literature), in which an inefficient recession can be self-fulfilled by the expectation of greater inefficiencies in the future.

Our theory thus uncovers self-fulfilled uncertainty-driven recessions that monetary policy has little power to prevent or tame. This process is inefficient and possesses a different character from standard indeterminacies in New Keynesian models.

In the latter half of the paper, we go beyond conventional interest rate policy and explore unconventional monetary policies. We first find that an asset-purchase program that commits to buy assets and support prices can have a profound effect. In particular, all the indeterminacies we document vanish in the starkest version of such a policy. However, we also find that this result hinges critically on both the aggressiveness and credibility of the policy. Asset-purchase programs must act aggressively enough to prevent price declines, and they must be trusted fully to deliver this outcome.

The importance of aggressiveness and credibility highlights the mechanism behind asset-purchase policies: they provide a floor for prices that works to raise expected capital returns. If the policy is both aggressive and credible, this price floor is rigid, so that the change in return dynamics constitutes an arbitrage opportunity. Existence of arbitrages are contrary to equilibrium, which effectively kills all indeterminacies in the model. No asset purchases ever need to be made, as the policy works solely through its effect on expectations.

But if the policy lacks either aggression or credibility, the price floor is no longer rigid. Not only can indeterminacies remain, they can be worsened in some cases. For example, we show that some less-aggressive, less-credible policies can increase the amount of uncertainty in equilibrium. The reasoning is that even sub-optimal asset purchases still work to increase expected capital returns, and this can only happen in a rational, frictionless financial market if risk premia rise as well. Thus, volatility must increase in our sub-optimal policies. As a corollary, we find that some sub-optimal policies can eliminate the deterministic multiplicities (the kind mostly studied by the literature) but fail to kill our fear-driven equilibria.

In summary, our analysis embodies the view that constraints on monetary policy can exacerbate recessions and crises, through self-fulfilling volatility. Our analysis also shows how unconventional policies can help, not necessarily through liquidity or balance-sheet effects but through beliefs. At the same time, our paper stresses the importance of policymakers using aggressive policies and maintaining their credibility to achieve the desired outcome.

Relative to the literature, Benhabib et al. (2001a) and Benigno and Fornaro (2018) are closest to our paper in studying indeterminacies in New Keynesian models. We differ by focusing on uncertainty, rather than inflation or investment which are both exogenous and constant in our setup. We discuss how our volatile equilibria contain slightly different intuition than standard deterministic multiplicities and how self-fulfilling volatility may be harder for policy to extinguish.

Our results are also distinct from the literature on monetary policy and asset price bubbles (Galí, 2014; Allen et al., 2018; Miao et al., 2019; Dong et al., 2020; Asriyan et al., 2021). The model we present does not admit bubbles, output is always weakly below potential, and optimal monetary policy has a clear directive to maximize output. Nevertheless, it would be interesting for future research to explore how rational bubbles interact with sentiment-driven volatility at the ZLB.

All New Keynesian models feature a well-known aggregate demand externality at the ZLB: privately lower demand reduces output and wealth, which induces others to cut demand as well. Ultimately, the externality manifests as a connection between asset prices and output efficiency. In a different framework, Khorrami and Mendo (2022) analyze a setting in which multiple equilibria also arise due to a price-output link that is the manifestation of an externality. There, an aggregate supply externality operates through fire sales that reduce allocative efficiency. This comparison suggests that the distinction between demand and supply externalities is immaterial to the existence of sunspot equilibria; what really matters is a price-output link, which can be achieved via financial frictions or via nominal rigidities.[1]

[1] A related interpretation, offered by Benhabib et al. (2020) in extending the model of Bacchetta et al.

2 Model

We present a complete-markets economy with nominal rigidities that supports self-fulfilling fluctuations. The setup is a simplified version of Caballero and Simsek (2020c), which the reader can consult for additional details.

Sunspot shocks. Our baseline model features no fundamental uncertainty in preferences or technologies. Nevertheless, we want to allow the possibility that economic objects evolve stochastically due to coordinated behavior. To do this, we introduce a standard Brownian motion Z that is extrinsic to all economic primitives. All random processes will be adapted to Z.[2] In Appendix B, we also allow for a fundamental shock to technology to illustrate how our results on multiplicity carry through in such a setting.

Preferences. The representative agent has rational expectations and time-separable logarithmic utility with discount rate ρ:

$$\sup_{C\geq 0} \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\log(C_t)dt\Big]. \tag{1}$$

Technology. There are two goods, a non-durable good (the numéraire, "consumption") and a durable good ("capital") that produces the consumption good. The aggregate supply of capital grows deterministically as

$$\dot{K}_t = gK_t, \tag{2}$$

where g is an exogenous constant. For simplicity, there is no investment in the model. As Caballero and Simsek (2020c) show, endogenous investment serves to amplify the effects of uncertainty on asset prices.

The relative price of capital, denoted by q_t, is determined in equilibrium. Since capital is the only positive net supply asset in the economy, aggregate wealth is q_tK_t. Conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t}dZ_t]. \tag{3}$$

(2012), is that asset prices should have a direct impact on the stochastic discount factor, which is exactly what happens with a price-output link. Certain OLG specifications, financial frictions, and (as we show here) nominal rigidities all connect asset prices to the SDF.

[2] In the background, the Brownian motion Z exists on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, assumed to be equipped with all the "usual conditions." All equalities and inequalities involving random variables are understood to hold almost-everywhere and/or almost-surely.

The term σ_q measures sunspot volatility that only exists because agents believe in it. Our core question is whether any equilibrium exists in which $\sigma_q \neq 0$ and what policy can do about it.

Producers employ capital in a linear production technology with productivity A. The assumption of a single productivity level is without loss of generality because of complete financial markets.[3] Producers' prices are fully rigid, which is a convenient assumption that also allows us to distinguish our results from the self-fulfilling deflation dynamics that can occur in New Keynesian models (Benhabib et al., 2001a,b). Here, inflation will always be equal to zero. As a result, note that the real riskless interest rate r_t is equal to the nominal rate, which is set by monetary policy.

The appendix of Caballero and Simsek (2020c) discusses a few auxiliary assumptions (lump sum profit taxes and linear capital subsidies, which we also implicitly adopt) designed to simplify the analysis, namely to ensure the market portfolio dividend equals aggregate output.

Monetary policy. Although the economy's potential output is AK_t, firms may not always operate at capacity because of nominal rigidities. We assume that monetary policy aims to achieve full utilization whenever possible, but they are subject to the ZLB $r_t \geq 0$.

In particular, let $\chi_t \leq 1$ denote firms' capital utilization, which will be determined in equilibrium. Aggregate output is $\chi_t A K_t$. Monetary authorities set the nominal rate (hence the real rate) to implement $\chi_t = 1$ whenever possible, subject to the ZLB. Under this rule, full utilization prevails whenever the real rate is positive, and inefficient utilization must arise at the ZLB:

$$0 = \min[1 - \chi_t, r_t]. \tag{4}$$

In the rest of the paper, we simply assume the central bank acts using a policy rule that implements (4). In Appendix A, we show that within the class of equilibria we study, (4) is actually the outcome of optimal discretionary monetary policy (i.e., monetary policy without commitment to future policies). More deeply, the implementation of $\chi_t = 1$ "whenever possible" itself requires some kind of commitment to off-equilibrium threats, for instance to reduce interest rates if χ_t ever fell below 1—this is the standard notion of "active" monetary policy that pervades the New Keynesian literature. In that sense, the rule (4) actually embeds some amount of commitment power.

[3] Indeed, in a heterogeneous-productivity world, all physical resources would be distributed among the most productive agents, who would then issue financial claims, so the economy would look as if there were a single productivity level.

Equilibrium definition. To define equilibrium, let N_t denote the net worth of the representative agent, and let M_t be the stochastic discount factor induced by asset prices (q_t, r_t).

Definition 1. An *equilibrium* is a set of stochastic processes $(N_t, K_t, C_t, q_t, r_t, \chi_t)_{t\geq 0}$, with $K_0 > 0$ given, such that

(i) Taking (q_t, r_t) as given, consumers maximize (1) subject to their lifetime budget and No-Ponzi constraints[4]

$$N_0 \geq \mathbb{E}\Big[\int_0^\infty M_t C_t dt\Big] \tag{5}$$

$$\lim_{T\to\infty} M_T N_T \geq 0. \tag{6}$$

(ii) Markets clear, namely $N_t = q_t K_t$ (asset market clearing) and $C_t = \chi_t A K_t$ (goods market clearing), where capital K_t evolves according to (2).

(iii) The central bank follows the interest rate rule (4).

Note that individual optimality will ensure that both (5)-(6) hold with equality. In what follows, we refer to a *fundamental equilibrium* as an equilibrium with no volatility, $\sigma_q \equiv 0$. A *sunspot equilibrium* is an equilibrium with volatility, $\sigma_q \neq 0$.

Equilibrium characterization. The consumer's problem is a relatively standard complete-markets consumption problem, and we provide a summary characterization of its implications that aids in solving the model and finding equilibria. First, log utility agents consume a fraction ρ of their wealth N_t, and aggregate wealth is $q_t K_t$, so goods market clearing can be written as

$$\rho q = \chi A. \tag{7}$$

We can think of (7) as a link between asset prices (q) and output efficiency (χ), which is a way to understand our mechanism. We also define $q^* := A/\rho$, which is the efficient capital valuation.

Consumption and portfolio choices are unconstrained and imply the Euler equation:

$$r = \frac{\chi A}{q} + g + \mu_q - \sigma_q^2. \tag{8}$$

[4] In addition, to prevent certain arbitrages such as "doubling strategies" that can emerge in a stochastic equilibrium, we must impose a lower bound on net worth, $n_t \geq -\underline{n}$, although the value of $\underline{n}$ can be arbitrarily large.

Note that $\chi A/q + g + \mu_q$ is the expected return-on-capital, and σ_q^2 is the risk premium in the economy. Using (7), we can substitute $\chi A/q$ with ρ.

Finally, note that agents' transversality condition is automatically satisfied in any equilibrium of this model. Indeed, optimal consumption implies marginal utility coincides with the stochastic discount factor, i.e., $M_t = e^{-\rho t} C_t^{-1}$. Consequently, $\mathbb{E}[M_T N_T] = \mathbb{E}[e^{-\rho T} C_T^{-1} N_T]$, which trivially converges to zero as $T \to \infty$, given consumption rule $C_T = \rho N_T$. We summarize the preceding characterization in a lemma that simplifies our search for equilibria.

Lemma 1. *Suppose processes $(q_t, \chi_t, r_t)_{t\geq 0}$ satisfy equations (4)-(8) for all $t \geq 0$. Put $K_t = e^{gt}K_0$, $N_t = q_t K_t$, and $C_t = \rho N_t$. Then, $(N_t, K_t, C_t, q_t, r_t, \chi_t)_{t\geq 0}$ constitutes an equilibrium of Definition 1.*

Remark 1. *We have chosen to write our equilibrium conditions in terms of the asset price q. But our model is a complete-markets model, and so everything could equally well be described by consumption C without reference to q. For example, since consumption dynamics are given by $dC_t = C_t[\mu_{C,t}dt + \sigma_{C,t}dZ_t] = C_t[(g + \mu_{q,t})dt + \sigma_{q,t}dZ_t]$ (this is understood by using $C = \rho q K$ and using the dynamics of q and K), the Euler equation (8) takes the standard log-utility form $r - \rho = \mu_C - \sigma_C^2$.*

It is convenient to describe equilibria in terms of q, because (i) q is a bounded variable, unlike C; (ii) in models with endogenous investment, the price of capital q measures endogenous investment incentives; (iii) volatility of asset prices is perhaps the most natural metric for forward-looking uncertainty, even in more complicated models with non-log preferences and financial constraints (in which C may not inherit the dynamics of q); (iv) when introducing and interpreting our unconventional monetary policy as asset purchases, it will be more natural to work the space of asset prices.

3 Fundamental equilibria

We start by describing equilibria without volatility, $\sigma_q \equiv 0$. First, we illustrate the basic indeterminacy. Second, for pedagogical purposes, we also demonstrate an analogy to the indeterminacies in a textbook discrete-time log-linearized New Keynesian economy.

3.1 Indeterminacy with a ZLB

There is always an efficient equilibrium with full utilization, $\chi = 1$. Using $\chi = 1$ in the goods market clearing condition (7), we obtain $q = q^*$, so $\mu_q = \sigma_q = 0$ must hold in this equilibrium. By equation (8), the interest rate is given by $r = \rho + g > 0$, which satisfies the monetary policy rule (4).

Can there exist an inefficient equilibrium with $\chi < 1$? By (4), under-utilization implies a binding ZLB. Using $r = 0$ and $\sigma_q = 0$ in the Euler equation (8) implies that $\mu_q = -(\rho + g)$, so q must be converging to zero asymptotically, at the exponential rate $\rho + g$. In other words, given any initial value $q_0 < q^*$, an equilibrium can be supported as long as it remains inefficient forever and furthermore converges eventually to complete shut-down.

Combining this with the existence of the efficient equilibrium, we see that any initial price $q_0 \in [0, q^*]$ can kick-start a fundamental equilibrium. We summarize these results in the following proposition.

Proposition 1. *Any initial price $q_0 \in [0, q^*]$ is consistent with a fundamental equilibrium. Except in the case $q_0 = q^*$, all other fundamental equilibria are inefficient in the sense that $\chi_t < 1$ at all times, and they feature asymptotic shut-down* $\lim_{T\to\infty} q_T = 0$.

3.2 Connection to textbook New Keynesian model

Consider the textbook log-linearized New Keynesian model (Woodford, 2003), with unitary EIS. Under an exogenous nominal interest rate, this model is characterized by the following Euler equation (IS curve) and inflation dynamics (Phillips curve):

$$x_t = \mathbb{E}_t[x_{t+1}] - (i_t - r_t^n - \mathbb{E}_t[\pi_{t+1}]) \tag{9}$$

$$\pi_t = \kappa x_t + e^{-\rho}\mathbb{E}_t[\pi_{t+1}] \tag{10}$$

where x is the (logarithm of the) output gap, i_t is the nominal interest rate, r^n is the "natural rate of interest," and π_t is the inflation rate. If prices are permanently sticky, as in our framework, then the parameter $\kappa = 0$, inflation is $\pi_t = 0$ for all t, and the nominal rate i_t can be replaced by the real rate r_t. In that case, the only relevant equation is the IS curve (9), which becomes

$$\mathbb{E}_t[x_{t+1}] = x_t + r_t - r_t^n. \tag{11}$$

That this dynamical system has a unit root illustrates the core indeterminacy of this class of models. Imagine there are no exogenous shocks and $r_t = r_t^n$ happens to hold (i.e., the exogenous interest rate happens to equal the natural rate). In that world, any constant solution $x_t = \bar{x}$ is allowed. In such a world but with additional shocks, x_t follows a random walk.

Introducing active monetary policy, suppose we add a Taylor-like rule

$$r_t = r_t^n + \phi x_t.$$

Substituting this rule into IS curve (11), we obtain a new equilibrium dynamical system

$$\mathbb{E}_t[x_{t+1}] = (1+\phi)x_t,$$

which is *unstable* as long as $\phi > 0$, in the sense that the unique non-explosive solution is $x_t = 0$ forever. In other words, with active monetary policy, equilibrium determinacy is restored. To implement this, monetary policy must commit to lower interest rates if the output gap turns negative.

With a ZLB, policy cannot lower rates, and indeterminacy re-emerges. Essentially, at the ZLB constraint, interest rates are back to following an exogenous process (namely, zero), which puts the economy back into the first case without active policy. Mathematically, suppose for simplicity the natural rate is constant at $r_t^n = 0$, and assume the following policy rule

$$r_t = \max[0, \phi x_t].$$

Substituting this rule into IS curve (11), the equilibrium dynamical system is

$$\mathbb{E}_t[x_{t+1}] = \max[x_t, (1+\phi)x_t] = \begin{cases} x_t, & \text{if } x_t \leq 0 \\ (1+\phi)x_t, & \text{if } x_t > 0 \end{cases}$$

which again supports any constant solution $x_t = \bar{x} \leq 0$ in a world without additional shocks. These results are all well-known, and so far essentially cover what we have shown in Proposition 1.[5]

[5]To be slightly more precise, suppose $r_t^n = \rho$, as in our model with $g = 0$ (which is the relevant benchmark as the textbook New Keynesian model has no growth). Let the policy rule be again constrained by the ZLB as

$$r_t = \max[0, \rho + \phi x_t].$$

Substituting this rule into IS curve (11), the equilibrium dynamical system is

$$\mathbb{E}_t[x_{t+1}] = x_t - \rho + \max[0, \rho + \phi x_t] = \begin{cases} x_t - \rho, & \text{if } x_t \leq -\rho/\phi \\ (1+\phi)x_t, & \text{if } x_t > -\rho/\phi. \end{cases}$$

This dynamical system supports any solution for x_t that declines by constant increments ρ, with those solutions indexed by the (endogenous and indeterminate) initial condition $x_0 < -\rho/\phi$. Since x is the log

4 Sunspot equilibria

In one sense, the multiplicity in Proposition 1 already suggests the existence of stochastic sunspot equilibria; one could imagine constructing them by "randomizing" over deterministic equilibria as in classic studies (Azariadis, 1981). On the other hand, the economics of stochastic equilibria, due to the presence of risk premia, differs somewhat from the economics behind multiple deterministic equilibria. As we will see later, this additional nuance distinguishes stochastic and deterministic equilibria again when thinking about which policies can restore determinacy.

4.1 Constructing volatile equilibria

Consider an inefficient, volatile economy: suppose $\chi < 1$, so that $r = 0$, but do not impose $\sigma_q = 0$. Using $r = 0$ in the Euler equation (8), we obtain $\mu_q = -(\rho + g) + \sigma_q^2$. As the ZLB is satisfied, an inefficient equilibrium places no further restrictions, except that (σ_q, μ_q) must keep $q_t \in (0, q^*]$. (Consumption cannot be zero in finite time, because the representative household would attain utility of $-\infty$.) This is a relatively modest requirement, because for any σ_q bounded and bounded away from zero, the dynamics of q_t will be approximately like a geometric Brownian motion, at least near the relevant boundary $q = 0$. As σ_q is indeterminate, there are many ways to do this. Essentially arbitrary levels of volatility are feasible at any price q.

The only other equilibrium consideration is what happens when q_t hits q^*, i.e., efficiency is restored. Here, again, there are many possibilities, as nothing pins down the speed at which the economy re-enters the inefficient region. The economy could remain efficient permanently (which is sensible given the non-volatile efficient equilibrium always exists), it could transition immediately back to inefficiency, or the economy could remain efficient for some period of time before stochastically re-entering the inefficient region. For this reason, the stationary probability of inefficiency π can be anything.[6] The next theorem accounts for the relevant technical details and proves the existence of an entire class of equilibria following the discussion above.

Theorem 1. *Let $v : \mathbb{R} \mapsto \mathbb{R}_+$ be any non-negative Lipschitz continuous function with $v(x) > 0$ for $x \in [0, q^*)$. A sunspot equilibrium exists, in which $\sigma_{q,t}^2 = v(q_t)$ and $\mu_{q,t} = -\rho - g + v(q_t)$*

output gap, this solution implies output levels decline geometrically by a factor $e^{-\rho}$, which exactly mirrors our Proposition 1, i.e., if $g = 0$ we had $\dot{q} = -\rho q$ when $q_0 < q^*$.

[6] Of course, the possibility of permanent inefficiency may be unrealistic and tied to the fact that goods prices are completely rigid in this stylized model. Given that prices eventually adjust, one may think of our result as saying short-run volatility and inefficiency can be highly transitory or somewhat more persistent, with the maximal degree of persistence likely related to the degree of price stickiness.

whenever $q_t < q^*$. Furthermore, the inefficiency in this equilibrium can be permanent, transitory, or anything in between, i.e., the stationary probability of inefficiency can be any $\pi \in [0,1]$.

(i) If $v(q^*) = 0$, then inefficiency is permanent: $\pi = 1$.

(ii) If $v(q^*) > 0$, inefficiency eventually subsides but can re-emerge: $\pi < 1$.

Proof of Theorem 1. Consider an auxiliary variable $x_t \in (0, q^* + b)$ for some $b > 0$. Write the evolution of x as $dx_t = x_t[\mu_{x,t}dt + \sigma_{x,t}dZ_t]$. We are letting x_t be the state variable in this equilibrium. Set $q_t = \min[x_t, q^*]$, and put $\sigma_{x,t}^2 = v(x_t)$ and $\mu_{x,t} = -\rho - g + v(x_t)$ when $x_t < q^*$. Nothing pins down (σ_x, μ_x) when $x_t > q^*$, and we may simply set them so that x_t never reaches the boundary $q^* + b$. Many such choices exist (e.g., σ_x vanishes as $x \to q^* + b$ while μ_x remains strictly negative).

To prove such an equilibrium exists, it remains to show that $(x_t)_{t \geq 0}$, hence $(q_t)_{t \geq 0}$, almost-surely never attains the boundary $\{0\}$. Given $v(0)$ is positive and bounded, x_t behaves like a geometric Brownian motion near $x = 0$, and $\log(x_t)$ has positive drift at $x = 0$ if and only if $v(0) > 2(\rho + g)$. No geometric Brownian motion ever attains the boundary $\{0\}$ in finite time, and furthermore if $v(0) > 2(\rho + g)$, such a process does not concentrate probability near $\{0\}$ asymptotically. A rigorous proof of this claim, using Feller's boundary classification for diffusions, is provided by Lemma 3 in Appendix A.

It remains to show any $\pi \in [0,1]$ is possible. If $v(q^*) = 0$, then $\mu_x(q^*-) < 0$, so that x_t never attains the point $x = q^*$ if started below it. Thus, $\pi = 1$ if $v(q^*) = 0$. One similarly shows that $x = q^*$ is attainable if $v(q^*) > 0$, since then $\sigma_x(x) > 0$ for all $x \in (0, q^*]$, whereas $\mu_x(x)$ is bounded. Thus, $\pi < 1$ if $v(q^*) > 0$. Furthermore, since (σ_x, μ_x) are not pinned down when $x_t > q^*$, appropriate choices of these dynamics can deliver any value of $\mathbb{P}[x_t \geq q^*] = 1 - \pi$.

Finally, to show that one can construct an equilibrium where inefficiency is completely transitory ($\pi = 0$), consider putting $v(q^*) > 0$, so that x_t eventually exceeds q^* with probability 1, and putting $\sigma_x(x) = 0$ and $\mu_x(x) > 0$ on $\{x \geq q^*\}$ so that x_t never leaves this region. $\square$

Figure 1 below displays a numerical example of an equilibrium from Theorem 1. We have chosen a functional form for σ_q to ensure the existence of a well-behaved stationary distribution for the economy; we discuss the role of σ_q in inducing stationarity below. In this example, the economy features inefficiency 16% of the time; inefficiency need not be permanent. While inefficient, the amount of non-fundamental volatility can be large, on the order of 15-30%. Asset prices can drop a significant amount, as seen in the long left tail of the stationary CDF.

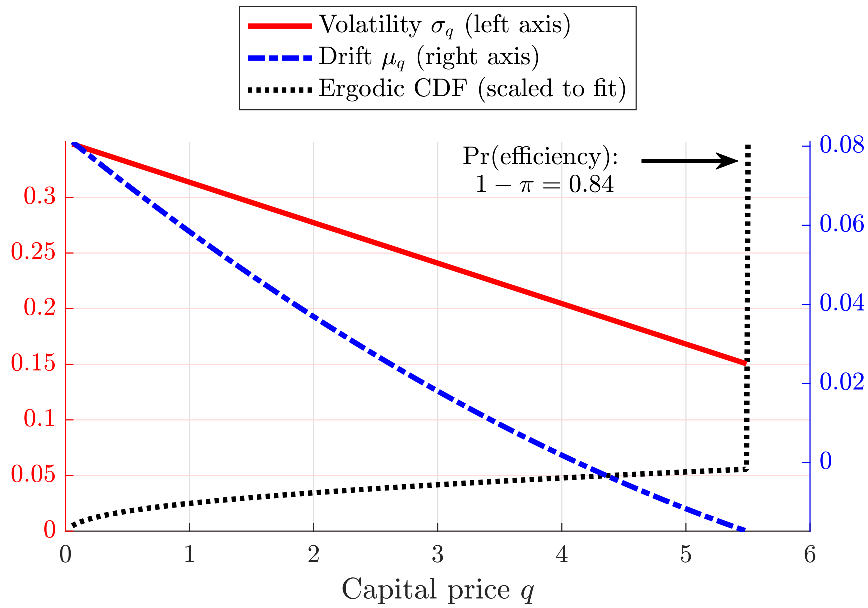

Figure 1: Equilibrium with nominal rigidities. We set volatility $\sigma_q(q) = (1 - q/q^*)\sqrt{\rho + g} + 0.15$ when $q < q^*$, which meets conditions of Theorem 1. To compute the stationary CDF, we specify the dynamics of an auxiliary diffusion x on domain $(0, 1.1q^*)$, and put $q = \min[x, q^*]$. When $x < q^*$, dynamics of x and q match, by definition. When $x > q^*$ (i.e., $q = q^*$), dynamics of x can be set arbitrarily, and they control how long q stays at q^*. The resulting stationary CDF features a mass point of size $1 - \pi = 0.84$ at $q = q^*$ (i.e., inefficiency occurs 16% of the time). Parameters: $A = 0.11$, $\rho = 0.02$, $g = 0.02$.

4.2 Contrast to deterministic multiplicity

What are the economics of these sunspot equilibria? Given the presence of asset-price volatility, and its induced risk premium, there are some differences from standard sunspot constructions that are simple randomizations over deterministic equilibria.

First, to understand the logistics of how sunspot equilibria arise, suppose agents are suddenly *fearful*, and they conjecture $\sigma_q > 0$. Is this justified? Fear leads to a precautionary savings motive, putting downward pressure on the "natural interest rate." Without a ZLB, the central bank has the power to lower r enough to match the natural rate and clear bond markets, with agents consuming and saving as before. Goods markets would be unaffected, utilization χ and price q would remain fixed, and agents' fear would be unsubstantiated. Forward-looking agents can think through this entire hypothetical sequence of events, and they will reject the feeling of fear as irrational. And as a result, the central bank would not actually have to do anything.

By contrast, suppose a ZLB exists. If fear and its associated precautionary savings pushes the natural rate below zero, the central bank cannot lower r enough to clear bond markets. Markets only clear if a counteracting force reduces savings, which is why wealth must fall. Due to wealth effects, current consumption also falls, and firms meet their lower demand by operating at less than full capacity in production ($\chi < 1$). Although this process is inefficient, nothing makes this sequence of hypothetical events irrational. Agents' fear will be justified, so long as it does not lead to unstable long-run behavior, which is guaranteed by bounded volatility σ_q.

The level of volatility helps distinguish the economics of sunspot equilibria from fundamental equilibria. Recall that inefficient fundamental equilibria necessitated long-run economic shut-down, via $\lim_{T\to\infty} q_T = 0$. Economically, inefficiency obtains because agents expect greater and greater inefficiency in the future, and this belief is self-fulfilled.

But in sunspot equilibria, it is not necessarily the case that $\lim_{T\to\infty} q_T = 0$. Volatility adds the risk premium σ_q^2, which now augments the capital price drift $\mu_q = -(\rho + g) + \sigma_q^2$, providing a force to buoy the equilibrium efficiency. In fact, one can show that the condition $\sigma_q^2(0+) > 2(\rho + g)$ suffices to guarantee that $\lim_{T\to\infty} q_T > 0$ almost-surely. Agents' belief in an inefficient and volatile equilibrium, in these cases, is justified by the fact that volatility is expected to vanish at some point in the future.[7] In a sense, volatility flips the intuition for multiplicity: an inefficient equilibrium can obtain partly because agents expect conditions to improve, rather than deteriorate, in the future.

While this economy above is stylized, the insights are general to the extent that the link between asset prices and output efficiency, captured in equation (7), is not severed. We can add other state variables, heterogeneous agents (e.g., some hand-to-mouth), or partial price flexibility, and the results will remain qualitatively unchanged.[8] In Appendix B, we also add fundamental risk in the form of shocks to capital; we show that sunspot equilibria still emerge and add additional fluctuations to the economy.

This entire discussion is conditional on a monetary policy rule that implements (4), which recall corresponds to optimal discretionary policy (see Lemma 2 of the Appendix A). The next section explores other policy options, namely unconventional monetary policies, that may be available to the central bank and relax policy constraints (the ZLB). Moreover, we will model such policies in a way that incorporates some commitment power, which is well-known to help in the New Keynesian literature.

[7] Going back to Theorem 1, one can see that volatility must decline eventually. Indeed, in case (i) of the theorem, inefficiency is permanent but volatility near q^* vanishes (and the dynamics ensure that the economy is recurrent, so it returns arbitrarily close to q^*). In case (ii) of the theorem, inefficiency eventually subsides, which clearly implies that volatility vanishes (i.e., when efficiency resumes). Therefore, in either case, future volatility is lower than current volatility.

[8] See Caballero and Simsek (2020a,b,c) for some of these extensions.

5 Unconventional monetary policy

Because of the ZLB constraint on interest rate policy, we also find it interesting to analyze a simple form of unconventional monetary policy. As we will see, unconventional policy will eliminate some undesirable equilibria that conventional policy cannot, depending on assumptions about policy credibility. Furthermore, this section will help clarify how our equilibria with volatility are somewhat more difficult to eliminate than the deterministic inefficiencies. In all the policy proposals that follow, some amount of commitment power is assumed, for instance to conduct unconventional monetary policy at a particular price or in a particular quantity.

5.1 Policy with full credibility

We suppose that, at some price $\bar{q} < q^*$, the central bank enters asset markets and purchases capital. To distinguish ourselves from the prevailing focus in the literature on fiscal policy, e.g., Benhabib et al. (2002a), we assume these asset purchases are budget-balanced: to make purchases, the central bank raises funds using lump-sum taxes, and it returns any profits from these open market operations to households lump-sum at some later date. A convenient consequence of this budget-balanced approach is that the representative household continues to have wealth $q_t K_t$ at all times, irrespective of any central bank purchases.

Enough capital is purchased to support asset prices at $\bar{q}$, and this pledge is viewed by agents as fully credible. In other words, the policy ensures that

$$q_t \geq \bar{q}. \tag{12}$$

Taking our model literally, such a stark price support will be feasible for the central bank, because beliefs and coordination will be central determinants of asset prices in our equilibria. Given enough credibility, the central bank could theoretically manipulate beliefs by taking very little action.

However, in the back of our minds, we think of these unconventional policies as costly in some unmodeled way. First, distortionary taxation or costly government borrowing may be required to finance such open market operations, rather than lump-sum taxation. Second, the assumption of full central bank commitment and credibility are obviously stretched. Third, while asset purchases undoubtably impact prices, the transmission may not operate through belief manipulation; various unmodeled liquidity frictions could also justify a link between “buying pressure” and asset prices, and if those issues are of first-order importance, one should account for the costs associated to central bank participation in illiquid markets. We abstract from these concerns, but the reader should think of $\bar{q}$ as being relatively low, such that these interventions are infrequent, as a simple way to account for the costs of unconventional policy.

In the analysis that follows, our equilibrium Definition 1, hence our characterization Lemma 1, is augmented by the condition that (12) holds at all times.

5.2 Restoring determinacy through policy credibility

We now show that (12) eliminates all equilibria except the efficient one. The crux of the analysis here is that unconventional monetary policy is regarded as credible, so all agents understand that q_t can never fall below $\bar{q}$. Intuitively, this should complicate the inefficient equilibria of Proposition 1, because those equilibria all required asymptotic shut-down, i.e., $q_T \to 0$. The intuition for killing the volatile sunspot equilibria of Theorem 1 is related but more subtle.

Killing deterministic inefficiency. The mechanics of unconventional policy are as follows. First, because of the policy's budget-balance assumptions (i.e., lump-sum taxation and rebates), the representative household maintains wealth of $q_t K_t$ at all times, irrespective of any potential asset purchases by the central bank. Thus, the Euler equation (8) must continue to hold at all times. Suppose q hits $\bar{q}$ at time τ, and purchases are made by the central bank. By (12), we must have $\mu_{q,\tau} \geq 0$. On the other hand, using $r_\tau = 0$ and $\sigma_\tau = 0$ in (8), we obtain $\mu_{q,\tau} = -(\rho + g) < 0$. The economics of this contradiction is that central bank asset purchases increase the rate of return on capital above the required rate, i.e., an arbitrage arises. This arbitrage implies non-equilibrium of the entire path leading to the asset purchases at $\bar{q}$. Since all of the inefficient equilibria (those featuring $q_0 < q^*$) will eventually hit a positive $\bar{q}$ in finite time, this shows how no inefficient equilibrium can arise! Intuitively, rational forward-looking agents expect asset prices to continuously fall in an inefficient equilibrium, but if the central bank can credibly commit to prevent that in the future, rational agents will rule out all such paths from consideration. This proves the following.

Proposition 2. *With any amount of unconventional monetary policy ($\bar{q} > 0$), the unique fundamental equilibrium coincides with the efficient equilibrium featuring full utilization at all times, $\chi_t = 1$.*

Killing stochastic inefficiency. The argument is similar to the deterministic case, but with some additional technical details. Most importantly, dynamics in stochastic equilibrium are always such that q_t reaches any $\bar{q} > 0$ in finite time. Indeed, in a stochastic equilibrium constructed as in Theorem 1, volatility $\sigma_q(q) > 0$ is positive for all $q \in (0, q^*)$, which guarantees q_t will eventually hit $\bar{q}$, due to the effect of recurring shocks. But even if σ_q were not strictly positive, a similar argument would still apply.[9] Given the policy guarantee $q_t \geq \bar{q}$ in equation (12), it must be that $\bar{q}$ is a reflecting boundary for the price process. (Note that $\bar{q}$ cannot be an absorbing boundary because shocks continue to hit the process through $\sigma_{q,t} dZ_t \neq 0$.)

Thus, the dynamics of asset prices are given by

$$\frac{dq_t}{q_t} = [-(\rho + g) + \sigma_{q,t}^2]dt + \sigma_{q,t} dZ_t + dR_t, \tag{13}$$

where R is the barrier process at $\bar{q}$, i.e., it increases if $q_t \leq \bar{q}$ and remains constant otherwise. Because the riskless rate is given by $r_t = 0$ by (4), the excess return on capital $(\rho + g)dt + \frac{dq_t}{q_t} - r_t dt = \sigma_{q,t}^2 dt + \sigma_{q,t} dZ_t + dR_t$ contains the barrier term. Appendix B of Karatzas and Shreve (1998) shows that such an excess return process admits arbitrage opportunities. Intuitively, the barrier process is a non-negative component to returns, so capital becomes effectively riskless at the policy barrier $\bar{q}$, and yet it earns a higher rate of return than riskless bonds. The fact that such an arbitrage arises again contradicts equilibrium. This proves the following.

Proposition 3. *With any amount of unconventional monetary policy ($\bar{q} > 0$), no sunspot equilibrium can exist.*

Putting together Propositions 2-3, we have the result that only the efficient equilibrium survives, with $q_t = q^*$ and $\chi_t = 1$ at all times. There are two interesting notes about Proposition 2. First of all, the central bank never needs to make any asset purchases, so unconventional monetary policy is simply an off-equilibrium threat here. Of course, the central bank must commit to making those purchases in the event they are needed. Secondly, the threat is powerful, in the sense that $\bar{q}$ can be arbitrarily close to zero: central banks need only commit to provide support in extreme future situations to have effects today. This latter characteristic is reminiscent of Obstfeld and Rogoff (1986),

[9] Indeed, if there existed any $q^\dagger \in (0, q^*)$ with $\sigma_q(q^\dagger) = 0$, the drift $\mu_q(q^\dagger) = -(\rho + g) < 0$ ensures that the economy never visits $[q^\dagger, q^*]$ in the stationary distribution. Consequently, it would suffice to consider the region $(0, q^\dagger)$ for a stochastic equilibrium featuring $\sigma_q > 0$. The arguments below would just be applied to this lower region. If policy happened to set $\bar{q} = q^\dagger$, then we would only need apply the arguments from the deterministic case above. And if policy happened to set $\bar{q} > q^\dagger$, then (by path-continuity of q_t) the economy would have already hit $\bar{q}$ before transitioning to this lower region, confirming that $\bar{q}$ is hit in finite time as required by the argument below.

who show how to rule out speculative hyperinflations by committing to a peg in extreme circumstances.

What critically matters for these results is that central bank policies are *aggressive* and *credible*. In particular, everyone understands that central banks will "do whatever it takes" to support asset prices (aggressive), agents believe central banks can actually maintain such price support (credible). Perhaps this is too stark. In the next two sections, we relax policy aggression and credibility and explore which aspects of equilibria survive.

5.3 Less aggressive policy

Now, we assume policy is less aggressive. Instead of supporting asset prices with a guarantee at some barrier $\bar{q}$, we suppose there is a smoother set of purchases and promises. Again, we do not model the exact transmission mechanism, but we suppose this smoother policy translates into a force that *tends to raise asset prices slowly and only on average*.

In terms of modeling, it is simplest to begin with the deterministic equilibria. Assume that policymakers seek to make purchases for $q \leq \bar{q}$ in such a way that μ_q is given by

$$\mu_{q,t} = -(\rho + g) + f(\bar{q} - q_t), \tag{14}$$

for some positive, increasing function $f : \mathbb{R} \mapsto \mathbb{R}_+$ that satisfies $f(x) = 0$ for $x < 0$. Policymakers start providing support at $\bar{q}$ and increase their level of support as q falls further below $\bar{q}$.

No inefficient deterministic equilibrium can survive. Indeed, if $\chi < 1$ so that $r = 0$, then the Euler equation (8) reads $\mu_q = -(\rho + g)$, which contradicts (14) for all $q < \bar{q}$. This is true even for f arbitrarily close to zero, implying a mild requirement to eliminate deterministic inefficiencies.

By contrast, a stochastic sunspot equilibrium can survive. Combining policy (14) and $r = 0$ with Euler equation (8), we see that

$$\sigma_{q,t}^2 = f(\bar{q} - q_t).$$

In other words, the less aggressive policy selects a particular volatility profile. On the bright side, volatility is eliminated for $q > \bar{q}$, since $f = 0$ in that region. Unfortunately, volatility still persists for $q \leq \bar{q}$, and nothing stops equilibrium from visiting this volatile region: if ever $q_t \in (\bar{q}, q^*)$ occurred, the drift $\mu_q = -(\rho + g) < 0$ would push the economy toward $\bar{q}$ until volatility arises, and the economy would reside in that inefficient region permanently.

Generalizing the policy (14) to become "more aggressive" does not necessarily help. In this step, we suppose policymakers understand that a stochastic equilibrium with variance $\sigma_{q,t}^2 = f(\bar{q} - q_t)$ will arise under their less aggressive regime. They attempt to correct this by increasing aggression to

$$\mu_{q,t} = -(\rho + g) + \alpha f(\bar{q} - q_t), \quad \alpha > 1. \tag{15}$$

The drift with $\alpha = 1$ is what arises in the stochastic equilibrium just analyzed in the previous paragraph, so policy is providing additional support relative to that baseline. If we take $\alpha \to \infty$, we recover the maximally aggressive policy (12). Unfortunately, for intermediate α, equilibrium volatility must be given by

$$\sigma_{q,t}^2 = \alpha f(\bar{q} - q_t),$$

so policymakers only succeed in coordinating agents on a more volatile equilibrium!

What's going on here? Attempts to engineer a higher drift μ_q must come from a risk premium in rational, frictionless financial markets. Mechanically, since σ_q comes from agents beliefs and coordination behavior, it is essentially a free variable to provide this risk premium.

From a different perspective, one could think that, holding volatility more fixed, a drop in prices provides such a risk premium. This is difficult to see from the present exercise, but imagine as policymakers increase their aggression α they also reduce their entry point $\bar{q}_\alpha$. Policymakers may think of this as substituting aggression for frequency of intervention. In this case, policy aggression has an ambiguous effect on volatility, but an unambiguous negative effect on asset prices. To see why the effect on volatility is ambiguous, recall that f is an increasing function, i.e., interventions are more aggressive for lower asset prices, which implies that variance $\sigma_q^2 = \alpha f(\bar{q}_\alpha - q)$ is decreasing in q. If the policy entry point $\bar{q}_\alpha$ is decreasing in α, as a way to substitute aggression for intervention frequency, then there are two effects: higher α raises volatility through a direct effect, while lower $\bar{q}_\alpha$ reduces volatility by delaying intervention through the term $f(\bar{q}_\alpha - q)$. Nevertheless, the reduction in $\bar{q}_\alpha$ necessarily lowers average asset prices that prevail in the stochastic equilibrium. If we think this way, policy generates a rise in drift μ_q through a drop in asset price levels, which emphasizes again a sense in which this policy may be counterproductive.

5.4 Policy with partial credibility

Finally, we relax the assumption that unconventional monetary policy is fully credible. To do this, we suppose agents do not believe that policymakers will be successful, at least not with probability 1, in providing the price support $q_t \geq \bar{q}$. Instead, everyone understands that asset purchases may not have the intended buoying effect. What can happen in this world?

To answer this question in an interesting way, we require more specific assumptions about what happens at $\bar{q}$. (For example, we cannot simply model a coin flip, once q_t reaches $\bar{q}$, that determines whether or not policy can support $q_t \geq \bar{q}$; such an assumption would be equivalent to the full-credibility setup and eliminate all inefficient equilibria, essentially because full-credibility emerges as an outcome of the coin-flip with some probability.) In particular, we need a set of assumptions that guarantees credibility is never fully recovered by the central bank.

We assume that, asset purchases succeed with probability $\xi \in (0,1)$, but if asset purchases fail, asset prices drop. Let $q^- < \bar{q}$ denote the post-failure asset price, given attempted intervention at $\bar{q} < q^*$. Let q^+ denote the post-success asset price. For simplicity, we assume (q^-, q^+) are known and non-random. At the intervention point $q = \bar{q}$, the Euler equation is now dominated by the terms involving (q^-, q^+) and reads[10]

$$1 = \xi \frac{\bar{q}}{q^+} + (1-\xi)\frac{\bar{q}}{q^-}.$$

The discussion above implies that we can never have $q^+ = q^- = \bar{q}$. Consequently, given a post-failure asset price $q^- < \bar{q}$, we must have the post-success price

$$q^+ = \frac{\xi \bar{q}}{1 - (1-\xi)\bar{q}/q^-}. \tag{16}$$

[10]To derive this, allow the jumps from q to either q^- or q^+ to take an amount of time $\tau \sim \exp(\lambda)$, and then take $\lambda \to \infty$. The limit is taken because the success/failure of the asset purchase is determined immediately. For fixed λ, the first-order condition for capital portfolio choice is

$$\mu_q + \rho + g - r + \lambda[\xi \frac{n}{n^+}\frac{q^+ - q}{q} + (1-\xi)\frac{n}{n^-}\frac{q^- - q}{q}] = \frac{qk}{n}\sigma_q^2,$$

where $n^+ := n + (q^+ - q)k$ and $n^- := n + (q^- - q)k$ are post-success and post-failure net worths of the agent. The representative agent always holds $n = qk$, so this equation becomes

$$\mu_q + \rho + g - r + \lambda[\xi \frac{q^+ - q}{q^+} + (1-\xi)\frac{q^- - q}{q^-}] = \sigma_q^2.$$

Taking $\lambda \to \infty$, this equation can only hold if the terms in square brackets are identically zero. Thus, we obtain equation (16).

This can only be an equilibrium if $\bar{q} < q^+ \leq q^*$, or equivalently,

$$\frac{(1-\xi)\bar{q}q^*}{q^* - \xi\bar{q}} \leq q^- < \bar{q}. \tag{17}$$

The interval in (17) is non-trivial for any ξ and any $\bar{q} < q^*$. Thus, if agents understand and coordinate on post-failure asset prices jumping to such a q^- satisfying (17), they necessarily expect post-success asset prices to jump up to some $q^+ \in (\bar{q}, q^*]$, given in (16).

This result demonstrates several points. First, partially-credible policy cannot eliminate equilibria that, absent the policy, were deterministically converging to shut-down, $q_T \to 0$. In partially-credible world with otherwise-deterministic asset prices, policy may succeed, but then the game will inevitably begin again in the near future. Eventually, any policy which is not fully credible will fail, and the path towards shut-down will be unabated.

Similarly, partially-credible policy will not eliminate the stochastic sunspot equilibria. In fact, we have shown that such policies necessarily add uncertainty into the economy. If we begin in an inefficient deterministic equilibrium, and policymakers attempt to eliminate it with a non-credible asset-purchase program, the result will be a stochastic equilibrium.

Lack of credibility thus mirrors lack of aggression, in the sense that such policies do actually eliminate pure deterministic inefficiencies, but they do so by adding uncertainty to the economy. One could argue, therefore, that stochastic sunspot equilibria of the type we document are more difficult to kill than multiple deterministic equilibria, as doing so requires a strong form of both aggression and credibility.

6 Conclusion

We have shown that macroeconomies with nominal rigidities—New Keynesian models—may inherently permit sunspot volatility if monetary policy is constrained. The volatility we document is distinct from inflation volatility and self-fulfilling beliefs about inflation; in particular, inflation is always zero in our model. Broadly speaking, our volatility is related to the crux of indeterminacy in New Keynesian models—that current demand depends on future demand—but with the additional subtlety that volatility adds a risk premium that can affect demand. As such, the reasoning behind self-fulfilling equilibria is modified: an inefficient, high-volatility equilibrium can emerge because agents expect volatility to decline and efficiency to rise in the future.

We show formally that unconventional monetary policy (e.g., asset purchases) can alleviate some of the indeterminacies we document. Most obviously, allowing additional policy instruments relaxes constraints imposed by the ZLB. However, we also show that the policy must be both aggressive and credible; without these features, policy becomes ineffective and possibly even counterproductive. In our analysis, we also learn that our volatile sunspot equilibria are somewhat more difficult to eliminate than more conventional types of multiplicity, in the sense that aggressive, credible policies are paramount in fighting volatile equilibria, but less critical for fighting deterministic inefficiencies.

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Appendix:
Fear and Volatility at the Zero Lower Bound
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July 19, 2022

A Additional lemmas

Lemma 2. *Optimal discretionary monetary policy—which maximizes (1) subject to $r_t \geq 0$, $\chi_t \leq 1$, optimal household and firm decisions, and its own future decisions—implements (4).*

Proof of Lemma 2. Optimal discretionary monetary policy seeks to pick a r_t to maximize (1), subject to (2), (3), (7), (8), $\chi_t \leq 1$, the ZLB $r_t \geq 0$, and subject to its own future decisions.

We will discretize the problem to time intervals of length Δ and later take $\Delta \to 0$. Noting that $C_t = \rho q_t K_t$, and using the fact that ρ and the time-path of K_t are exogenous, the time-t household utility is proportional to

$$\begin{aligned}
&\mathbb{E}_t\Big[\int_0^\infty \rho e^{-\rho s}\log(q_{t+s})ds\Big] \\
&\quad\approx \rho\log(q_t)\Delta + \mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s}\log(q_{t+s})ds\Big] \\
&\quad\approx -\rho\Delta\mathbb{E}_t[\log(q_{t+\Delta}) - \log(q_t)] + \underbrace{\mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s}\log(q_{t+s})ds\Big] + \rho\Delta\mathbb{E}_t[\log(q_{t+\Delta})]}_{\text{terms taken as given by discretionary central bank}}.
\end{aligned}$$

The term with brackets underneath is taken as given by the time-t discretionary central bank, because it involves expectations of future variables that the future central bank can influence.

Thus, taking $\Delta \to 0$, the time-t central bank solves

$$\min_{r_t \geq 0} \mathbb{E}_t[d\log(q_t)]$$

subject to the constraints

$$\begin{aligned}
r_t &= \rho + g + \mu_{q,t} - \sigma_{q,t}^2 \\
\mu_{q,t} &= 0 \quad \text{if} \quad q_t = q^* \\
\sigma_{q,t} &= 0 \quad \text{if} \quad q_t = q^*.
\end{aligned}$$

Note that $\sigma_{q,t}$ is independent of policy when $q_t \neq q^*$. There are two cases. If $q_t = q^*$, then the constraints imply that $r_t = \rho + g$. If $q_t \neq q^*$, we may substitute the dynamics of $\log(q_t)$ (using Itô's lemma and replacing μ_q from the first constraint) to re-write the problem as

$$\min_{r_t \geq 0} [r_t - \rho - g + \frac{1}{2}\sigma_{q,t}^2].$$

Since σ_q is taken as given, the optimal solution is $r_t = 0$. Thus, the discretionary central bank optimally sets

$$r_t = (\rho + g)\mathbf{1}_{\{q_t = q^*\}} = (\rho + g)\mathbf{1}_{\{\chi_t = 1\}}.$$

In other words, the complementary slackness condition $(1 - \chi_t) r_t = 0$ holds, which together with $r_t \geq 0$ implies (4). $\square$

Lemma 3 (Boundary classification)**.** *Let $(x_t)_{t\geq 0}$ be a one-dimensional diffusion satisfying $dx_t/x_t = [-(\rho + g) + v(x_t)]dt + \sqrt{v(x_t)}dZ_t$ with $v(\cdot)$ strictly positive on $(0, q^*)$. If v is Lipschitz, bounded, and bounded away from zero, then $x_t > 0$ for all t. Furthermore, if $v(0) > 2(\rho + g)$, then a stationary distribution exists for x_t and $\lim_{t\to\infty} x_t > 0$ almost-surely.*

B Adding fundamental uncertainty

In this section, we add fundamental risk to verify the robustness of our results. Some additional insights emerge. First, the efficient equilibrium is eliminated with high enough fundamental risk, suggesting that even more sophisticated monetary policy (e.g., with some commitment power) cannot necessarily eliminate self-fulfilling volatility in truly adverse states. Second, the source of volatility becomes indeterminate: fundamental shocks could be amplified or volatility could be attached to sunspot shocks.

Suppose aggregate capital has some fundamental risk s. Assume additionally that at a random time $\tau \sim \exp(\lambda)$, this fundamental risk reverts to zero permanently. At that time, we will suppose the economy transitions into the efficient equilibrium forever after (i.e., $\chi_t = 1$, $q_t = q^*$, and $r_t = \rho + g$ for all $t \geq \tau$). Prior to time τ, the capital evolution equation (2) is modified to

$$dK_t = K_t[g dt + s dB_t], \quad \text{for} \quad t < \tau, \tag{2'}$$

where B is a standard Brownian motion independent of Z. We will assume that s^2 is sufficiently high, so that the efficient equilibrium ceases to exist, but not so high as to prevent any equilibrium.[11]

Assumption 1. *Assume $\rho + g < s^2 < \rho + g + \lambda$.*

In this extension, volatility can either be connected to fundamentals, with possible amplification so that endogenous fluctuations are greater than the fundamental shock, or related to sentiments. Mathematically, prior to the transition to efficiency, the dynamics of q in (3) are now modified to read

$$dq_t = q_t\left[\mu_{q,t}dt + \sigma_{q,t} \cdot \begin{pmatrix} dB_t \\ dZ_t \end{pmatrix}\right], \quad \text{for} \quad t < \tau. \tag{3'}$$

We continue to assume monetary policy attempts to implement (4). The price-output link (7) also still holds in this setting, but the asset-pricing equation (8) is modified to read

$$r = \rho + g + \mu_q + s\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \sigma_q - \left|s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sigma_q\right|^2 + \lambda\frac{q^* - q}{q^*}. \tag{6'}$$

Similar to the baseline model, an equilibrium satisfies (4), (7), (6′), and $q_t > 0$ for all t. The fundamental equilibria, described below, include the one studied in Caballero and Simsek (2020c), as well as others that diverge asymptotically to an asymptotic shut-down (as in Proposition 1).

Proposition 4 (Fundamental equilibria)**.** *Under Assumption 1, there exists an equilibrium with $\mu_q = 0$ and $\sigma_q = 0$ at all times. When exogenous volatility is high (i.e., when $t < \tau$), this equilibrium features $\chi < 1$, $r = 0$, and $q = q^{ss} := q^*(\rho + g + \lambda - s^2)/\lambda$. Among equilibria having $\mu_q = 0$ and $\sigma_q = 0$, this equilibrium is unique. There cannot be any equilibrium featuring $q_t > q^{ss}$ when $t < \tau$. Finally, there exist a continuum of equilibria featuring $q_0 < q^{ss}$, in which* $\lim_{T\to\infty} q_T = 0$.

[11] If $s^2 > \rho + g + \lambda$, no equilibrium could exist. A proof sketch of this argument is as follows. Without endogenous volatility ($\sigma_q = 0$), the drift of q would be $\mu_q = -(\rho + g) + s^2 - \lambda(q^* - q)/q^* > -(\rho + g + \lambda) + s^2 > 0$ in this situation, so that q_t would eventually attain q^*, if unabated. With endogenous volatility ($\sigma_q \neq 0$), q_t would eventually attain q^* simply due to shocks. Either way, equilibrium requires efficiency at some point in time. But the efficient equilibrium cannot be supported for any amount of time, because $s^2 > \rho + g$, so another force must arise to prevent q_t from ever attaining q^*. In particular, there must be a predictable negative movement in q_t, which cannot be absolutely continuous with respect to time, either at or before hitting q^* (for example, a reflecting boundary at $q^* - \epsilon$). In such case, no-arbitrage requires that the riskless bond have a singular return equal to $r_t dt - dL_t$, where L is the singular process keeping $q_t \leq q^*$ (see Karatzas and Shreve (1998), Appendix B). The ZLB disallows this riskless bond return, and thus no equilibrium can exist.

Proof of Proposition 4. Plug in $r = 0$, $\mu_q = 0$, and $\sigma_q = 0$ into (6′) to solve uniquely for $q = q^{ss}$ under volatility s. Note that $q^{ss} < q^*$, so $\chi < 1$ and thus (4) holds. Uniqueness within the class of equilibria having $\mu_q = 0$ and $\sigma_q = 0$ can be established by using $s^2 > \rho + g$ and equation (6′) to show that $\chi = 1$ is impossible. The fact that $q > q^{ss}$ is not possible is due to the same logic as in footnote 11: in such a candidate equilibrium, $q \to q^*$ in finite time. But τ is exponentially distributed, so the efficient equilibrium is ruled out (by the parameter restriction $s^2 > \rho + g$ in Assumption 1) for an arbitrarily long amount of time. Finally, the fact that $q < q^{ss}$ is possible follows the same construction as in Proposition 1: $\mu_q < 0$ for all $q < q^{ss}$, so that $q_T \to 0$ asymptotically is required, but nothing rules this out as it takes infinitely long. □

As in the baseline model, there are also sunspot equilibria in this setting, with the following properties. First, at the ZLB, these equilibria can feature excess volatility, and this level of price volatility is essentially arbitrary (i.e., σ_q is essentially arbitrary). Second, when efficiency fails, an arbitrary fraction of total return variance $|s\binom{1}{0} + \sigma_q|^2$ can be connected to fundamental shocks versus sentiment shocks. The reasoning for this latter indeterminacy is that agents only care about total capital return variance when trading capital; the source of the shocks is irrelevant. Finally, the inefficiency in the sunspot equilibrium is worse than the fundamental equilibrium of Proposition 4, in the sense that q and χ are always lower. One can think of this situation as a *volatility trap*: beliefs about endogenous volatility will produce stability in the equilibrium dynamics, which is enough to keep the endogenous volatility around until exogenous risk disappears. We formalize this discussion in the following theorem, a generalization of Theorem 1. In this theorem, the function v corresponds to total return variance and f to the variance share associated to the fundamental shock.

Theorem 2 (Sunspot equilibria)**.** *Let Assumption 1 hold. Let $f, v : \mathbb{R} \mapsto \mathbb{R}_+$ be any two Lipschitz continuous functions satisfying $v \geq 0$, $0 \leq f \leq 1$, and boundary conditions*

(i) $v(0) > \rho + g + \lambda + s\sqrt{f(0)v(0)}$;

(ii) $v(\frac{q^{ss}}{M}) = s^2$ *and* $f(\frac{q^{ss}}{M}) = 1$, *where* $q^{ss} := q^*(\rho + g + \lambda - s^2)/\lambda$ *and* $M > 1$ *is any number.*

An equilibrium exists with $r_t = 0$, $\chi_t < 1$, and volatility

$$\sigma_{q,t} = \left[\frac{\sqrt{f(q_t)v(q_t)} - s}{\sqrt{(1 - f(q_t))v(q_t)}} \right], \quad \textit{for} \quad t < \tau.$$

In this equilibrium, q_t is always strictly below the fundamental equilibrium price q^{ss}, for $t < \tau$.

Proof of Theorem 2. Substitute all the proposed equilibrium objects into (6′) to find the price drift

$$\mu_{q,t} = s^2 - \left(\rho + g + s\sqrt{f(q_t)v(q_t)}\right) + v(q_t) - \lambda(1 - q_t/q^*), \quad \text{for} \quad t < \tau.$$

These dynamics must prevent q_t from ever reaching zero, which is guaranteed by condition (i). Indeed, this condition implies that $\lim_{q\to 0}\mu_q(q;s) > 0$, so that q_t behaves locally near zero as a geometric Brownian motion with positive drift (the formal argument is identical to that of Lemma 3, which was used in the proof of Theorem 1).

The dynamics also must prevent q_t from ever reaching q^{ss}, which is guaranteed by condition (ii). Indeed, this condition implies that $\lim_{q\to q^{ss}/M}\sigma_q(q;s) = 0$ as well as $\lim_{q\to q^{ss}/M}\mu_q(q;s) = s^2 - \rho - g - \lambda + \lambda\frac{q^{ss}}{Mq^*}$. By plugging in q^{ss}, we see that the drift expression is negative at this boundary for any $M > 1$. Together with the vanishing volatility (and the Lipschitz continuity assumption on σ_q), this implies that q_t cannot reach q^{ss}/M (hence it cannot reach $q^{ss} > q^{ss}/M$).

As a result of these dynamics, any initial price $q_0 \in (0, \frac{q^{ss}}{M})$ is consistent with equilibrium. Thus, q_t is always below q^{ss} for $t < \tau$, as desired. Since $q^{ss} < q^*$, this also verifies that $\chi < 1$ and hence $r = 0$ by equation (4). $\square$